3.2 The Momentum Principles

In Parts I and II, the basic dynamics principles used were Newton's Laws, and these are equivalent to force equilibrium and moment equilibrium. For example, they were used to derive the stress transformation equations in Part I, §3.4 and the Equations of Motion in Part II, §1.1. Newton's laws there were applied to differential material elements.

An alternative but completely equivalent set of dynamics laws are **Euler's Laws**; these are more appropriate for finite-sized collections of moving particles, and can be used to express the force and moment equilibrium in terms of integrals. Euler's Laws are also called the **Momentum Principles**: the **principle of linear momentum** (Euler's first law) and the **principle of angular momentum** (Euler's second law).

3.2.1 The Principle of Linear Momentum

Momentum is a measure of the tendency of an object to keep moving once it is set in motion. Consider first the particle of rigid body dynamics: the (linear) momentum $\mathbf{p}$ is defined to be its mass times velocity, $\mathbf{p} = m\mathbf{v}$. The rate of change of momentum $\dot{\mathbf{p}}$ is

$$\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \qquad (3.2.1)$$

and use has been made of the fact that $dm/dt = 0$. Thus Newton's second law, $\mathbf{F} = m\mathbf{a}$, can be rewritten as

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \qquad (3.2.2)$$

This equation, formulated by Euler, states that *the rate of change of momentum is equal to the applied force*. It is called the **principle of linear momentum**, or **balance of linear momentum**. If there are no forces applied to a system, the total momentum of the system remains constant; the law in this case is known as the **law of conservation of (linear) momentum**.

Eqn. 3.2.2 as applied to a particle can be generalized to the mechanics of a continuum in one of two ways. One could consider a differential element of material, of mass dm and velocity $\mathbf{v}$. Alternatively, one can consider a finite portion of material, a control mass in the current configuration with spatial mass density $\rho(\mathbf{x},t)$ and spatial velocity field $\mathbf{v}(\mathbf{x},t)$. The total linear momentum of this mass of material is

$$\boxed{\mathbf{L}(t) = \int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv} \quad \textbf{Linear Momentum} \qquad (3.2.3)$$

The principle of linear momentum states that

$$\dot{\mathbf{L}}(t) = \frac{d}{dt}\int_v \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv = \mathbf{F}(t) \qquad (3.2.4)$$

where $\mathbf{F}(t)$ is the resultant of the forces acting on the portion of material.

Note that the volume over which the integration in Eqn. 3.2.4 takes place is not fixed; the integral is taken over a *fixed portion of material particles*, and the space occupied by this matter may change over time.

By virtue of the Transport theorem relation 3.1.31, this can be written as

$$\dot{\mathbf{L}}(t) = \int_v \rho(\mathbf{x},t)\dot{\mathbf{v}}(\mathbf{x},t)dv = \mathbf{F}(t) \qquad (3.2.5)$$

The resultant force acting on a body is due to the surface tractions **t** acting over surface elements and body forces **b** acting on volume elements, Fig. 3.2.1:

$$\boxed{\mathbf{F}(t) = \int_s \mathbf{t}ds + \int_v \mathbf{b}dv, \qquad F_i = \int_s t_i\, ds + \int_v b_i\, dv} \qquad \textbf{Resultant Force} \qquad (3.2.6)$$

and so the principle of linear momentum can be expressed as

$$\boxed{\int_s \mathbf{t}ds + \int_v \mathbf{b}dv = \int_v \rho\dot{\mathbf{v}}dv} \qquad \textbf{Principle of Linear Momentum} \qquad (3.2.7)$$

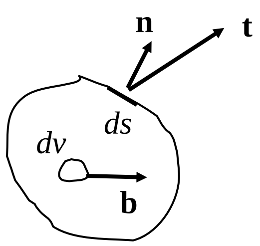

Figure 3.2.1: surface and body forces acting on a finite volume of material

The principle of linear momentum, Eqns. 3.2.7, will be used to prove Cauchy's Lemma and Cauchy's Law in the next section and, in §3.6, to derive the Equations of Motion.

3.2.2 The Principle of Angular Momentum

Considering again the mechanics of a single particle: the **angular momentum** is the moment of momentum about an axis, in other words, it is the product of the linear momentum of the particle and the perpendicular distance from the axis of its line of action. In the notation of Fig. 3.2.2, the angular momentum **h** is

$$\mathbf{h} = \mathbf{r} \times m\mathbf{v} \qquad (3.2.8)$$

which is the vector with magnitude $d \times m|\mathbf{v}|$ and perpendicular to the plane shown.

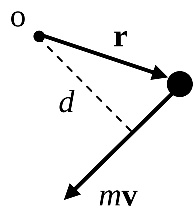

Figure 3.2.2: surface and body forces acting on a finite volume of material

Consider now a collection of particles. The **principle of angular momentum** states that the resultant moment of the external forces acting on the system of particles, $\mathbf{M}$, equals the rate of change of the total angular momentum of the particles:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt} \tag{3.2.9}$$

Generalising to a continuum, the angular momentum is

$$\boxed{\mathbf{H} = \int_v \mathbf{r} \times \rho \mathbf{v} dv} \quad \textbf{Angular Momentum} \tag{3.2.10}$$

and the principle of angular momentum is

$$\boxed{\begin{gathered} \int_s \mathbf{r} \times \mathbf{t}^{(\mathbf{n})} ds + \int_v \mathbf{r} \times \mathbf{b} dv = \frac{d}{dt} \int_v \mathbf{r} \times \rho \mathbf{v} dv \\ \int_s \varepsilon_{ijk} x_j t_k^{(\mathbf{n})} ds + \int_v \varepsilon_{ijk} x_j b_k \, dv = \frac{d}{dt} \int_V \varepsilon_{ijk} x_j \rho v_k \, dv \end{gathered}} \quad \textbf{Principle of Angular Momentum}$$

$$(3.2.11)$$

The principle of angular momentum, 3.2.11, will be used, in §3.6, to deduce the symmetry of the Cauchy stress.