4.5 Material Anisotropy

4.5.1 Material Symmetry

The isotropic material was defined as one whose material response was unaffected by rigid body rotations of the reference configuration. Other material symmetries are possible; to generalise the notion, instead of considering orthogonal transformations, consider an arbitrary deformation $\mathbf{F}_0$ of the reference configuration $S_0$ bringing it to a new configuration $S^\circ$, Fig. 4.5.1 (compare with the isotropic case, Fig. 2.8.6).

Figure 4.5.1: a deformation of the reference configuration

Considering the Cauchy-elastic material, if the deformation $\mathbf{F}_0$ has no effect on the response of the material, then

$$\sigma(\mathbf{F}^\circ) = \sigma(\mathbf{F}) \rightarrow \sigma(\mathbf{FF}_0^{-1}) = \sigma(\mathbf{F})$$ (4.5.1)

When $\mathbf{F}_0 = \mathbf{Q}$, one has the isotropic material. Setting $\mathbf{G} = \mathbf{F}_0^{-1}$, 4.5.1 can be cast in the most usual form:

$$\sigma(\mathbf{F}) = \sigma(\mathbf{FG})$$ (4.5.2)

Note that the restriction $\det \mathbf{G} = \pm 1$ is assumed, since otherwise arbitrary dilatations could occur with no change in material response, which seems physically unreasonable.

Note that the set of all tensors $\mathbf{G}$ which satisfy 4.5.2 forms a group (see the Appendix to this Chapter, §4.A.2) and hence is called the symmetry group of the material (with respect to the configuration $S_0$).

Apart from isotropy, the two most important practical cases of material symmetry are transverse isotropy and orthotropy.
4.5.2 Transverse Isotropy

Consider first the transversely isotropic material. Such a material has a single preferred direction, defined by a unit vector \( \mathbf{a}_0 \) in the reference configuration. Such a vector is illustrated in Fig. 4.5.2, showing also the unit vectors \( \hat{n}_2, \hat{n}_3 \) completing an orthonormal set. The symmetry group of the transversely isotropic material is the set of orthogonal tensors \( \mathbf{Q} \) which transform the set \( \{\mathbf{a}_0, \mathbf{n}_2, \mathbf{n}_3\} \) into the new orthonormal set \( \{\pm \mathbf{a}_0, \mathbf{n}'_2, \mathbf{n}'_3\} \). In particular,

\[
\mathbf{Q} \mathbf{a}_0 = \pm \mathbf{a}_0 \quad (4.5.3)
\]

In order to ensure that the sense of \( \mathbf{Q} \mathbf{a}_0 \) is immaterial, it is best to introduce the structural tensor \( \mathbf{a}_0 \otimes \mathbf{a}_0 \), which transforms as the axes change according to

\[
\mathbf{Q} \mathbf{a}_0 \otimes \mathbf{Q} \mathbf{a}_0 = \pm \mathbf{a}_0 \otimes \pm \mathbf{a}_0 \quad (4.5.4)
\]

or

\[
\mathbf{Q}(\mathbf{a}_0 \otimes \mathbf{a}_0)\mathbf{Q}^T = \mathbf{a}_0 \otimes \mathbf{a}_0 \quad (4.5.5)
\]

![Figure 4.5.2: an orthonormal set of vectors](image)

The strain energy can now be taken to be a function of \( \mathbf{C} \), as in the isotropic case, and \( \mathbf{a}_0 \otimes \mathbf{a}_0 \), which characterises the structure of the material:

\[
W = W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) \quad (4.5.6)
\]

Allowing for transformations of the undeformed configuration,

\[
W(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = S(\mathbf{Q} \mathbf{C} \mathbf{Q}^T, \mathbf{Q} \mathbf{a}_0 \otimes \mathbf{a}_0 \mathbf{Q}^T) \quad (4.5.7)
\]

with \( \mathbf{Q} \) here restricted to the symmetry group defined by 4.5.3. Then \( W \) is an isotropic scalar function of two symmetric tensors and so, from Table 4.A.1, takes the form
\[ W = W\left( \text{tr} C, \text{tr} C^2, \text{tr} C^3, \text{tr} (a_0 \otimes a_0), \text{tr} (a_0 \otimes a_0)^2, \text{tr} (a_0 \otimes a_0)^3, \text{tr} C(a_0 \otimes a_0), \text{tr} C^2(a_0 \otimes a_0), \text{tr} C(a_0 \otimes a_0)^2, \text{tr} C^2(a_0 \otimes a_0)^2 \right) \] (4.5.8)

Since \{ ▲ Problem 1 \}
\[ \text{tr} (a \otimes a) = 1, \quad \text{tr} (C(a \otimes a)) = a C a, \quad \text{tr} (C^2(a \otimes a)) = a C^2 a \] (4.5.9)

one arrives at the representation
\[ W = W(I_1(C), I_2(C), I_3(C), I_4(C, a_0), I_5(C, a_0)) \] (4.5.10)

where the fourth and fifth scalar (pseudo-) invariants \( I_4, I_5 \) are defined by
\[ I_4 = a C a, \quad I_5 = a C^2 a \] (4.5.11)

Note also that, from the definition of the stretch, Eqn. 2.2.17,
\[ I_4 = a_0 C a_0 = \lambda_a^2 \] (4.5.12)

where \( \lambda_a \) is the stretch of the unit line element \( a_0 \).

If the preferred direction is \( e_3 \), then the fourth and fifth invariants in terms of components are
\[ I_4 = a_0 C a_0 = C_{33} \]
\[ I_5 = a_0 C^2 a_0 = C_{13}^2 + C_{23}^2 + C_{33}^2 \] (4.5.13)

in which case the five invariants can be taken as \{ \( I_1, I_2, I_3, C_{33}, C_{13}^2 + C_{23}^2 \) \}.

Using the relations \{ ▲ Problem 2 \}
\[ \frac{\partial I_4}{\partial C} = a_0 \otimes a_0, \quad \frac{\partial I_5}{\partial C} = a_0 \otimes C a_0 + a_0 C \otimes a_0 \] (4.5.14)

the PK2 stresses for a hyperelastic material are then
\[ S = 2 \sum_{i=1}^{5} \frac{\partial W(C, a_0)}{\partial I_i} \frac{\partial I_i}{\partial C} \]
\[ = 2 \left[ \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right] - \frac{\partial W}{\partial I_2} C + I_3 \frac{\partial W}{\partial I_3} C^{-1} \]
\[ + \frac{\partial W}{\partial I_4} a_0 \otimes a_0 + \frac{\partial W}{\partial I_5} (a_0 \otimes C a_0 + a_0 C \otimes a_0) \] (4.5.15)
Let \( \mathbf{a} \) be a unit vector in the current configuration, in the direction of \( \mathbf{F}_a \), that is,

\[
\lambda_a \mathbf{a} = \mathbf{F}_a \tag{4.5.16}
\]

Then, using Eqn. 3.5.7, \( \mathbf{R} = J^{-1} \mathbf{F} \mathbf{S}^T \mathbf{F}^T \), with \( \mathbf{F} \mathbf{I}^T = \mathbf{b} \), \( \mathbf{F} \mathbf{C}^T = \mathbf{b}^2 \) (see Eqn.2.2.14), \( \mathbf{F}^{-1} \mathbf{F}^T = \mathbf{I} \) and noting that \( \mathbf{C} \) and \( \mathbf{b} \) have the same principal invariants, 4.5.13 becomes

\[
\mathbf{R} = 2J^{-1} \left[ I_3 \frac{\partial W}{\partial I_3} \mathbf{I} + \left( \frac{\partial W}{\partial I_1} + I_3 \frac{\partial W}{\partial I_2} \right) \mathbf{b} - \frac{\partial W}{\partial I_2} \mathbf{b}^2 \right. \\
\left. + I_4 \frac{\partial W}{\partial I_4} \mathbf{a} \otimes \mathbf{a} + I_4 \frac{\partial W}{\partial I_5} (\mathbf{a} \otimes \mathbf{b} + \mathbf{ab} \otimes \mathbf{a}) \right] \tag{4.5.17}
\]

Using the Cayley-Hamilton theorem allows one to re-write the Cauchy stress as

\[
\mathbf{R} = 2J^{-1} \left[ I_2 \frac{\partial W}{\partial I_2} + I_3 \frac{\partial W}{\partial I_3} \right] \mathbf{I} + \frac{\partial W}{\partial I_1} \mathbf{b} - I_3 \frac{\partial W}{\partial I_2} \mathbf{b}^{-1} \\
+ I_4 \frac{\partial W}{\partial I_4} \mathbf{a} \otimes \mathbf{a} + I_4 \frac{\partial W}{\partial I_5} (\mathbf{a} \otimes \mathbf{b} + \mathbf{ab} \otimes \mathbf{a}) \tag{4.5.18}
\]

Expressing the scalar invariants in terms of \( \mathbf{b} \) rather than \( \mathbf{C} \), the coefficients of the tensors in 4.5.17-18 are functions of the set

\[
\{ \text{tr} \mathbf{b}, \text{tr} \mathbf{b}^2, \text{tr} \mathbf{b}^3, \lambda_a^2 \mathbf{a} \otimes \mathbf{a}, \lambda_a^2 \mathbf{a} \mathbf{b} \mathbf{a} \} \tag{4.5.19}
\]

**Transversely Isotropic Materials with Constraints**

For an incompressible material, \( I_3 = 1 \), and, analogous to 4.2.40, the strain energy takes the form

\[
W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)) - \frac{1}{2} p(I_3 - 1) \tag{4.5.20}
\]

For a material which is inextensible in the direction of \( \mathbf{a}_0 \), from 4.5.13, \( I_4 = 0 \), and the strain energy takes the form

\[
W(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0)) - \frac{1}{2} q(I_4 - 1) \tag{4.5.21}
\]
4.5.3 Orthotropy

Consider now a material which is dependent on two characteristic directions, \( \mathbf{a}_o \) and \( \mathbf{b}_o \); again the sense of these directions is immaterial. The strain energy is now of the form

\[
W = W(\mathbf{C}, \mathbf{a}_o \otimes \mathbf{a}_o, \mathbf{b}_o \otimes \mathbf{b}_o)
\]

As isotropic scalar function of three symmetric tensors depends on the following traces (see Table 4.A.1)

\[
\begin{align*}
&\text{tr} \mathbf{C}, \text{ tr} \mathbf{C}^2, \text{ tr} \mathbf{C}^3, \\
&\text{tr}(\mathbf{a}_o \otimes \mathbf{a}_o), \text{ tr}(\mathbf{a}_o \otimes \mathbf{a}_o)^2, \text{ tr}(\mathbf{a}_o \otimes \mathbf{a}_o)^3, \text{ tr}(\mathbf{b}_o \otimes \mathbf{b}_o), \text{ tr}(\mathbf{b}_o \otimes \mathbf{b}_o)^2, \text{ tr}(\mathbf{b}_o \otimes \mathbf{b}_o)^3 \\
&\text{tr} \mathbf{C}(\mathbf{a}_o \otimes \mathbf{a}_o), \text{ tr} \mathbf{C}^2(\mathbf{a}_o \otimes \mathbf{a}_o), \text{ tr} \mathbf{C}(\mathbf{a}_o \otimes \mathbf{a}_o)^2, \text{ tr} \mathbf{C}^2(\mathbf{a}_o \otimes \mathbf{a}_o)^3 \\
&\text{tr} \mathbf{C}(\mathbf{b}_o \otimes \mathbf{b}_o), \text{ tr} \mathbf{C}^2(\mathbf{b}_o \otimes \mathbf{b}_o), \text{ tr} \mathbf{C}(\mathbf{b}_o \otimes \mathbf{b}_o)^2, \text{ tr} \mathbf{C}^2(\mathbf{b}_o \otimes \mathbf{b}_o)^3 \\
&\text{tr}(\mathbf{a}_o \otimes \mathbf{a}_o)(\mathbf{b}_o \otimes \mathbf{b}_o), \text{ tr}(\mathbf{a}_o \otimes \mathbf{a}_o)^2(\mathbf{b}_o \otimes \mathbf{b}_o), \\
&\text{tr}(\mathbf{a}_o \otimes \mathbf{a}_o)(\mathbf{b}_o \otimes \mathbf{b}_o)^2, \text{ tr}(\mathbf{a}_o \otimes \mathbf{a}_o)^2(\mathbf{b}_o \otimes \mathbf{b}_o)^2 \\
&\text{tr}(\mathbf{C}(\mathbf{a}_o \otimes \mathbf{a}_o)(\mathbf{b}_o \otimes \mathbf{b}_o))
\end{align*}
\]

Using 4.5.9 (see Eqn. 1.9.10e) this reduces to the set of nine invariants

\[
\begin{align*}
&\text{tr} \mathbf{C}, \text{ tr} \mathbf{C}^2, \text{ tr} \mathbf{C}^3, \\
&\mathbf{a}_o \mathbf{C}_a \mathbf{a}_o, \mathbf{a}_o \mathbf{C}_a^2 \mathbf{a}_o, \mathbf{b}_o \mathbf{C}_b \mathbf{b}_o, \mathbf{b}_o \mathbf{C}_b^2 \mathbf{b}_o, (\mathbf{a}_o \cdot \mathbf{b}_o)^2, (\mathbf{a}_o \cdot \mathbf{b}_o)\mathbf{a}_o \mathbf{C}_b \mathbf{b}_o
\end{align*}
\]

with \( \mathbf{a}_o \mathbf{C}_b \mathbf{b}_o = \mathbf{b}_o \mathbf{C}_a \mathbf{a}_o \). The term \( \mathbf{a}_o \cdot \mathbf{b}_o \) is the cosine of the angle between the two characteristic directions; this does not change during the deformation and so this term can be omitted, leaving

\[
\begin{align*}
&\text{tr} \mathbf{C}, \text{ tr} \mathbf{C}^2, \text{ tr} \mathbf{C}^3, \mathbf{a}_o \mathbf{C}_a \mathbf{a}_o, \mathbf{a}_o \mathbf{C}_a^2 \mathbf{a}_o, \mathbf{b}_o \mathbf{C}_b \mathbf{b}_o, \mathbf{b}_o \mathbf{C}_b^2 \mathbf{b}_o, (\mathbf{a}_o \cdot \mathbf{b}_o)\mathbf{a}_o \mathbf{C}_b \mathbf{b}_o
\end{align*}
\]

An orthotropic material is one for which \( \mathbf{a}_o \) and \( \mathbf{b}_o \) are perpendicular, \( \mathbf{a}_o \cdot \mathbf{b}_o = 0 \), making the last term here zero. This also then defines a third preferred direction, \( \mathbf{c}_o \), orthogonal to both \( \mathbf{a}_o \) and \( \mathbf{b}_o \), which introduces extra terms \( \mathbf{c}_o \mathbf{C}_c \mathbf{c}_c \mathbf{c}_o \). But

\[
\begin{align*}
\text{tr} \mathbf{C} &= \mathbf{a}_o \mathbf{C}_a \mathbf{a}_o + \mathbf{b}_o \mathbf{C}_b \mathbf{b}_o + \mathbf{c}_o \mathbf{C}_c \mathbf{c}_c \\
\text{tr} \mathbf{C}^2 &= \mathbf{a}_o \mathbf{C}_a^2 \mathbf{a}_o + \mathbf{b}_o \mathbf{C}_b^2 \mathbf{b}_o + \mathbf{c}_o \mathbf{C}_c^2 \mathbf{c}_c
\end{align*}
\]

so that \( \mathbf{c}_o \mathbf{C}_c \mathbf{c}_c \mathbf{c}_c \) are redundant. Finally, the strain energy is of the form

\[
W = W\left(\text{tr} \mathbf{C}, \text{ tr} \mathbf{C}^2, \text{ tr} \mathbf{C}^3, \mathbf{a}_o \mathbf{C}_a \mathbf{a}_o, \mathbf{a}_o \mathbf{C}_a^2 \mathbf{a}_o, \mathbf{b}_o \mathbf{C}_b \mathbf{b}_o, \mathbf{b}_o \mathbf{C}_b^2 \mathbf{b}_o\right)
\]
As before, the stresses in a hyperelastic material can now be obtained by differentiation.

### 4.5.4 Problems

1. Show that, for unit vector $a$,
   
   (i) $(a \otimes a)^2 = a \otimes a$,
   
   (ii) $\text{tr}(a \otimes a) = 1$
   
   (iii) $\text{tr}(C(a \otimes a)) = aCa$
   
   (iv) $\text{tr}(C^2(a \otimes a)) = aC^2a$

2. Show that
   
   $$\frac{\partial I_4}{\partial C} = a_0 \otimes a_0, \quad \frac{\partial I_5}{\partial C} = a_0 \otimes Ca_0 + a_0 C \otimes a_0$$

3. Show that
   
   (i) $F(a_0 \otimes a_0)F^T = \lambda^2 a \otimes a$
   
   (ii) $F(a_0 \otimes Ca_0)F^T = \lambda^2 a \otimes ba$

   For (ii), it might help to note the following relations (for vector $b$ and second-order tensors $A, B$):

   $$(Ab)B \neq A(bb)$$

   $$(AB)b = A(Bb)$$

   $$(A^Tb)A^T = (AA^T)b$$