2.12 Pull Back, Push Forward and Lie Time Derivatives

This section is in the main concerned with the following issue: an observer attached to a fixed, say Cartesian, coordinate system will see a material move and deform over time, and will observe various vectorial and tensorial quantities to change also. However, a hypothetical observer attached to the deforming material, and moving and deforming with the material, will see something different. The question is: what quantities will be seen to change from this embedded observer's viewpoint?

2.12.1 Time Derivatives of Spatial Fields

In terms of the spatial basis, a spatial vector **v** can be expressed in terms of the covariant components and contravariant components,

$$\mathbf{v} = v_i \mathbf{g}^i, \qquad \mathbf{v} = v^i \mathbf{g}_i \tag{2.12.1}$$

We want to distinguish between two quantities. The first is the material time derivative of the vector **v**:

$$\dot{\mathbf{v}} = \dot{\overline{v_i \mathbf{g}^i}} = \dot{v}_i \mathbf{g}^i + v_i \dot{\mathbf{g}}^i, \qquad \mathbf{v} = \dot{\overline{v^i \mathbf{g}_i}} = \dot{v}^i \mathbf{g}_i + v^i \dot{\mathbf{g}}_i \tag{2.12.2}$$

The second is the time derivative *holding the base vectors fixed*,

$$\dot{v}_i \mathbf{g}^i, \qquad \dot{v}^i \mathbf{g}_i \tag{2.12.3}$$

This latter is called the **convected derivative** and is the rate of the change as seen by an observer attached to the deforming bases.

From Eqn. 2.12.1, the components of **v** can be expressed as

$$v_i = \mathbf{v} \cdot \mathbf{g}_i, \quad v^i = \mathbf{v} \cdot \mathbf{g}^i \tag{2.12.4}$$

Taking the material time derivative, and using Eqns. 2.11.11, 2.11.13,

$$\begin{aligned} \dot{v}_i &= \dot{\overline{\mathbf{v} \cdot \mathbf{g}_i}} & \dot{v}^i &= \dot{\overline{\mathbf{v} \cdot \mathbf{g}^i}} \\ &= \dot{\mathbf{v}} \cdot \mathbf{g}_i + \mathbf{v} \cdot \dot{\mathbf{g}}_i\,, & &= \dot{\mathbf{v}} \cdot \mathbf{g}^i + \mathbf{v} \cdot \dot{\mathbf{g}}^i \\ &= \left(\dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v}\right) \cdot \mathbf{g}_i & &= \left(\dot{\mathbf{v}} - \mathbf{l}\mathbf{v}\right) \cdot \mathbf{g}^i \end{aligned} \tag{2.12.5}$$

Thus there are two convected derivatives of a vector, depending on whether one is using covariant or contravariant components:

$$\dot{v}_i \mathbf{g}^i = \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}}\mathbf{v}$$
$$\dot{v}^i \mathbf{g}_i = \dot{\mathbf{v}} - \mathbf{l}\mathbf{v} \tag{2.12.6}$$

As will be seen below, these quantities are **Lie derivatives** of the vector **v**.

The time derivative of the components can be expressed in an alternative way, by expressing the spatial base vectors $\mathbf{g}_i, \mathbf{g}^i$ in terms of the material base vectors $\mathbf{G}_i, \mathbf{G}^i$; using Eqns. 2.10.23:

$$\begin{aligned} \dot{v}_i &= \overline{\mathbf{v}\cdot\mathbf{g}_i}^{\cdot} & \dot{v}^i &= \overline{\mathbf{v}\cdot\mathbf{g}^i}^{\cdot} \\ &= \overline{\mathbf{v}\cdot\mathbf{F}\mathbf{G}_i}^{\cdot}, & &= \overline{\mathbf{v}\cdot\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i}^{\cdot} \\ &= \overline{\mathbf{F}^{\mathrm{T}}\mathbf{v}}^{\cdot}\mathbf{G}^j & &= \overline{\mathbf{F}^{-1}\mathbf{v}}^{\cdot}\mathbf{G}^i \end{aligned} \tag{2.12.7}$$

So, as an alternative to Eqns. 2.12.6,

$$\dot{v}_i \mathbf{g}^i = \overline{\mathbf{F}^{\mathrm{T}}\mathbf{v}}^{\cdot}$$
$$\dot{v}^i \mathbf{g}_i = \overline{\mathbf{F}^{-1}\mathbf{v}}^{\cdot} \tag{2.12.8}$$

As will be seen further below, the quantities on the right are the material time derivatives of the **pull-back** of the vector **v**.

Repeating the above, now for a spatial tensor **a**: in terms of the spatial basis, **a** can be expressed in terms of the covariant components and contravariant components as

$$\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \qquad \mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \tag{2.12.9}$$

The material time derivative of the tensor **a** is

$$\begin{aligned} \dot{\mathbf{a}} &= \overline{a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j}^{\cdot} = \dot{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j + a_{ij}\dot{\mathbf{g}}^i \otimes \mathbf{g}^j + a_{ij}\mathbf{g}^i \otimes \dot{\mathbf{g}}^j \\ &= \overline{a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j}^{\cdot} = \dot{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j + a^{ij}\dot{\mathbf{g}}_i \otimes \mathbf{g}_j + a^{ij}\mathbf{g}_i \otimes \dot{\mathbf{g}}_j \end{aligned} \tag{2.12.10}$$

and the convected derivative is the first term:

$$\dot{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \qquad \dot{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \tag{2.12.11}$$

The components of **a** can be expressed as

$$a_{ij} = \mathbf{g}_i \mathbf{A} \mathbf{g}_j, \quad a^{ij} = \mathbf{g}^i \mathbf{A} \mathbf{g}^j \tag{2.12.12}$$

Taking the material time derivative, and again using Eqns. 2.11.11, 2.11.13,

$$\begin{aligned} \dot{a}_{ij} &= \dot{\overline{\mathbf{g}_i \mathbf{a} \mathbf{g}_j}} \\ &= \dot{\mathbf{g}}_i \mathbf{a} \mathbf{g}_j + \mathbf{g}_i \dot{\mathbf{a}} \mathbf{g}_j + \mathbf{g}_i \mathbf{a} \dot{\mathbf{g}}_j \,, \\ &= \mathbf{g}_i \left(\dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}} \mathbf{a} \right) \mathbf{g}_j \end{aligned} \qquad \begin{aligned} \dot{a}^{ij} &= \dot{\overline{\mathbf{g}^i \mathbf{a} \mathbf{g}^j}} \\ &= -\mathbf{l}^{\mathrm{T}} \mathbf{g}^i \mathbf{a} \mathbf{g}^j + \mathbf{g}^i \dot{\mathbf{a}} \mathbf{g}^j - \mathbf{g}^i \mathbf{a} \mathbf{l}^{\mathrm{T}} \mathbf{g}^j \\ &= \mathbf{g}^i \left(\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} \right) \mathbf{g}^j \end{aligned} \tag{2.12.13}$$

The convected derivatives are thus

$$\begin{aligned} \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j &= \dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}} \mathbf{a} \\ \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j &= \dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} \end{aligned} \tag{2.12.14}$$

As will be seen below, these quantities are **Lie derivatives** of the tensor **a**.

The time derivative of the components can be expressed in an alternative way, by expressing the spatial base vectors $\mathbf{g}_i, \mathbf{g}^i$ in terms of the material base vectors $\mathbf{G}_i, \mathbf{G}^i$; using Eqns. 2.10.23:

$$\begin{aligned} \dot{a}_{ij} &= \dot{\overline{\mathbf{g}_i \mathbf{a} \mathbf{g}_j}} \\ &= \dot{\overline{\mathbf{F}\mathbf{G}_i \mathbf{a} \mathbf{F}\mathbf{G}_i}} \quad , \\ &= \mathbf{G}_i \, \dot{\overline{\mathbf{F}^{\mathrm{T}} \mathbf{a} \mathbf{F}}} \, \mathbf{G}_i \end{aligned} \qquad \begin{aligned} \dot{a}^{ij} &= \dot{\overline{\mathbf{g}^i \mathbf{a} \mathbf{g}^j}} \\ &= \dot{\overline{\mathbf{F}^{-\mathrm{T}} \mathbf{G}^i \mathbf{a} \mathbf{F}^{-\mathrm{T}} \mathbf{G}^j}} \\ &= \mathbf{G}^i \, \dot{\overline{\mathbf{F}^{-1} \mathbf{a} \mathbf{F}^{-\mathrm{T}}}} \, \mathbf{G}^j \end{aligned} \tag{2.12.15}$$

So, as an alternative to Eqns. 2.12.14,

$$\begin{aligned} \dot{a}_{ij} &= \dot{\overline{\mathbf{F}^{\mathrm{T}} \mathbf{a} \mathbf{F}}} \\ \dot{a}^{ij} &= \dot{\overline{\mathbf{F}^{-1} \mathbf{a} \mathbf{F}^{-\mathrm{T}}}} \end{aligned} \tag{2.12.16}$$

As will be seen next, the quantities on the right are the material time derivatives of the **pull-back** of the tensor **a**.

Example

Considering again Example 1 which was worked through in detail in §2.10, suppose we have a shearing deformation as shown in Fig. 2.12.1 (this is Fig. 2.10.3).

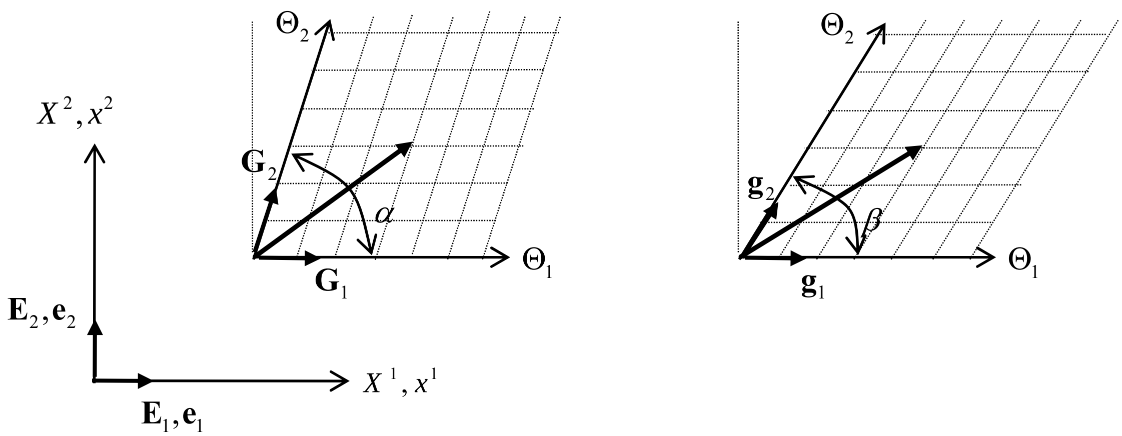

Figure 2.12.1: A pure shear deformation of one parallelogram into another

Let the shear angle β in Fig. 2.12.1 evolve over time according to

$$\beta = \alpha + \gamma t \tag{2.12.17}$$

From Eqns. 2.10.7, 2.10.11, the rates of change of the base vectors are

$$\begin{aligned} &\frac{d}{dt}\mathbf{g}_1 = \frac{d}{dt}\mathbf{e}_1 = 0, \qquad \frac{d}{dt}\mathbf{g}_2 = \frac{d}{dt}\left(\frac{1}{\tan(\alpha+\gamma t)}\mathbf{e}_1 + \mathbf{e}_2\right) = -\frac{\gamma}{\sin^2(\alpha+\gamma t)}\mathbf{e}_1 \\ &\frac{d}{dt}\mathbf{g}^1 = \frac{d}{dt}\left(\mathbf{e}_1 - \frac{1}{\tan(\alpha+\gamma t)}\mathbf{e}_2\right) = +\frac{\gamma}{\sin^2(\alpha+\gamma t)}\mathbf{e}_2, \qquad \frac{d}{dt}\mathbf{g}^2 = \frac{d}{dt}\mathbf{e}_2 = 0 \end{aligned} \tag{2.12.18}$$

The velocity gradient is, from Eqn. 2.11.5,

$$\begin{aligned} \mathbf{l} &= \dot{\mathbf{g}}_1 \otimes \mathbf{g}^1 + \dot{\mathbf{g}}_2 \otimes \mathbf{g}^2 \\ &= -\frac{\gamma}{\sin^2\beta}\mathbf{e}_1 \otimes \mathbf{e}_2 \\ &= \dot{\Pi}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (\mathbf{e}_i) \end{aligned} \tag{2.12.19}$$

where Π is given by Eqn. 2.10.26, and

$$\dot{\Pi}(t) = \frac{d}{dt}\left(\frac{1}{\tan(\alpha+\gamma t)} - \frac{1}{\tan(\alpha)}\right) = -\frac{\gamma}{\sin^2(\alpha+\gamma t)} \tag{2.12.20}$$

Considering again the vector $\mathbf{V}$ of Eqn. 2.10.44, $\mathbf{V} = \begin{bmatrix} V_x & V_y \end{bmatrix}^{\mathrm{T}} (\mathbf{E}_i)$, and its corresponding deformed vector $\mathbf{v}$ of Eqn. 2.10.47, $\mathbf{v} = \begin{bmatrix} V_x + \Pi V_y & V_y \end{bmatrix} (\mathbf{e}_i)$,

$$\dot{\mathbf{v}} = \dot{\Pi} \begin{bmatrix} V_y \\ 0 \end{bmatrix} \quad (\mathbf{e}_i), \tag{2.12.21}$$

The contravariant and covariant components of $\dot{\mathbf{v}}$ are

$$\dot{\mathbf{v}} = \widehat{v}^i \mathbf{g}_i, \quad \widehat{v}^i = \dot{\Pi} \begin{bmatrix} V_y \\ 0 \end{bmatrix}, \qquad \dot{\mathbf{v}} = \widehat{v}_i \mathbf{g}^i, \quad \widehat{v}_i = \dot{\Pi} \begin{bmatrix} V_y \\ \dfrac{1}{\tan\beta} V_y \end{bmatrix} \tag{2.12.22}$$

The "hat" on the $\widehat{v}$ is to emphasise that (see Eqns. 2.12.5)

$$\widehat{v}^i = \dot{\mathbf{v}} \cdot \mathbf{g}^i \neq \dot{v}^i = \dot{\overline{\mathbf{v} \cdot \mathbf{g}^i}}, \qquad \widehat{v}_i = \dot{\mathbf{v}} \cdot \mathbf{g}_i \neq \dot{v}_i = \dot{\overline{\mathbf{v} \cdot \mathbf{g}_i}} \tag{2.12.23}$$

From Eqns. 2.12.6, the convected derivatives are

$$\begin{aligned} \dot{\mathbf{v}} - \mathbf{l}\mathbf{v} &= \dot{\Pi} \left\{ \begin{bmatrix} V_y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_x + \Pi V_y \\ V_y \end{bmatrix} \right\} \\ &= \dot{\Pi} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}, \qquad \begin{aligned} \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}}\mathbf{v} &= \dot{\Pi} \left\{ \begin{bmatrix} V_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_x + \Pi V_y \\ V_y \end{bmatrix} \right\} \\ &= \dot{\Pi} \begin{bmatrix} V_y \\ V_x + \Pi V_y \end{bmatrix} \end{aligned} \tag{2.12.23}$$

Thus $\dot{\mathbf{v}} - \mathbf{l}\mathbf{v} = 0$, which, from Eqn. 2.12.6, implies that $\dot{v}^i = 0$. This is the expected result: the contravariant components do not change over time. They are always $\begin{bmatrix} V_x - V_y / \tan\alpha & V_y \end{bmatrix}$, as given by Eqn. 2.10.47b.

Consider now an example tensor

$$\mathbf{A} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \quad (\mathbf{E}_i) \tag{2.12.24}$$

The covariant and contravariant components are

$$\mathbf{A} = \begin{bmatrix} A_{xx} - \frac{1}{\tan\alpha}\left(A_{xy} + A_{yx}\right) + \frac{1}{\tan^2\alpha}A_{yy} & A_{xy} - \frac{1}{\tan\alpha}A_{yy} \\ A_{yx} - \frac{1}{\tan\alpha}A_{yy} & A_{yy} \end{bmatrix} \quad (\mathbf{G}_i)$$
$$\mathbf{A} = \begin{bmatrix} A_{xx} & A_{xy} + A_{xx}\frac{1}{\tan\alpha} \\ A_{yx} + A_{xx}\frac{1}{\tan\alpha} & A_{yy} + \frac{1}{\tan\alpha}\left(A_{xy} + A_{yx}\right) + \frac{1}{\tan^2\alpha}A_{xx} \end{bmatrix} \quad (\mathbf{G}^i) \tag{2.12.25}$$

This deforms to (with **F** given by Eqn. 2.10.25)

$$\mathbf{a} = \begin{bmatrix} 1 & \Pi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} = \begin{bmatrix} A_{xx} + \Pi A_{yx} & A_{xy} + \Pi A_{yy} \\ A_{yx} & A_{yy} \end{bmatrix} \quad (\mathbf{e}_i) \tag{2.12.25}$$

Now

$$\begin{aligned} \mathbf{FA} &= \left(\mathbf{g}_i \otimes \mathbf{G}^i\right) A^{mn}\mathbf{G}_m \otimes \mathbf{G}_n \\ &= A^{ij}\mathbf{g}_i \otimes \mathbf{G}_j \end{aligned} \tag{2.12.26}$$

Converting between the various convected base vectors using Eqns. 2.10.7-8, 2.10.11-12, the contravariant and covariant and components are $\mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$, $\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$:

$$a^{ij} = \begin{bmatrix} A_{xx} - \frac{1}{\tan\beta}A_{xy} - \frac{1}{\tan\alpha}A_{yx} + \frac{1}{\tan\alpha}\frac{1}{\tan\beta}A_{yy} & A_{xy} - A_{yy}\frac{1}{\tan\alpha} \\ A_{yx} - \frac{1}{\tan\beta}A_{yy} & A_{yy} \end{bmatrix}$$
$$a_{ij} = \begin{bmatrix} A_{xx} + A_{yx}\Pi & A_{xy} + A_{yy}\Pi + \frac{1}{\tan\beta}\left(A_{xx} + A_{yx}\Pi\right) \\ A_{yx} + \frac{1}{\tan\beta}\left(A_{xx} + A_{yx}\Pi\right) & A_{yy} + \frac{1}{\tan\beta}A_{yx} + \frac{1}{\tan\beta}\left(A_{xy} + A_{yy}\Pi\right) + \frac{1}{\tan^2\beta}\left(A_{xx} + A_{yx}\Pi\right) \end{bmatrix} \tag{2.12.27}$$

Also,

$$\dot{\mathbf{a}} = \dot{\Pi}\begin{bmatrix} A_{yx} & A_{yy} \\ 0 & 0 \end{bmatrix} \quad (\mathbf{e}_i), \tag{2.12.28}$$

and the contravariant and covariant components are

$$\dot{\mathbf{a}} = \hat{\dot{a}}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j, \quad \hat{\dot{a}}^{ij} = \dot{\Pi}\begin{bmatrix} A_{yx} - \dfrac{1}{\tan\beta}A_{yy} & A_{yy} \\ 0 & 0 \end{bmatrix}$$
$$\dot{\mathbf{a}} = \hat{\dot{a}}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \quad \hat{\dot{a}}_{ij} = \dot{\Pi}\begin{bmatrix} A_{yx} & \dfrac{1}{\tan\beta}A_{yx} + A_{yy} \\ \dfrac{1}{\tan\beta}A_{yx} & \dfrac{1}{\tan\beta}\left(\dfrac{1}{\tan\beta}A_{yx} + A_{yy}\right) \end{bmatrix} \tag{2.12.29}$$

Again, the "hat" emphasises that (see Eqns. 2.12.13)

$$\hat{\dot{a}}^{ij} = \mathbf{g}^i\dot{\mathbf{a}}\mathbf{g}^j \neq \dot{a}^{ij} = \overline{\mathbf{g}^i\mathbf{a}\mathbf{g}^j}^{\,\cdot}, \qquad \hat{\dot{a}}_{ij} = \mathbf{g}_i\dot{\mathbf{a}}\mathbf{g}_j \neq \dot{a}_{ij} = \overline{\mathbf{g}_i\mathbf{a}\mathbf{g}_j}^{\,\cdot} \tag{2.12.30}$$

Now

$$\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} = \dot{\Pi}\begin{bmatrix} -A_{xy} - A_{yy}\Pi & 0 \\ -A_{yy} & 0 \end{bmatrix}$$
$$\dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}}\mathbf{a} = \dot{\Pi}\begin{bmatrix} A_{yx} & A_{xx} + A_{yx}\Pi + A_{yy} \\ A_{xx} + A_{yx}\Pi & A_{xy} + A_{yx} + A_{yy}\Pi \end{bmatrix} \tag{2.12.31}$$

Thus $\dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} = 0$, i.e. $\dot{a}^{ij} = 0$, only when $A_{xy} = A_{yy} = 0$, which is consistent with Eqn. 2.12.27a (only constant terms, independent of β remain in that case).

2.12.2 Push-Forward and Pull-Back

Next are defined the push-forward and pull-back of vectors and tensors, which will lead into the concept of Lie derivatives, which relate back to what was just discussed above regarding convected derivatives.

Vectors

Consider a vector **V** given in terms of the reference configuration base vectors:

$$\begin{aligned}\mathbf{V} &= V_i\left(\Theta^j\right)\mathbf{G}^i \\ &= V^i\left(\Theta^j\right)\mathbf{G}_i\end{aligned} \tag{2.12.32}$$

The **push-forward**, symbolised by $\chi_*(\bullet)$, is defined to be the vector with *the same components*, but with respect to the current configuration base vectors. There are 2 push-forward operations, depending on the type of components used; the symbol b is used for covariant components V_i and the symbol # for contravariant components V^i; using 2.10.23,

$$\boxed{\begin{aligned} \chi_*(\mathbf{V})^b &\equiv V_i\mathbf{g}^i = V_i\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i = \mathbf{F}^{-\mathrm{T}}\mathbf{V} \\ \chi_*(\mathbf{V})^\# &\equiv V^i\mathbf{g}_i = V^i\mathbf{F}\mathbf{G}_i = \mathbf{F}\mathbf{V} \end{aligned}}$$

Push-forward of Vector (2.12.33)

Eqn. 2.12.33b says that the push forward of the contravariant form of **V** is simply **FV**. In other words, the push forward here is the actual corresponding vector in the deformed configuration, $\mathbf{v} = \mathbf{F}\mathbf{V} = v^i\left(\Theta^j\right)\mathbf{g}_i$, and, as a consequence of the definitions, $V^i = v^i$, as illustrated in Fig. 2.12.2.

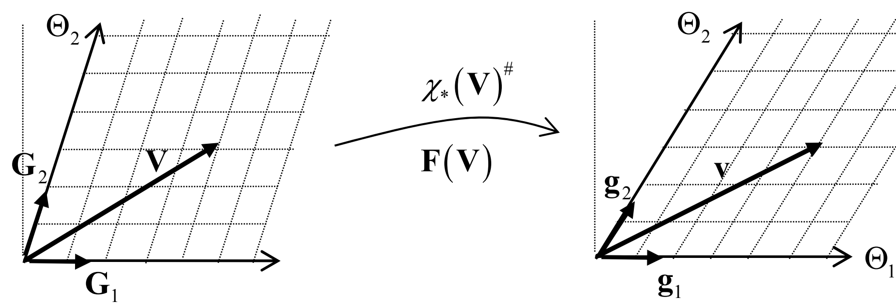

Figure 2.12.2: The push-forward of a vector V

A special case of Eqn. 2.12.33b is the push forward of a line element in the reference configuration, giving the corresponding line element in the current configuration:

$$\chi_*(d\mathbf{X})^\# = d\Theta^i\mathbf{g}_i = d\mathbf{x}\,. \tag{2.12.34}$$

Similarly, consider a vector **v** given in terms of the current configuration basis:

$$\mathbf{v} = v_i\mathbf{g}^i = v^i\mathbf{g}_i \tag{2.12.35}$$

The **pull-back** of **v**, $\chi_*^{-1}(\mathbf{v})$, is defined to be the vector with components v_i (or v^i) with respect to the reference configuration base vectors $\mathbf{G}^i$ (or $\mathbf{G}_i$). Using 2.10.23,

$$\boxed{\begin{aligned} \chi_*^{-1}(\mathbf{v})^b &= v_i\mathbf{G}^i = v_i\mathbf{F}^{\mathrm{T}}\mathbf{g}^i = \mathbf{F}^{\mathrm{T}}\mathbf{v} \\ \chi_*^{-1}(\mathbf{v})^\# &= v^i\mathbf{G}_i = v^i\mathbf{F}^{-1}\mathbf{g}_i = \mathbf{F}^{-1}\mathbf{v} \end{aligned}}$$

Pull-back of a vector (2.12.36)

and, for a line element in the current configuration,

$$\chi_*^{-1}(d\mathbf{x})^\# = dx^i\mathbf{G}_i = \mathbf{F}^{-1}d\mathbf{x} = d\mathbf{X}\,. \tag{2.12.37}$$

Note that a push-forward and pull-back applied successively to a vector with the same component type will result in the initial vector.

From the above, for two material vectors **U** and **V** and two spatial vectors **u** and **v**,

$$\begin{aligned}\mathbf{U}\cdot\mathbf{V} &= \chi_*(\mathbf{U})^b \cdot \chi_*(\mathbf{V})^\# = \chi_*(\mathbf{U})^\# \cdot \chi_*(\mathbf{V})^b \\ \mathbf{u}\cdot\mathbf{v} &= \chi_*^{-1}(\mathbf{u})^b \cdot \chi_*^{-1}(\mathbf{v})^\# = \chi_*^{-1}(\mathbf{u})^\# \cdot \chi_*^{-1}(\mathbf{v})^b\end{aligned} \tag{2.12.38}$$

For example, as a special case of this, in the reference configuration, $\mathbf{G}_1$ and $\mathbf{G}^2$ are perpendicular: $\mathbf{G}_1 \cdot \mathbf{G}^2 = 0$. Pushing forward these vectors, we get from Eqn. 2.12.33: $\mathbf{F}\mathbf{G}_1 = \mathbf{g}_1$ and $\mathbf{F}^{-\mathrm{T}}\mathbf{G}^2 = \mathbf{g}^2$, and again $\chi_*(\mathbf{G}_1)^\# \cdot \chi_*(\mathbf{G}^2)^b = \mathbf{g}_1 \cdot \mathbf{g}^2 = 0$.

Tensors

Consider a material tensor **A**:

$$\mathbf{A} = A_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = A^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = A^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = A_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j \tag{2.12.39}$$

As for the vector, the push-forward of **A**, $\chi_*(\mathbf{A})$, is defined to be the tensor with the same components, but with respect to the deformed base vectors. Thus, using 2.10.23,

$$\boxed{\begin{aligned}\chi_*(\mathbf{A})^b &= A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = A_{ij}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{F}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{-1} \\ \chi_*(\mathbf{A})^\# &= A^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = A^{ij}\left(\mathbf{F}\mathbf{G}_i \otimes \mathbf{F}\mathbf{G}_j\right) = \mathbf{F}\mathbf{A}\mathbf{F}^{\mathrm{T}} \\ \chi_*(\mathbf{A})^\backslash &= A^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = A^i_{\cdot j}\left(\mathbf{F}\mathbf{G}_i \otimes \mathbf{F}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{F}\mathbf{A}\mathbf{F}^{-1} \\ \chi_*(\mathbf{A})^/ &= A_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = A_i^{\cdot j}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{F}\mathbf{G}_j\right) = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{\mathrm{T}}\end{aligned}}$$

Push-forward of Tensor (2.12.40)

Similarly, consider a spatial tensor **a**:

$$\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = a^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = a_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j \tag{2.12.41}$$

The pull-back is

$$\boxed{\begin{aligned} \chi_*^{-1}(\mathbf{a})^b &= a_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = a_{ij}\left(\mathbf{F}^{\mathrm{T}}\mathbf{g}^i \otimes \mathbf{F}^{\mathrm{T}}\mathbf{g}^j\right) &&= \mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F} \\ \chi_*^{-1}(\mathbf{a})^\# &= a^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = a^{ij}\left(\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{-1}\mathbf{g}_j\right) &&= \mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}} \\ \chi_*^{-1}(\mathbf{a})^\backslash &= a^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = a^i_{\cdot j}\left(\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{\mathrm{T}}\mathbf{g}^j\right) &&= \mathbf{F}^{-1}\mathbf{a}\mathbf{F} \\ \chi_*^{-1}(\mathbf{a})^/ &= a_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = a_i^{\cdot j}\left(\mathbf{F}^{\mathrm{T}}\mathbf{g}^i \otimes \mathbf{F}^{-1}\mathbf{g}_j\right) &&= \mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}^{-\mathrm{T}} \end{aligned}}$$

Pull-back of Tensor (2.12.42)

The first of these, $\mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}$, is called the **covariant pull-back**, whereas the second, $\mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}}$, is called the **contravariant pull-back**.

Time Derivatives

It will be recognised that the expressions for the pull backs of a spatial covariant tensor and spatial contravariant tensor in Eqns. 2.12.42a,b are those appearing in Eqns. 2.12.16. Keeping in mind Eqn. 2.12.14, one sees that, for a spatial tensor in terms of covariant components, $\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, and contravariant components, $\mathbf{a} = a^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$,

$$\begin{aligned} \dot{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j &= \left(\overline{\mathbf{F}^{\mathrm{T}}\mathbf{a}\mathbf{F}}^{\,\cdot}\right)\mathbf{g}^i \otimes \mathbf{g}^j &&= \dot{\mathbf{a}} + \mathbf{a}\mathbf{l} + \mathbf{l}^{\mathrm{T}}\mathbf{a} \\ \dot{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j &= \left(\overline{\mathbf{F}^{-1}\mathbf{a}\mathbf{F}^{-\mathrm{T}}}^{\,\cdot}\right)\mathbf{g}_i \otimes \mathbf{g}_j &&= \dot{\mathbf{a}} - \mathbf{l}\mathbf{a} - \mathbf{a}\mathbf{l}^{\mathrm{T}} \end{aligned} \tag{2.12.43}$$

Other Push-Forward and Pull-Back relations for Vectors and Tensors

Here follow some relations involving the push-forward and pull-backs of tensors.

For two material tensors **A** and **B** and two spatial tensors **a** and **b**, the scalar product is

$$\begin{aligned} \mathbf{A}:\mathbf{B} &= A_{ij}B^{ij} = A^{ij}B_{ij} = A^i_{\cdot j}B_i^{\cdot j} = A_i^{\cdot j}B^i_{\cdot j} \\ \mathbf{a}:\mathbf{b} &= a_{ij}b^{ij} = a^{ij}b_{ij} = a^i_{\cdot j}b_i^{\cdot j} = a_i^{\cdot j}b^i_{\cdot j} \end{aligned} \tag{2.12.44}$$

This scalar product then push-forwards and pull-backs as {▲Problem 1}

$$\begin{aligned} \mathbf{A}:\mathbf{B} &= \chi_*(\mathbf{A})^b : \chi_*(\mathbf{B})^\# = \chi_*(\mathbf{A})^\# : \chi_*(\mathbf{B})^b \\ &= \chi_*(\mathbf{A})^/ : \chi_*(\mathbf{B})^\backslash = \chi_*(\mathbf{A})^\backslash : \chi_*(\mathbf{B})^/ \\ \mathbf{a}:\mathbf{b} &= \chi_*^{-1}(\mathbf{a})^b : \chi_*^{-1}(\mathbf{b})^\# = \chi_*^{-1}(\mathbf{a})^\# : \chi_*^{-1}(\mathbf{b})^b \\ &= \chi_*^{-1}(\mathbf{a})^/ : \chi_*^{-1}(\mathbf{b})^\backslash = \chi_*^{-1}(\mathbf{a})^\backslash : \chi_*^{-1}(\mathbf{b})^/ \end{aligned} \tag{2.12.45}$$

For material tensor $\mathbf{A}$ and material vectors $\mathbf{U}, \mathbf{V}$, and spatial tensor $\mathbf{a}$ and spatial vectors $\mathbf{u}, \mathbf{v}$,

$$
\begin{aligned}
\mathbf{UAV} &= U_i A^{ij} V_j = U^i A_{ij} V^j = U_i A^i_{\cdot j} V^j = U^i A_i^{\cdot j} V_j \\
\mathbf{uav} &= u_i a^{ij} v_j = u^i a_{ij} v^j = u_i a^i_{\cdot j} v^j = u^i a_i^{\cdot j} v_j
\end{aligned}
\tag{2.12.46}
$$

Then

$$
\begin{aligned}
\mathbf{UAV} &= \chi_*(\mathbf{U})^\flat \chi_*(\mathbf{A})^\sharp \chi_*(\mathbf{V})^\flat = \chi_*(\mathbf{U})^\sharp \chi_*(\mathbf{A})^\flat \chi_*(\mathbf{V})^\sharp \\
&= \chi_*(\mathbf{U})^\flat \chi_*(\mathbf{A})^{\backslash} \chi_*(\mathbf{V})^\sharp = \chi_*(\mathbf{U})^\sharp \chi_*(\mathbf{A})^{/} \chi_*(\mathbf{V})^\flat \\
\mathbf{uav} &= \chi_*^{-1}(\mathbf{u})^\flat \chi_*^{-1}(\mathbf{a})^\sharp \chi_*^{-1}(\mathbf{v})^\flat = \chi_*^{-1}(\mathbf{u})^\sharp \chi_*^{-1}(\mathbf{a})^\flat \chi_*^{-1}(\mathbf{v})^\sharp \\
&= \chi_*^{-1}(\mathbf{u})^\flat \chi_*^{-1}(\mathbf{a})^{\backslash} \chi_*^{-1}(\mathbf{v})^\sharp = \chi_*^{-1}(\mathbf{u})^\sharp \chi_*^{-1}(\mathbf{a})^{/} \chi_*^{-1}(\mathbf{v})^\flat
\end{aligned}
\tag{2.12.47}
$$

For material tensor $\mathbf{A}$ and material vector $\mathbf{V}$, and spatial tensor $\mathbf{a}$ and spatial vector $\mathbf{v}$, the contractions $\mathbf{AV}$ and $\mathbf{av}$ are

$$
\begin{aligned}
\mathbf{AV} &= A_{ij} V^j = A_i^{\cdot j} V_j = A^i_{\cdot j} V^j = A^{ij} V_j \\
\mathbf{av} &= a_{ij} v^j = a_i^{\cdot j} v_j = a^i_{\cdot j} v^j = a^{ij} v_j
\end{aligned}
\tag{2.12.48}
$$

and so transform as

$$
\begin{aligned}
\chi_*(\mathbf{AV})^\flat &= \chi_*(\mathbf{A})^\flat \chi_*(\mathbf{V})^\sharp = \chi_*(\mathbf{A})^{/} \chi_*(\mathbf{V})^\flat \\
\chi_*(\mathbf{AV})^\sharp &= \chi_*(\mathbf{A})^\sharp \chi_*(\mathbf{V})^\flat = \chi_*(\mathbf{A})^{\backslash} \chi_*(\mathbf{V})^\sharp \\
\\
\chi_*^{-1}(\mathbf{av})^\flat &= \chi_*^{-1}(\mathbf{a})^\flat \chi_*^{-1}(\mathbf{v})^\sharp = \chi_*^{-1}(\mathbf{a})^{/} \chi_*^{-1}(\mathbf{v})^\flat \\
\chi_*^{-1}(\mathbf{av})^\sharp &= \chi_*^{-1}(\mathbf{a})^\sharp \chi_*^{-1}(\mathbf{v})^\flat = \chi_*^{-1}(\mathbf{a})^{\backslash} \chi_*^{-1}(\mathbf{v})^\sharp
\end{aligned}
\tag{2.12.49}
$$

Finally, for material tensors $\mathbf{A}$, $\mathbf{B}$ and spatial tensors $\mathbf{a}$, $\mathbf{b}$,

$$
\begin{aligned}
\mathbf{AB} &= A_{ik} B^{kj} \mathbf{G}^i \otimes \mathbf{G}_j = A_i^{\cdot k} B_k^{\cdot j} \mathbf{G}^i \otimes \mathbf{G}_j = A_i^{\cdot k} B_{kj} \mathbf{G}^i \otimes \mathbf{G}^j = A_{ik} B^k_{\cdot j} \mathbf{G}^i \otimes \mathbf{G}^j = \cdots \\
\mathbf{ab} &= a_{ik} b^{kj} \mathbf{g}^i \otimes \mathbf{g}_j = a_i^{\cdot k} b_k^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j = a_i^{\cdot k} b_{kj} \mathbf{g}^i \otimes \mathbf{g}^j = a_{ik} b^k_{\cdot j} \mathbf{g}^i \otimes \mathbf{g}^j = \cdots
\end{aligned}
\tag{2.12.50}
$$

and so

$$
\begin{aligned}
&\chi_*(\mathbf{AB})^/ = \chi_*(\mathbf{A})^b \chi_*(\mathbf{B})^\# = \chi_*(\mathbf{A})^/ \chi_*(\mathbf{B})^/ \\
&\chi_*(\mathbf{AB})^b = \chi_*(\mathbf{A})^/ \chi_*(\mathbf{B})^b = \chi_*(\mathbf{A})^b \chi_*(\mathbf{B})^\backslash \\
&\vdots \\
&\chi_*^{-1}(\mathbf{ab})^/ = \chi_*^{-1}(\mathbf{a})^b \chi_*^{-1}(\mathbf{b})^\# = \chi_*^{-1}(\mathbf{a})^/ \chi_*^{-1}(\mathbf{b})^/ \\
&\chi_*^{-1}(\mathbf{ab})^b = \chi_*^{-1}(\mathbf{a})^b \chi_*^{-1}(\mathbf{b})^\# = \chi_*^{-1}(\mathbf{a})^/ \chi_*^{-1}(\mathbf{b})^/
\end{aligned}
\tag{2.12.51}
$$

Push-Forward and Pull-Back operations for Strain Tensors

The push-forward of the covariant right Cauchy-Green strain and its contravariant inverse are

$$
\begin{aligned}
\chi_*(\mathbf{C})^b &= C_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{F}^{-\mathrm{T}}\mathbf{C}\mathbf{F}^{-1} \\
\chi_*\left(\mathbf{C}^{-1}\right)^\# &= \left(C^{-1}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{F}\mathbf{C}\mathbf{F}^{\mathrm{T}}
\end{aligned}.
\tag{2.12.52}
$$

From 2.10.39, $C_{ij} = g_{ij}$, the covariant components of the identity tensor expressed in terms of the convected base vectors in the current configuration, i.e. the spatial metric tensor, $\mathbf{g} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^i$, and $\left(C^{-1}\right)^{ij} = g^{ij}$, the contravariant components of **g**. Thus the push-forward of covariant **C** is **g** and the pull-back of covariant **g** is **C**, and the push-forward of contravariant $\mathbf{C}^{-1}$ is **g** and the pull-back of contravariant **g** is $\mathbf{C}^{-1}$:

$$
\boxed{
\begin{aligned}
\chi_*(\mathbf{C})^b &= \mathbf{g}, \quad \chi_*^{-1}(\mathbf{g})^b = \mathbf{C} \\
\chi_*\left(\mathbf{C}^{-1}\right)^\# &= \mathbf{g}, \quad \chi_*^{-1}(\mathbf{g})^\# = \mathbf{C}^{-1}
\end{aligned}
}
\tag{2.12.53}
$$

Push-forward of the right Cauchy-Green strain

Similarly, the pull-back of covariant $\mathbf{b}^{-1}$ is **G** and the push-forward of covariant **G** is $\mathbf{b}^{-1}$**,** and the pull-back of contravariant **b** is **G** and the push-forward of contravariant **G** is **b**.

$$
\boxed{
\begin{aligned}
\chi_*(\mathbf{G})^b &= \mathbf{b}^{-1}, \quad \chi_*^{-1}\left(\mathbf{b}^{-1}\right)^b = \mathbf{G} \\
\chi_*(\mathbf{G})^\# &= \mathbf{b}, \qquad \chi_*^{-1}(\mathbf{b})^\# = \mathbf{G}
\end{aligned}
}
\tag{2.12.54}
$$

Pull-back of the left Cauchy-Green strain

For the covariant form of the Green-Lagrange strain, the push-forward is

$$
\chi_*(\mathbf{E})^b = E_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{F}^{-\mathrm{T}}\mathbf{E}\mathbf{F}^{-1}.
\tag{2.12.55}
$$

From 2.10.43, $E_{ij} = e_{ij}$, the covariant components of the Euler-Almansi strain tensor, and so the push-forward of covariant **E** is **e** and the pull-back of covariant **e** is **E**.

$$\boxed{\chi_*(\mathbf{E})^b = \mathbf{e}, \qquad \chi_*^{-1}(\mathbf{e})^b = \mathbf{E}}. \tag{2.12.56}$$

Push-forward of the Green-Lagrange strain
Pull-back of the Euler-Almansi strain

Push-Forward and Pull-Back with Polar Decomposition Intermediate Configurations

Pull backs and push-forwards can be defined relative to any two configurations. Consider the polar decomposition and the intermediate configurations discussed in §2.10 (see Fig. 2.10.11). Effectively, we are replacing **F** with **R**: pushing forward a material tensor **A** from the reference configuration $\{\mathbf{G}_i\}$ to the configuration $\{\hat{\mathbf{G}}_i\}$ leads to

$$\begin{aligned}
\chi_*(\mathbf{A})^b{}_{\mathbf{R}(\mathbf{G})} &= A_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j = A_{ij}\left(\mathbf{R}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{R}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{R}^{-\mathrm{T}}\mathbf{A}\mathbf{R}^{-1} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^\#{}_{\mathbf{R}(\mathbf{G})} &= A^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j = A^{ij}\left(\mathbf{R}\mathbf{G}_i \otimes \mathbf{R}\mathbf{G}_j\right) = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^\backslash{}_{\mathbf{R}(\mathbf{G})} &= A^i_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j = A^i_{\cdot j}\left(\mathbf{R}\mathbf{G}_i \otimes \mathbf{R}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{R}\mathbf{A}\mathbf{R}^{-1} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}} \\
\chi_*(\mathbf{A})^/{}_{\mathbf{R}(\mathbf{G})} &= A_i^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j = A_i^{\cdot j}\left(\mathbf{R}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{R}\mathbf{G}_j\right) = \mathbf{R}^{-\mathrm{T}}\mathbf{A}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{A}\mathbf{R}^{\mathrm{T}}
\end{aligned}. \tag{2.12.57}$$

Note that the result is the same regardless of whether one is using the covariant, contravariant or mixed forms.

Similarly, the pull back of a tensor $\hat{\mathbf{A}}$ from the intermediate configuration $\{\hat{\mathbf{G}}_i\}$ to the reference configuration $\{\mathbf{G}_i\}$ is

$$\begin{aligned}
\chi_*^{-1}(\hat{\mathbf{A}})^b{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^\#{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^\backslash{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R} \\
\chi_*^{-1}(\hat{\mathbf{A}})^/{}_{\mathbf{R}(\hat{\mathbf{G}})} &= \hat{A}_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{A}}\mathbf{R}
\end{aligned} \tag{2.12.58}$$

The push-forward of a tensor $\hat{\mathbf{a}}$ from $\{\hat{\mathbf{g}}_i\}$ to $\{\mathbf{g}\}$ and the corresponding pull-back of a spatial tensor **a** is

$$\begin{aligned}
\chi_*(\hat{\mathbf{a}})^b{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^b{}_{\mathbf{R}(\mathbf{g})} &= a_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^\#{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^\#{}_{\mathbf{R}(\mathbf{g})} &= a^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^\backslash{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}}, & \chi_*^{-1}(\mathbf{a})^\backslash{}_{\mathbf{R}(\mathbf{g})} &= a^i_{\cdot j}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R} \\
\chi_*(\hat{\mathbf{a}})^/{}_{\mathbf{R}(\hat{\mathbf{g}})} &= \hat{a}_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \mathbf{R}\hat{\mathbf{a}}\mathbf{R}^{\mathrm{T}} & \chi_*^{-1}(\mathbf{a})^/{}_{\mathbf{R}(\mathbf{g})} &= a_i^{\cdot j}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R}
\end{aligned} \tag{2.12.59}$$

The push-forwards and pull-backs due to the stretch tensors are

$$\left.\begin{aligned}
\chi_*(\mathbf{A})^b{}_{\mathbf{U}(\mathbf{G})} &= A_{ij}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j = A_{ij}\left(\mathbf{U}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{U}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{U}^{-\mathrm{T}}\mathbf{A}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}^{-1} \\
\chi_*(\mathbf{A})^\#{}_{\mathbf{U}(\mathbf{G})} &= A^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = A^{ij}\left(\mathbf{U}\mathbf{G}_i \otimes \mathbf{U}\mathbf{G}_j\right) = \mathbf{U}\mathbf{A}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\mathbf{A}\mathbf{U} \\
\chi_*(\mathbf{A})^\backslash{}_{\mathbf{U}(\mathbf{G})} &= A^i_{\cdot j}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j = A^i_{\cdot j}\left(\mathbf{U}\mathbf{G}_i \otimes \mathbf{U}^{-\mathrm{T}}\mathbf{G}^j\right) = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} \\
\chi_*(\mathbf{A})^/{}_{\mathbf{U}(\mathbf{G})} &= A_i^{\cdot j}\hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j = A_i^{\cdot j}\left(\mathbf{U}^{-\mathrm{T}}\mathbf{G}^i \otimes \mathbf{U}\mathbf{G}_j\right) = \mathbf{U}^{-\mathrm{T}}\mathbf{A}\mathbf{U}^{\mathrm{T}} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}
\end{aligned}\right. . \qquad (2.12.60)$$

$$\begin{aligned}
\chi_*^{-1}(\hat{\mathbf{a}})^b{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \mathbf{U}\hat{\mathbf{a}}\mathbf{U} \\
\chi_*^{-1}(\hat{\mathbf{a}})^\#{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \mathbf{U}^{-1}\hat{\mathbf{a}}\mathbf{U}^{-1} \\
\chi_*^{-1}(\hat{\mathbf{a}})^\backslash{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j = \mathbf{U}^{-1}\hat{\mathbf{a}}\mathbf{U} \\
\chi_*^{-1}(\hat{\mathbf{a}})^/{}_{\mathbf{U}(\hat{\mathbf{g}})} &= \hat{a}_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \mathbf{U}\hat{\mathbf{a}}\mathbf{U}^{-1}
\end{aligned} \qquad (2.12.61)$$

and

$$\begin{aligned}
\chi_*(\hat{\mathbf{A}})^b{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{v}^{-1}\hat{\mathbf{A}}\mathbf{v}^{-1} \\
\chi_*(\hat{\mathbf{A}})^\#{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \mathbf{v}\hat{\mathbf{A}}\mathbf{v} \\
\chi_*(\hat{\mathbf{A}})^\backslash{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{v}\hat{\mathbf{A}}\mathbf{v}^{-1} \\
\chi_*(\hat{\mathbf{A}})^/{}_{\mathbf{v}(\hat{\mathbf{G}})} &= \hat{A}_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \mathbf{v}^{-1}\hat{\mathbf{A}}\mathbf{v}
\end{aligned}, \quad
\begin{aligned}
\chi_*^{-1}(\mathbf{a})^b{}_{\mathbf{v}(\mathbf{g})} &= a_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j = \mathbf{v}\mathbf{a}\mathbf{v} \\
\chi_*^{-1}(\mathbf{a})^\#{}_{\mathbf{v}(\mathbf{g})} &= a^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j = \mathbf{v}^{-1}\mathbf{a}\mathbf{v}^{-1} \\
\chi_*^{-1}(\mathbf{a})^\backslash{}_{\mathbf{v}(\mathbf{g})} &= a^i_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j = \mathbf{v}^{-1}\mathbf{a}\mathbf{v} \\
\chi_*^{-1}(\mathbf{a})^/{}_{\mathbf{v}(\mathbf{g})} &= a_i^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j = \mathbf{v}\mathbf{a}\mathbf{v}^{-1}
\end{aligned} \qquad (2.12.62)$$

Push-forwards and pull-backs can also be defined using $\mathbf{F}^{\mathrm{T}}$ (in the place of **F**) and these move between the intermediate configurations, $\hat{\mathbf{G}} \Leftrightarrow \hat{\mathbf{g}}$.

Recall Eqn. 2.10.64, which state that the covariant components of $\mathbf{U}, \mathbf{v}, \mathbf{U}^{-1}, \mathbf{v}^{-1}$ with respect to the bases $\mathbf{G}^i, \hat{\mathbf{G}}^i, \hat{\mathbf{g}}^i, \mathbf{g}^i$ respectively, are equal. This can be explained also in terms of push-forwards and pull-backs. For example, with $\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}$ and $\mathbf{v}^{-1} = \mathbf{R}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}}$, one can write (in fact these relations are valid for all component types)

$$\mathbf{v} = \chi_*(\mathbf{U})_{\mathbf{R}(\mathbf{G})}, \qquad \mathbf{v}^{-1} = \chi_*\left(\mathbf{U}^{-1}\right)_{\mathbf{R}(\hat{\mathbf{g}})} \qquad (2.12.63)$$

The first of these shows that the components of **U** with respect to $\mathbf{G}$ are the same as those of **v** with respect to $\hat{\mathbf{G}}$ (for all component types). The second shows that the components of $\mathbf{U}^{-1}$ with respect to $\hat{\mathbf{g}}$ are the same as those of $\mathbf{v}^{-1}$ with respect to **g**.

As another example, with $\mathbf{C} = \mathbf{U}^2$,

$$\mathbf{C} = \chi_*^{-1}(\hat{\mathbf{g}})^b{}_{\mathbf{U}(\hat{\mathbf{g}})}, \qquad \mathbf{C}^{-1} = \chi_*^{-1}(\hat{\mathbf{g}})^\#{}_{\mathbf{U}(\hat{\mathbf{g}})} \qquad (2.12.64)$$

2.12.3 The Lie Time Derivative

The **Lie (time) derivative** is a concept of tensor analysis which is used to distinguish between the change in some quantity, and the change in that quantity excluding changes due to the motion/configuration changes. As mentioned in the introduction to this section, we can imagine a hypothetical observer attached to the deforming material, who moves and deforms with the material. This observer will see no change in the configuration itself, $\dot{\mathbf{g}}_i = \dot{\mathbf{g}}^i = 0$. However, they will still see changes to vectors and tensors. These changes are measured using the Lie Derivative, which will be seen to be none other than the convected derivative discussed above.

Vectors

First, the **Lie (time) derivative** $\mathrm{L}_v \mathbf{v}$ of a vector $\mathbf{v}$ is the material derivative *holding the deformed basis constant*, that is, Eqns. 2.12.3:

$$\begin{aligned} \mathrm{L}_v^b\,\mathbf{v} &= \dot{v}_i \mathbf{g}^i \\ \mathrm{L}_v^\#\,\mathbf{v} &= \dot{v}^i \mathbf{g}_i \end{aligned} \qquad (2.12.65)$$

Formally, it is defined in terms of the pull-back and push-forward,

$$\boxed{\mathrm{L}_v \mathbf{v} = \chi_*\left(\frac{d}{dt}\left[\chi_*^{-1}(\mathbf{v})\right]\right)} \qquad \textbf{The Lie Time Derivative} \qquad (2.12.66)$$

This is illustrated in the Fig. 2.12.3. The spatial vector is first pulled back to the reference configuration, there the differentiation is carried out, where the base vectors are constant, then the vector is pushed forward again to the spatial description.

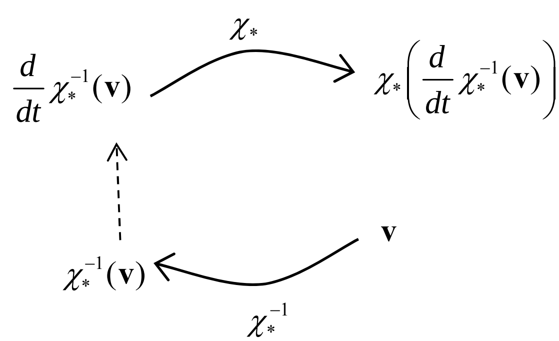

Figure 2.12.3: The Lie Derivative

For covariant components, one first pulls back the vector $v_i \mathbf{g}^i$ to $v_i \mathbf{G}^i$, the derivative is taken, $\dot{v}_i \mathbf{G}^i$, and then it is pushed forward to $\dot{v}_i \mathbf{g}^i$, which is consistent with the definition 2.12.65a. The definition 2.12.51 allows one to calculate the Lie derivative in absolute notation: using 2.12.36a, 2.12.33a, 2.11.9,

$$\begin{aligned} \mathrm{L}_v^b \mathbf{v} &= \chi_* \left(\frac{d}{dt} \left[\chi_*^{-1} \left(\mathbf{v} \right)^b \right] \right)^b = \mathbf{F}^{-\mathrm{T}} \left(\frac{d}{dt} \left[\mathbf{F}^{\mathrm{T}} \mathbf{v} \right] \right) \\ &= \mathbf{F}^{-\mathrm{T}} \left(\dot{\mathbf{F}}^{\mathrm{T}} \mathbf{v} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{v}} \right) \\ &= \mathbf{F}^{-\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}} \mathbf{l}^{\mathrm{T}} \mathbf{v} + \mathbf{F}^{\mathrm{T}} \dot{\mathbf{v}} \right) \\ &= \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v} \end{aligned} \tag{2.12.67}$$

The Lie derivative for the contravariant components can be calculated in a similar way, and in summary (these are simply Eqns. 2.12.6): {▲Problem 2}

$$\boxed{\begin{aligned} \mathrm{L}_v^b \mathbf{v} &= \dot{v}_i \mathbf{g}^i = \dot{\mathbf{v}} + \mathbf{l}^{\mathrm{T}} \mathbf{v} \\ \mathrm{L}_v^\# \mathbf{v} &= \dot{v}^i \mathbf{g}_i = \dot{\mathbf{v}} - \mathbf{l} \mathbf{v} \end{aligned}} \qquad \textbf{Lie Derivatives of Vectors} \tag{2.12.68}$$

Tensors

The material time derivative of a spatial tensor **a** is

$$\begin{aligned} \dot{\mathbf{a}} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j + a_{ij} \dot{\mathbf{g}}^i \otimes \mathbf{g}^j + a_{ij} \mathbf{g}^i \otimes \dot{\mathbf{g}}^j \\ &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j + a^{ij} \dot{\mathbf{g}}_i \otimes \mathbf{g}_j + a^{ij} \mathbf{g}_i \otimes \dot{\mathbf{g}}_j \\ &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j + a^i_{\cdot j} \dot{\mathbf{g}}_i \otimes \mathbf{g}^j + a^i_{\cdot j} \mathbf{g}_i \otimes \dot{\mathbf{g}}^j \\ &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j + a_i^{\cdot j} \dot{\mathbf{g}}^i \otimes \mathbf{g}_j + a_i^{\cdot j} \mathbf{g}^i \otimes \dot{\mathbf{g}}_j \end{aligned} \tag{2.12.69}$$

The Lie (time) derivative $\mathrm{L}_v \mathbf{a}$ is then

$$\begin{aligned} \mathrm{L}_v^b \mathbf{a} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \\ \mathrm{L}_v^\# \mathbf{a} &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \\ \mathrm{L}_v^\backslash \mathbf{a} &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j \\ \mathrm{L}_v^/ \mathbf{a} &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j \end{aligned} \tag{2.12.70}$$

For example, for covariant components, one first pulls back the tensor $a_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$ to $a_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$, the derivative is taken, $\dot{a}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$, and then it is pushed forward to $\dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$. Thus, using 2.12.42a, 2.12.42a, 2.11.9,

$$
\begin{aligned}
\mathrm{L}_v^b \mathbf{a} &= \chi_* \left(\frac{d}{dt} \left[\chi_*^{-1} (\mathbf{a})^b \right] \right)^b \\
&= \mathbf{F}^{-\mathrm{T}} \left(\dot{\mathbf{F}}^\mathrm{T} \mathbf{a} \mathbf{F} + \mathbf{F}^\mathrm{T} \dot{\mathbf{a}} \mathbf{F} + \mathbf{F}^\mathrm{T} \mathbf{a} \dot{\mathbf{F}} \right) \mathbf{F}^{-1} \\
&= \mathbf{F}^{-\mathrm{T}} \left(\mathbf{F}^\mathrm{T} \mathbf{l}^\mathrm{T} \mathbf{a} \mathbf{F} + \mathbf{F}^\mathrm{T} \dot{\mathbf{a}} \mathbf{F} + \mathbf{F}^\mathrm{T} \mathbf{a} \mathbf{l} \mathbf{F} \right) \mathbf{F}^{-1} \\
&= \mathbf{l}^\mathrm{T} \mathbf{a} + \dot{\mathbf{a}} + \mathbf{a} \mathbf{l}
\end{aligned}
\tag{2.12.71}
$$

The Lie derivative for the other components can be calculated in a similar way, and in summary (these are Eqns. 2.12.14): {▲Problem 3}

$$
\boxed{
\begin{aligned}
\mathrm{L}_v^b \mathbf{a} &= \dot{a}_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = \dot{\mathbf{a}} + \mathbf{l}^\mathrm{T} \mathbf{a} + \mathbf{a} \mathbf{l} \\
\mathrm{L}_v^\# \mathbf{a} &= \dot{a}^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = \dot{\mathbf{a}} - \mathbf{l} \mathbf{a} - \mathbf{a} \mathbf{l}^\mathrm{T} \\
\mathrm{L}_v^\backslash \mathbf{a} &= \dot{a}^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j = \dot{\mathbf{a}} - \mathbf{l} \mathbf{a} + \mathbf{a} \mathbf{l} \\
\mathrm{L}_v^/ \mathbf{a} &= \dot{a}_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j = \dot{\mathbf{a}} + \mathbf{l}^\mathrm{T} \mathbf{a} - \mathbf{a} \mathbf{l}^\mathrm{T}
\end{aligned}
}
\qquad \textbf{Lie Derivatives of Tensors}
\tag{2.12.72}
$$

The first of these, $\dot{\mathbf{a}} + \mathbf{l}^\mathrm{T} \mathbf{a} + \mathbf{a} \mathbf{l}$, is called the **Cotter-Rivlin rate**. The second of these, $\dot{\mathbf{a}} - \mathbf{l} \mathbf{a} - \mathbf{a} \mathbf{l}^\mathrm{T}$, is also called the **Oldroyd rate**.

Lie Derivatives of Strain Tensors

From 2.5.18,

$$
\begin{aligned}
&\mathbf{d} = \dot{\mathbf{e}} + \mathbf{l}^\mathrm{T} \mathbf{e} + \mathbf{e} \mathbf{l} \\
&\dot{\mathbf{b}} - \mathbf{l} \mathbf{b} - \mathbf{b} \mathbf{l}^\mathrm{T} = \mathbf{0}
\end{aligned}
\tag{2.12.73}
$$

and so the Lie derivative of the covariant form of the Euler-Almansi strain is the rate of deformation and the Lie derivative of the contravariant form of the left Cauchy-Green tensor is zero. Further, from 2.12.53a, the Lie derivative of the metric tensor is the push forward of the material time derivative of the right Cauchy-Green strain:

$$
\mathrm{L}_v^b \mathbf{g} = \chi_* \left(\dot{\mathbf{C}} \right)^b, \tag{2.12.74}
$$

Also, directly from 2.11.15,

$$
\mathrm{L}_v^b \mathbf{g} = 2\mathbf{d} \tag{2.12.75}
$$

Corotational Rates

The Lie derivatives in 2.12.72 were derived using pull-backs and push-forwards between the reference configuration and the current configuration. If, instead, we relate quantities to the rotated intermediate configuration, in other words use **R** instead of **F** in the calculations, we find that, using Eqn. 2.6.1, $\mathbf{\Omega_R} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$,

$$\begin{aligned} \mathrm{L}_v\mathbf{a} &= \chi_*\left(\frac{d}{dt}\left[\chi_*^{-1}(\mathbf{a})\right]\right) \\ &= \mathbf{R}\left(\frac{d}{dt}\left[\mathbf{R}^{\mathrm{T}}\mathbf{a}\mathbf{R}\right]\right)\mathbf{R}^{\mathrm{T}} \\ &= \dot{\mathbf{a}} - \mathbf{\Omega_R}\mathbf{a} + \mathbf{a}\mathbf{\Omega_R} \end{aligned} \qquad (2.12.76)$$

This is called the **Green-Naghdi rate**.

Rather than pulling back from the intermediate configuration to the reference configuration, we can choose the current configuration to be the reference configuration. Rotating from this configuration (see section 2.6.3), $\mathbf{\Omega_R} = \mathbf{w}$, the spin tensor, and one obtains the **Jaumann rate**, $\dot{\mathbf{a}} - \mathbf{w}\mathbf{a} + \mathbf{a}\mathbf{w}$.

Lie Derivatives and Objective Rates

The concept of objectivity was discussed in section 2.8. Essentially, if two observers are rotating relative to each other with rotation $\mathbf{Q}(t)$ and both are observing some spatial tensor, **T** as measured by one observer and $\mathbf{T}^*$ as measured by the other, then this tensor is objective provided $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}$ for all **Q**, i.e. the measurement of the deformation would be independent of the observer. One of the most important uses of the Lie derivative is that *Lie derivatives of objective spatial tensors are objective spatial tensors*. Thus the rates given in 2.12.72 are all objective.

For example, suppose we have an objective spatial tensor **a**, i.e. so that $\mathbf{a}^* = \mathbf{Q}\mathbf{a}\mathbf{Q}^{\mathrm{T}}$. The velocity gradient is not objective, and instead satisfies the relation 2.8.27: $\mathbf{l}^* = \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. Using the properties of the transpose, the orthogonality of **Q**, and the identity $\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}$, one has for Eqns. 2.12.72a,b,

$$
\begin{aligned}
\left(\mathrm{L}_v^b \mathbf{a}\right)^* &= \dot{\overline{\mathbf{a}^*}} + \mathbf{l}^{*\mathrm{T}}\mathbf{a}^* + \mathbf{a}^*\mathbf{l}^* \\
&= \dot{\overline{\mathbf{QaQ}^\mathrm{T}}} + \left(\mathbf{QlQ}^\mathrm{T} + \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T}\right)^\mathrm{T}\left(\mathbf{QaQ}^\mathrm{T}\right) + \left(\mathbf{QaQ}^\mathrm{T}\right)\left(\mathbf{QlQ}^\mathrm{T} + \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T}\right) \\
&= \dot{\mathbf{Q}}\mathbf{aQ}^\mathrm{T} + \mathbf{Q}\dot{\mathbf{a}}\mathbf{Q}^\mathrm{T} + \mathbf{Qa}\dot{\mathbf{Q}}^\mathrm{T} + \mathbf{Ql}^\mathrm{T}\mathbf{Q}^\mathrm{T}\mathbf{QaQ}^\mathrm{T} + \mathbf{Q}\dot{\mathbf{Q}}^\mathrm{T}\mathbf{QaQ}^\mathrm{T} \\
&\qquad + \mathbf{QaQ}^\mathrm{T}\mathbf{QlQ}^\mathrm{T} + \mathbf{QaQ}^\mathrm{T}\dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T} \\
&= \mathbf{Q}\left(\dot{\mathbf{a}} + \mathbf{l}^\mathrm{T}\mathbf{a} + \mathbf{al}\right)\mathbf{Q}^\mathrm{T} \\
\left(\mathrm{L}_v^\# \mathbf{a}\right)^* &= \dot{\overline{\mathbf{a}^*}} - \mathbf{l}^*\mathbf{a}^* - \mathbf{a}^*\mathbf{l}^{*\mathrm{T}} \\
&= \dot{\overline{\mathbf{QaQ}^\mathrm{T}}} - \left(\mathbf{QlQ}^\mathrm{T} + \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T}\right)\left(\mathbf{QaQ}^\mathrm{T}\right) - \left(\mathbf{QaQ}^\mathrm{T}\right)\left(\mathbf{QlQ}^\mathrm{T} + \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T}\right)^\mathrm{T} \\
&= \dot{\mathbf{Q}}\mathbf{aQ}^\mathrm{T} + \mathbf{Q}\dot{\mathbf{a}}\mathbf{Q}^\mathrm{T} + \mathbf{Qa}\dot{\mathbf{Q}}^\mathrm{T} - \mathbf{QlQ}^\mathrm{T}\mathbf{QaQ}^\mathrm{T} - \dot{\mathbf{Q}}\mathbf{Q}^\mathrm{T}\mathbf{QaQ}^\mathrm{T} \\
&\qquad - \mathbf{QaQ}^\mathrm{T}\mathbf{Ql}^\mathrm{T}\mathbf{Q}^\mathrm{T} - \mathbf{QaQ}^\mathrm{T}\mathbf{Q}\dot{\mathbf{Q}}^\mathrm{T} \\
&= \mathbf{Q}\left(\dot{\mathbf{a}} - \mathbf{la} - \mathbf{al}^\mathrm{T}\right)\mathbf{Q}^\mathrm{T}
\end{aligned}
\tag{2.12.77}
$$

showing that these rates are indeed objective.

Further, any linear combination of them is objective, for example,

$$
\frac{1}{2}\left[\left(\dot{\mathbf{a}} + \mathbf{l}^\mathrm{T}\mathbf{a} + \mathbf{al}\right) + \left(\dot{\mathbf{a}} - \mathbf{la} - \mathbf{al}^\mathrm{T}\right)\right] = \dot{\mathbf{a}} + \frac{1}{2}\left[-\left(\mathbf{l} - \mathbf{l}^\mathrm{T}\right)\mathbf{a} + \mathbf{a}\left(\mathbf{l} - \mathbf{l}^\mathrm{T}\right)\right] = \dot{\mathbf{a}} - \mathbf{wa} + \mathbf{aw} \tag{2.12.78}
$$

is objective, provided $\mathbf{a}$ is. This is the **Jaumann rate** introduced in Eqn. 2.8.36 and mentioned after Eqn. 2.12.76 above. Further, as mentioned after Eqn. 2.12.72, the **Cotter-Rivlin rate** of Eqn. 2.8.37 is equivalent to $\mathrm{L}_v^b \mathbf{a}$.

The Lie Derivative and the Directional Derivative

Recall that the material time derivative of a tensor can be written in terms of the directional derivative, §2.6.5. Hence the Lie derivative can also be expressed as

$$
\mathrm{L}_v \mathbf{T} = \chi_*\left(\partial_\mathbf{f}\left(\chi_*^{-1}(\mathbf{T})\right)[\mathbf{v}]\right) \tag{2.12.79}
$$

and hence the subscript v on the L. Thus one can say that the Lie derivative is the push forward of the directional derivative of the material field $\chi_*^{-1}(\mathbf{T})$ in the direction of the velocity vector.

2.12.4 Problems

1. Eqns. 2.12.30 follow immediately from 2.12.29. However, use Eqns. 2.12.40, 2.12.42, i.e. $\chi_*(\mathbf{A})^b = \mathbf{F}^{-\mathrm{T}}\mathbf{A}\mathbf{F}^{-1}$, etc., directly, to verify relations 2.12.45.
2. Derive the Lie derivatives of a vector **v**, Eqns. 2.12.68.
3. Derive the Lie derivatives of a tensor **a**, Eqns. 2.12.72.