2.3 Viscous Fluids

2.3.1 A General Class of Viscous Fluid

In a viscous fluid none of the applied work is stored, it is all dissipated, so that in the basic equation for isothermal deformations, 1.6.24,

$$\sigma : d = \dot{\Psi} + \Phi$$  \hspace{1cm} (2.3.1)

the free energy rate \( \dot{\Psi} = 0 \). As an example, in a simple shear flow (see the example at the end of Part III, §2.5.3) one can take the dissipation function to be

$$\Phi = 2\mu \dot{\gamma}^2$$  \hspace{1cm} (2.3.2)

where \( \mu \) is the fluid viscosity, and \( \dot{\gamma} = \frac{1}{2}(\partial v_1 / \partial x_2) \) is the shear strain rate. Note that this dissipation satisfies the required inequality \( \Phi \geq 0 \). Since for such a flow the stress power is \( \sigma : d = \tau \dot{\gamma} \), one arrives at the standard constitutive equation for a Newtonian viscous fluid:

$$\tau = 2\mu \dot{\gamma} = \mu \frac{\partial v_1}{\partial x_2}$$  \hspace{1cm} (2.3.3)

In the general 3- dimensional situation, one can generate viscous fluid models by assuming the dissipation is a function of the rate of deformation:

$$\sigma : d = \Phi(d)$$  \hspace{1cm} (2.3.4)

Homogeneous Functions

One cannot proceed with 2.3.4 as for the elastic material because the \textit{dissipation rate function is not the time derivative of a state function}. Instead, to generate a class of fluids, one can assume that the dissipation is a \textbf{homogeneous function} of degree \( n \) in \( d \), by which is meant that

$$\Phi(kd) = k^n \phi(d)$$  \hspace{1cm} (2.3.5)

for scalar \( k \). Advantage can then be taken of \textbf{Euler’s theorem}, which states that (see the Appendix to this Chapter, §2.A), for any homogeneous function of degree \( n \),

$$\frac{\partial \Phi}{\partial d_y} d_y = n\Phi , \hspace{1cm} \frac{\partial \Phi(d)}{\partial d} : d = n\Phi(d)$$  \hspace{1cm} (2.3.6)

From 2.3.4,

$$\sigma : d = \frac{1}{n} \frac{\partial \Phi(d)}{\partial d} : d$$  \hspace{1cm} (2.3.7)
or

\[
\left( \sigma - \frac{1}{n} \frac{\partial \Phi(d)}{\partial d} \right) : d = 0
\]

(2.3.8)

The Orthogonality Hypothesis

One cannot simply “cancel” the \( d \)’s in Eqn. 2.3.8. It is not the same situation as the one used in elasticity theory, where the strain rates were cancelled, e.g. 2.2.9-10; in that case the free energy did not depend on \( \dot{\varepsilon} \). Here, however, the dissipation function does depend on \( d \).

Eqn. 2.3.8 can be interpreted geometrically. With the double contraction playing the same role for tensors as the dot product does for vectors, one can say that the tensor \( d \) is orthogonal to the tensor \( \sigma - \left( \partial \Phi(d) / \partial d \right) / n \), Fig. 2.3.1. If this latter tensor was orthogonal to all possible \( d \) (and was independent of \( d \)) then it would have to be zero. However, as one changes \( d \), one changes \( \sigma - \left( \partial \Phi(d) / \partial d \right) / n \).

\[\sigma - \left( \partial \Phi(d) / \partial d \right) / n\]

Figure 2.3.1: Geometric representation of Eqn. 2.3.8

In thermomechanics, one almost universally assumes that the \( d \) in Eqn. 2.3.8 can indeed be “cancelled”, and that similar equations which arise in thermomechanics can be dealt with in the same manner (see later). This statement is known as the (Ziegler’s) Orthogonality Hypothesis. It is a constitutive hypothesis, in that it restricts the class of viscous fluids to ones for which \( \sigma - \left( \partial \Phi(d) / \partial d \right) / n \) is always orthogonal to \( d \). However, the hypothesis includes the vast majority of material models. Identifying the precise implications of this hypothesis are a matter of current research.

Thus, one has the constitutive law

\[\sigma = \frac{1}{n} \frac{\partial \Phi(d)}{\partial d}\]

(2.3.9)
Maximising the Dissipation

The constitutive equation 2.3.9 was deduced from 2.3.7-8 using the orthogonality hypothesis. It can be derived in a different way, using what is called the principle of maximum entropy production, which can be stated here as (Srinivasa, 2001, Rajagopal and Srinivasa, 2003):

The rate of deformation $d$ is that which maximises the dissipation $\Phi(d)$, subject to the “constraint” 2.3.4, $\Phi = \sigma : d$.

To apply this principle, one can use the Method of Lagrange Multipliers. Then one must maximise the new function

$$\Phi^*(d) = \Phi(d) + \lambda (\sigma : d - \Phi(d))$$  \hspace{1cm} (2.3.10)

where $\lambda$ is a Lagrange multiplier. Setting

$$\frac{\partial \Phi^*(d)}{\partial d} = \frac{\partial \Phi(d)}{\partial d} + \lambda \left( \sigma - \frac{\partial \Phi(d)}{\partial d} \right) = 0$$  \hspace{1cm} (2.3.11)

then leads to

$$\sigma = \frac{\lambda - 1}{\lambda} \frac{\partial \Phi(d)}{\partial d}$$  \hspace{1cm} (2.3.12)

For $\Phi(d)$ homogeneous of degree $n$, from 2.3.7 and 2.3.12,

$$\frac{\lambda - 1}{\lambda} \frac{\partial \Phi(d)}{\partial d} : d = \frac{1}{n} \frac{\partial \Phi(d)}{\partial d} : d$$  \hspace{1cm} (2.3.13)

and so the Lagrange multiplier is related to the degree of the homogeneous function and 2.3.12 again leads to Eqn. 2.3.9.

Note that the application of the method of Lagrange multipliers only indicates that Eqn. 2.3.9 leads to a stationary value of the dissipation. It is assumed that the dissipation function is convex so that dissipation (which must be greater than or equal to zero) is actually maximised.

2.3.2 The Linear Fluid

Let the rate of dissipation function be the following quadratic function of the basic invariants $D_1, D_2, D_3$ of $d$,

$$\Phi = \lambda D_1^2 + 2\mu D_2$$  \hspace{1cm} (2.3.14)
where $\mu$ (the viscosity) and $\lambda$ are material parameters. Since $D_1^2 = (\text{tr} d)^2$ and $D_2 = \text{tr} d^2$, this dissipation function is homogeneous of degree 2. It follows that

$$\sigma = \frac{1}{2} \varepsilon \frac{\partial \Phi}{\partial \sigma} = \lambda D_1 I + 2\mu d$$

(2.3.15)

In the case of a liquid there is no volume change, so $D_1 \equiv \text{div} \ v$ is zero, and the mean pressure is not determined by the constitutive law. One has

$$\sigma' \equiv \sigma + p I = 2\mu d$$

(2.3.16)

where $\sigma'$ is the deviatoric stress tensor. These are the equations for the standard linear **Newtonian** liquid.

### 2.3.3 The Reiner-Rivlin Fluid

All fluids are isotropic, in which case the rate of deformation is a function of the invariants, $d = d(D_1, D_2, D_3)$, and, analogous to the isotropic Hyperelasticity of Part III, \$4.4, the general form of the constitutive law for a fluid can be written

$$\sigma = \frac{1}{n} \left( \frac{\partial \Phi}{\partial D_1} \ I + 2 \frac{\partial \Phi}{\partial D_2} \ d + 3 \frac{\partial \Phi}{\partial D_3} \ d^2 \right)$$

(2.3.17)

A general “fluid” satisfying such an equation is known as a **Reiner-Rivlin** fluid. It is a reasonable model for the steady state creep of metals, ice and other crystalline materials, but most departures from standard linear Newtonian behavior involve memory effects, which require viscoelastic models.