3.4 Properties of the Stress Tensor

3.4.1 Stress Transformation

Let the components of the Cauchy stress tensor in a coordinate system with base vectors $\mathbf{e}_i$ be σ_{ij}. The components in a second coordinate system with base vectors $\mathbf{e}'_j$, σ'_{ij}, are given by the tensor transformation rule 1.10.5:

$$\sigma'_{ij} = Q_{pi} Q_{qj} \sigma_{pq} \tag{3.4.1}$$

where Q_{ij} are the direction cosines, $Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$.

Isotropic State of Stress

Suppose the state of stress in a body is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$$

One finds that the application of the tensor transformation rule yields the very same components no matter what the coordinate system. This is termed an **isotropic** state of stress, or a **spherical** state of stress (see §1.13.3). One example of isotropic stress is the stress arising in fluid at rest, which cannot support shear stress, in which case

$$\boldsymbol{\sigma} = -p\mathbf{I} \tag{3.4.2}$$

where the scalar *p* is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic** state of stress.

A note on the Transformation Formula

Using the vector transformation rule 1.5.5, the traction and normal transform according to $[\mathbf{t}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{t}]$, $[\mathbf{n}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$. Also, Cauchy's law transforms according to $[\mathbf{t}'] = [\boldsymbol{\sigma}'][\mathbf{n}']$ which can be written as $[\mathbf{Q}^{\mathrm{T}}][\mathbf{t}] = [\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{n}]$, so that, pre-multiplying by $[\mathbf{Q}]$, and since $[\mathbf{Q}]$ is orthogonal, $[\mathbf{t}] = \{[\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]\}[\mathbf{n}]$, so $[\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}]$, which is the inverse tensor transformation rule 1.13.6a, showing the internal consistency of the theory.

In Part I, Newton's law was applied to a material element to derive the two-dimensional stress transformation equations, Eqn. 3.4.7 of Part I. Cauchy's law was proved in a similar way, using the principle of momentum. In fact, Cauchy's law and the stress transformation equations are equivalent. Given the stress components in one coordinate system, the stress transformation equations give the components in a new coordinate system; particularising this, they give the stress components, and thus the traction vector, acting on new surfaces, oriented in some way with respect to the original axes, which is what Cauchy's law does.

3.4.2 Principal Stresses

Since the stress $\boldsymbol{\sigma}$ is a symmetric tensor, it has three real eigenvalues $\sigma_1, \sigma_2, \sigma_3$, called **principal stresses**, and three corresponding orthonormal eigenvectors called **principal directions**. The eigenvalue problem can be written as

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n} \tag{3.4.3}$$

where $\mathbf{n}$ is a principal direction and σ is a scalar principal stress. Since the traction vector is a multiple of the unit normal, σ is a normal stress component. Thus a principal stress is a stress which acts on a plane of zero shear stress, Fig. 3.4.1.

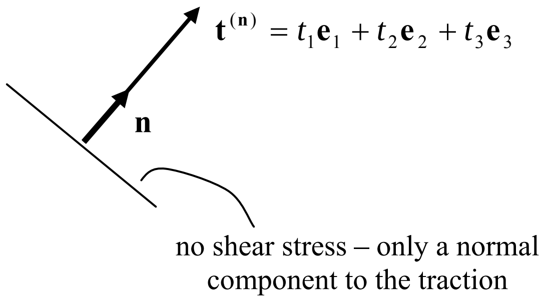

Figure 3.4.1: traction acting on a plane of zero shear stress

The principal stresses are the roots of the characteristic equation 1.11.5,

$$\sigma^3 - \mathbf{I}_1\sigma^2 + \mathbf{I}_2\sigma - \mathbf{I}_3 = 0 \tag{3.4.4}$$

where, Eqn. 1.11.6-7, 1.11.17,

$$\begin{aligned}
I_1 &= \mathrm{tr}\boldsymbol{\sigma} \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \tfrac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\boldsymbol{\sigma}^2\right] \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
&= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\
I_3 &= \tfrac{1}{3}\left[\mathrm{tr}\boldsymbol{\sigma}^3 - \tfrac{3}{2}\mathrm{tr}\boldsymbol{\sigma}\mathrm{tr}\boldsymbol{\sigma}^2 + \tfrac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})^3\right] \\
&= \det\boldsymbol{\sigma} \\
&= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{32} \\
&= \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{3.4.5}$$

The principal stresses and principal directions are properties of the stress tensor, and do not depend on the particular axes chosen to describe the state of stress., and the **stress invariants** I_1, I_2, I_3 are invariant under coordinate transformation. *c.f.* §1.11.1.

If one chooses a coordinate system to coincide with the three eigenvectors, one has the spectral decomposition 1.11.11 and the stress matrix takes the simple form 1.11.12,

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{3.4.6}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions.

3.4.3 Maximum Stresses

Directly from §1.11.3, the three principal stresses include the maximum and minimum normal stress components acting at a point. This result is re-derived here, together with results for the maximum shear stress

Normal Stresses

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions* and consider an arbitrary unit normal vector $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, Fig. 3.4.2. From 3.3.8 and Cauchy's law, the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n} \tag{3.4.7}$$

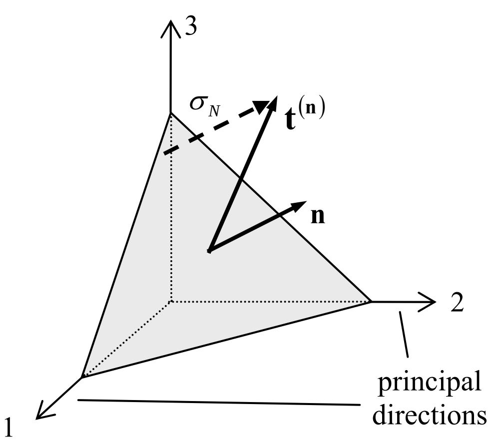

Figure 3.4.2: normal stress acting on a plane defined by the unit normal n

With respect to the principal stresses, using 3.4.6,

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n} = \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \tag{3.4.8}$$

and the normal stress is

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{3.4.9}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{3.4.10}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3\left(n_1^2 + n_2^2 + n_3^2\right) \geq \sigma_3 \tag{3.4.11}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses.

From 3.3.39, 3.4.8 and 3.4.9, the shear stress on the plane is

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{3.4.12}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right)n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right)n_2^2 + \sigma_3^2 - \left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2 + \sigma_3\right]^2 \tag{3.4.13}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned}
\frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1\left(\sigma_1 - \sigma_3\right)\left\{\sigma_1 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0 \\
\frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2\left(\sigma_2 - \sigma_3\right)\left\{\sigma_2 - \sigma_3 - 2\left[\left(\sigma_1 - \sigma_3\right)n_1^2 + \left(\sigma_2 - \sigma_3\right)n_2^2\right]\right\} = 0
\end{aligned} \tag{3.4.14}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 3.4.14 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|
\end{aligned} \tag{3.4.15}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{3.4.16}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 3.4.3.

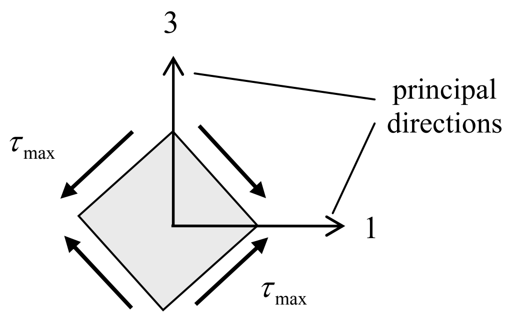

Figure 3.4.3: maximum shear stress at apoint

Example (maximum shear stress)

Consider the stress state

$$\left[\sigma_{ij}\right]=\begin{bmatrix}5 & 0 & 0\\ 0 & -6 & -12\\ 0 & -12 & 1\end{bmatrix}$$

This is the same tensor considered in the example of §1.11.1. Using the results of that example, the principal stresses are $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress at that point is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{25}{2}$$

The planes and direction upon which they act are shown in Fig. 3.4.4.

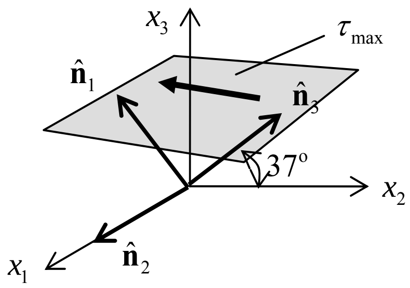

Figure 3.4.4: maximum shear stress

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