2.2 Deformation and Strain

A number of useful ways of describing and quantifying the deformation of a material are discussed in this section.

Attention is restricted to the reference and current configurations. No consideration is given to the particular sequence by which the current configuration is reached from the reference configuration and so the deformation can be considered to be independent of time. In what follows, particles in the reference configuration will often be termed "undeformed" and those in the current configuration "deformed".

In a change from Chapter 1, lower case letters will now be reserved for both vector- *and* tensor- functions of the spatial coordinates **x**, whereas upper-case letters will be reserved for functions of material coordinates **X**. There will be exceptions to this, but it should be clear from the context what is implied.

2.2.1 The Deformation Gradient

The **deformation gradient F** is the fundamental measure of deformation in continuum mechanics. It is the second order tensor which maps line elements in the reference configuration into line elements (consisting of the *same* material particles) in the current configuration.

Consider a line element $d\mathbf{X}$ emanating from position **X** in the reference configuration which becomes $d\mathbf{x}$ in the current configuration, Fig. 2.2.1. Then, using 2.1.3,

$$\begin{aligned} d\mathbf{x} &= \boldsymbol{\chi}(\mathbf{X} + d\mathbf{X}) - \boldsymbol{\chi}(\mathbf{X}) \\ &= (\mathrm{Grad}\,\boldsymbol{\chi})d\mathbf{X} \end{aligned} \tag{2.2.1}$$

A capital G is used on "Grad" to emphasise that this is a gradient with respect to the material coordinates[1], the **material gradient**, $\partial\boldsymbol{\chi}/\partial\mathbf{X}$.

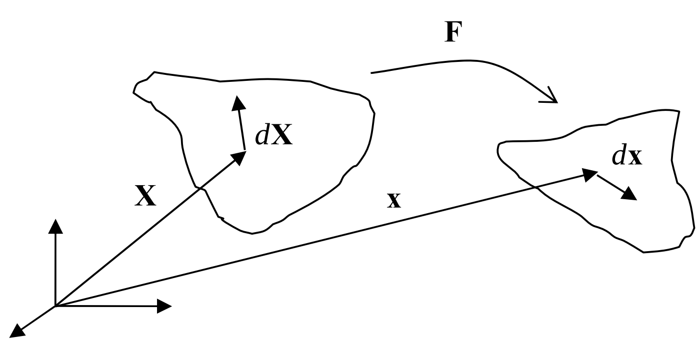

Figure 2.2.1: the Deformation Gradient acting on a line element

[1] one can have material gradients and spatial gradients of material or spatial fields – see later

The motion vector-function $\boldsymbol{\chi}$ in 2.1.3, 2.2.1, is a function of the variable **X**, but it is customary to denote this simply by **x**, the value of $\boldsymbol{\chi}$ at **X**, i.e. $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, so that

$$\boxed{\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \operatorname{Grad}\mathbf{x}, \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}} \quad \textbf{Deformation Gradient} \qquad (2.2.2)$$

with

$$\boxed{d\mathbf{x} = \mathbf{F}\, d\mathbf{X}, \qquad dx_i = F_{iJ}\, dX_J} \quad \textbf{action of F} \qquad (2.2.3)$$

Lower case indices are used in the index notation to denote quantities associated with the spatial basis $\{\mathbf{e}_i\}$ whereas upper case indices are used for quantities associated with the material basis $\{\mathbf{E}_I\}$.

Note that

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} d\mathbf{X}$$

is a differential quantity and this expression has some error associated with it; the error (due to terms of order $(d\mathbf{X})^2$ and higher, neglected from a Taylor series) tends to zero as the differential $d\mathbf{X} \to 0$. The deformation gradient (whose components are finite) thus characterises the deformation in the *neighbourhood* of a point **X**, mapping infinitesimal line elements $d\mathbf{X}$ emanating from **X** in the reference configuration to the infinitesimal line elements $d\mathbf{x}$ emanating from **x** in the current configuration, Fig. 2.2.2.

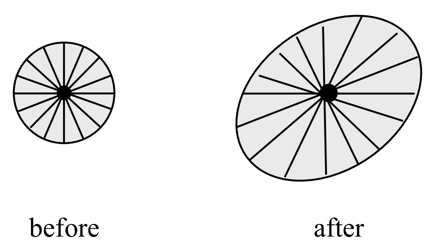

Figure 2.2.2: deformation of a material particle

Example

Consider the cube of material with sides of unit length illustrated by dotted lines in Fig. 2.2.3. It is deformed into the rectangular prism illustrated (this could be achieved, for example, by a continuous rotation and stretching motion). The material and spatial coordinate axes are coincident. The material description of the deformation is

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = -6X_2\mathbf{e}_1 + \frac{1}{2}X_1\mathbf{e}_2 + \frac{1}{3}X_3\mathbf{e}_3$$

and the spatial description is

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = 2x_2\mathbf{E}_1 - \frac{1}{6}x_1\mathbf{E}_2 + 3x_3\mathbf{E}_3$$

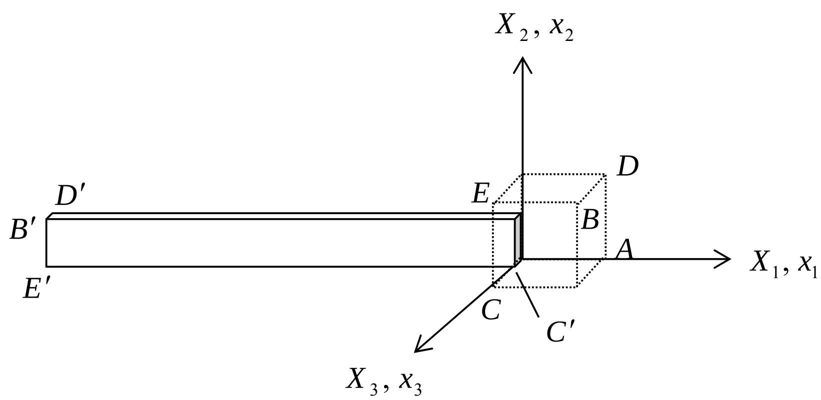

Figure 2.2.3: a deforming cube

Then

$$\mathbf{F} = \frac{\partial x_i}{\partial X_J} = \begin{bmatrix} 0 & -6 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Once **F** is known, the position of any element can be determined. For example, taking a line element $d\mathbf{X} = [da, 0, 0]^{\mathrm{T}}$, $d\mathbf{x} = \mathbf{F}d\mathbf{X} = [0, da/2, 0]^{\mathrm{T}}$.

■

Homogeneous Deformations

A **homogeneous deformation** is one where the deformation gradient is uniform, i.e. independent of the coordinates, and the associated motion is termed **affine**. Every part of the material deforms as the whole does, and straight parallel lines in the reference configuration map to straight parallel lines in the current configuration, as in the above example. Most examples to be considered in what follows will be of homogeneous deformations; this keeps the algebra to a minimum, but homogeneous deformation analysis is very useful in itself since most of the basic experimental testing of materials, e.g. the uniaxial tensile test, involve homogeneous deformations.

Rigid Body Rotations and Translations

One can add a constant vector **c** to the motion, $\mathbf{x} = \mathbf{x} + \mathbf{c}$, without changing the deformation, $\mathrm{Grad}(\mathbf{x} + \mathbf{c}) = \mathrm{Grad}\,\mathbf{x}$. Thus the deformation gradient does not take into account rigid-body **translations** of bodies in space. If a body only translates as a rigid body in space, then $\mathbf{F} = \mathbf{I}$, and $\mathbf{x} = \mathbf{X} + \mathbf{c}$ (again, note that **F** does not tell us where in space a particle is, only how it has deformed locally). If there is *no* motion, then not only is $\mathbf{F} = \mathbf{I}$, but $\mathbf{x} = \mathbf{X}$.

If the body rotates as a rigid body (with no translation), then $\mathbf{F} = \mathbf{R}$, a rotation tensor (§1.10.8). For example, for a rotation of θ about the X_2 axis,

$$\mathbf{F} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}$$

Note that different particles of the same material body can be translating only, rotating only, deforming only, or any combination of these.

The Inverse of the Deformation Gradient

The inverse deformation gradient $\mathbf{F}^{-1}$ carries the spatial line element d**x** to the material line element d**X**. It is defined as

$$\boxed{\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \operatorname{grad}\mathbf{X}, \qquad F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j}} \quad \textbf{Inverse Deformation Gradient} \qquad (2.2.4)$$

so that

$$\boxed{d\mathbf{X} = \mathbf{F}^{-1}\, d\mathbf{x}, \qquad dX_I = F_{Ij}^{-1}\, dx_j} \quad \textbf{action of } \mathbf{F}^{-1} \qquad (2.2.5)$$

with (see Eqn. 1.15.2)

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{F}\mathbf{F}^{-1} = \mathbf{I} \qquad F_{iM} F_{Mj}^{-1} = \delta_{ij} \qquad (2.2.6)$$

Cartesian Base Vectors

Explicitly, in terms of the material and spatial base vectors (see 1.14.3),

$$\begin{aligned} \mathbf{F} &= \frac{\partial \mathbf{x}}{\partial X_J} \otimes \mathbf{E}_J = \frac{\partial x_i}{\partial X_J} \mathbf{e}_i \otimes \mathbf{E}_J \\ \mathbf{F}^{-1} &= \frac{\partial \mathbf{X}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial X_I}{\partial x_j} \mathbf{E}_I \otimes \mathbf{e}_j \end{aligned} \qquad (2.2.7)$$

so that, for example, $\mathbf{F}d\mathbf{X} = (\partial x_i / \partial X_J)\mathbf{e}_i \otimes \mathbf{E}_J (dX_M \mathbf{E}_M) = (\partial x_i / \partial X_J) dX_J \mathbf{e}_i = d\mathbf{x}$.

Because **F** and $\mathbf{F}^{-1}$ act on vectors in one configuration to produce vectors in the other configuration, they are termed **two-point tensors**. They are defined in both configurations. This is highlighted by their having both reference and current base vectors **E** and **e** in their Cartesian representation 2.2.7.

Here follow some important relations which relate scalar-, vector- and second-order tensor-valued functions in the material and spatial descriptions, the first two relating the material and spatial gradients {▲Problem 1}.

$$\begin{aligned} \text{grad}\phi &= \text{Grad}\phi\,\mathbf{F}^{-1} \\ \text{grad}\mathbf{v} &= \text{Grad}\mathbf{V}\,\mathbf{F}^{-1} \\ \text{div}\mathbf{a} &= \text{Grad}\mathbf{A} : \mathbf{F}^{-\mathrm{T}} \end{aligned} \tag{2.2.8}$$

Here, ϕ is a scalar; $\mathbf{V}$ and $\mathbf{v}$ are the *same* vector, the former being a function of the material coordinates, the material description, the latter a function of the spatial coordinates, the spatial description. Similarly, $\mathbf{A}$ is a second order tensor in the material form and $\mathbf{a}$ is the equivalent spatial form.

The first two of 2.2.8 relate the material gradient to the spatial gradient: the gradient of a function is a measure of how the function changes as one moves through space; since the material coordinates and the spatial coordinates differ, the change in a function with respect to a unit change in the material coordinates will differ from the change in the *same* function with respect to a unit change in the spatial coordinates (see also §2.2.7 below).

Example

Consider the deformation

$$\begin{aligned} \mathbf{x} &= (2X_2 - X_3)\mathbf{e}_1 + (-X_2)\mathbf{e}_2 + (X_1 + 3X_2 + X_3)\mathbf{e}_3 \\ \mathbf{X} &= (x_1 + 5x_2 + x_3)\mathbf{E}_1 + (-x_2)\mathbf{E}_2 + (-x_1 - 2x_2)\mathbf{E}_3 \end{aligned}$$

so that

$$\mathbf{F} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \qquad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ 0 & -1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$

Consider the vector $\mathbf{v}(\mathbf{x}) = (2x_1 - x_2)\mathbf{e}_1 + (-3x_2^2 + x_3)\mathbf{e}_2 + (x_1 + x_3)\mathbf{e}_3$ which, in the material description, is

$$\mathbf{V}(\mathbf{X}) = (5X_2 - 2X_3)\mathbf{E}_1 + (X_1 + 3X_2 + X_3 - 3X_2^2)\mathbf{E}_2 + (X_1 + 5X_2)\mathbf{E}_3$$

The material and spatial gradients are

$$\text{Grad}\mathbf{V} = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 3 - 6X_2 & 1 \\ 1 & 5 & 0 \end{bmatrix}, \qquad \text{grad}\mathbf{v} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and it can be seen that

$$\text{Grad}\mathbf{V}\,\mathbf{F}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 6X_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -6x_2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \text{grad}\,\mathbf{v}$$

■

2.2.2 The Cauchy-Green Strain Tensors

The deformation gradient describes how a line element in the reference configuration maps into a line element in the current configuration. It has been seen that the deformation gradient gives information about deformation (change of shape) and rigid body rotation, but does not encompass information about possible rigid body translations. The deformation and rigid rotation will be separated shortly (see §2.2.5). To this end, consider the following **strain** tensors; these tensors give direct information about the deformation of the body. Specifically, the **Left Cauchy-Green Strain** and **Right Cauchy-Green Strain** tensors give a measure of how the lengths of line elements and angles between line elements (through the vector dot product) change between configurations.

The Right Cauchy-Green Strain

Consider two line elements in the reference configuration $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$ which are mapped into the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$ in the current configuration. Then, using 1.10.3d,

$$\begin{aligned} d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} &= \left(\mathbf{F} d\mathbf{X}^{(1)}\right) \cdot \left(\mathbf{F} d\mathbf{X}^{(2)}\right) \\ &= d\mathbf{X}^{(1)} \left(\mathbf{F}^{\mathrm{T}} \mathbf{F}\right) d\mathbf{X}^{(2)} \\ &= d\mathbf{X}^{(1)} \mathbf{C}\, d\mathbf{X}^{(2)} \end{aligned} \qquad \textbf{action of C} \qquad (2.2.9)$$

where, by definition, $\mathbf{C}$ is the right Cauchy-Green Strain[2]

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \qquad C_{IJ} = F_{kI} F_{kJ} = \frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J} \qquad \textbf{Right Cauchy-Green Strain} \qquad (2.2.10)$$

It is a symmetric, positive definite (which will be clear from Eqn. 2.2.17 below), tensor, which implies that it has real positive eigenvalues (*cf.* §1.11.2), and this has important consequences (see later). Explicitly in terms of the base vectors,

$$\mathbf{C} = \left(\frac{\partial x_k}{\partial X_I} \mathbf{E}_I \otimes \mathbf{e}_k\right)\left(\frac{\partial x_m}{\partial X_J} \mathbf{e}_m \otimes \mathbf{E}_J\right) = \frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J} \mathbf{E}_I \otimes \mathbf{E}_J \,. \qquad (2.2.11)$$

Just as the line element $d\mathbf{X}$ is a vector defined in and associated with the reference configuration, $\mathbf{C}$ is defined in and associated with the reference configuration, acting on vectors in the reference configuration, and so is called a **material tensor**.

[2] "right" because **F** is on the right of the formula

The inverse of **C**, $\mathbf{C}^{-1}$, is called the **Piola deformation tensor**.

The Left Cauchy-Green Strain

Consider now the following, using Eqn. 1.10.18c:

$$\boxed{\begin{aligned} d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} &= \left(\mathbf{F}^{-1} d\mathbf{x}^{(1)}\right) \cdot \left(\mathbf{F}^{-1} d\mathbf{x}^{(2)}\right) \\ &= d\mathbf{x}^{(1)} \left(\mathbf{F}^{-\mathrm{T}} \mathbf{F}^{-1}\right) d\mathbf{x}^{(2)} \\ &= d\mathbf{x}^{(1)} \mathbf{b}^{-1} d\mathbf{x}^{(2)} \end{aligned}} \quad \textbf{action of } \mathbf{b}^{-1} \qquad (2.2.12)$$

where, by definition, **b** is the left Cauchy-Green Strain, also known as the **Finger tensor**:

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \qquad b_{ij} = F_{iK} F_{jK} = \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_K}} \quad \textbf{Left Cauchy-Green Strain} \qquad (2.2.13)$$

Again, this is a symmetric, positive definite tensor, only here, **b** is defined in the current configuration and so is called a **spatial tensor**.

The inverse of **b**, $\mathbf{b}^{-1}$, is called the **Cauchy deformation tensor**.

It can be seen that the right and left Cauchy-Green tensors are related through

$$\mathbf{C} = \mathbf{F}^{-1}\mathbf{b}\mathbf{F}, \qquad \mathbf{b} = \mathbf{F}\mathbf{C}\mathbf{F}^{-1} \qquad (2.2.14)$$

Note that tensors can be material (e.g. **C**), two-point (e.g. **F**) or spatial (e.g. **b**). Whatever type they are, they can always be described using material or spatial coordinates through the motion mapping 2.1.3, that is, using the material or spatial descriptions. Thus one distinguishes between, for example, a spatial tensor, which is an intrinsic property of a tensor, and the spatial description of a tensor.

The Principal Scalar Invariants of the Cauchy-Green Tensors

Using 1.10.10b,

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right) = \mathrm{tr}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = \mathrm{tr}\mathbf{b} \qquad (2.2.15)$$

This holds also for arbitrary powers of these tensors, $\mathrm{tr}\mathbf{C}^n = \mathrm{tr}\mathbf{b}^n$, and therefore, from Eqn. 1.11.17, the invariants of **C** and **b** are equal.

2.2.3 The Stretch

The **stretch** (or the **stretch ratio**) λ is defined as the ratio of the length of a deformed line element to the length of the corresponding undeformed line element:

$$\boxed{\lambda = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}} \qquad \textbf{The Stretch} \tag{2.2.16}$$

From the relations involving the Cauchy-Green Strains, letting $d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} \equiv d\mathbf{X}$, $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)} \equiv d\mathbf{x}$, and dividing across by the square of the length of $d\mathbf{X}$ or $d\mathbf{x}$,

$$\lambda^2 = \left(\frac{|d\mathbf{x}|}{|d\mathbf{X}|}\right)^2 = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}, \qquad \lambda^{-2} = \left(\frac{|d\mathbf{X}|}{|d\mathbf{x}|}\right)^2 = d\hat{\mathbf{x}}\mathbf{b}^{-1}d\hat{\mathbf{x}} \tag{2.2.17}$$

Here, the quantities $d\hat{\mathbf{X}} = d\mathbf{X}/|d\mathbf{X}|$ and $d\hat{\mathbf{x}} = d\mathbf{x}/|d\mathbf{x}|$ are unit vectors in the directions of $d\mathbf{X}$ and $d\mathbf{x}$. Thus, through these relations, **C** and **b** determine how much a line element stretches (and, from 2.2.17, **C** and **b** can be seen to be indeed positive definite).

One says that a line element is **extended**, **unstretched** or **compressed** according to $\lambda > 1$, $\lambda = 1$ or $\lambda < 1$.

Stretching along the Coordinate Axes

Consider three line elements lying along the three coordinate axes[3]. Suppose that the material deforms in a special way, such that these line elements undergo a **pure stretch**, that is, they change length with no change in the right angles between them. If the stretches in these directions are λ_1, λ_2 and λ_3, then

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3 \tag{2.2.18}$$

and the deformation gradient has only diagonal elements in its matrix form:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad F_{iJ} = \lambda_i \delta_{iJ} \quad \text{(no sum)} \tag{2.2.19}$$

Whereas material undergoes pure stretch along the coordinate directions, line elements off-axes will in general stretch/contract *and* rotate relative to each other. For example, a line element $d\mathbf{X} = [\alpha, \alpha, 0]^{\mathrm{T}}$ stretches by $\lambda = \sqrt{d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}}} = \sqrt{\left(\lambda_1^2 + \lambda_2^2\right)/2}$ with $d\mathbf{x} = [\lambda_1\alpha, \lambda_2\alpha, 0]^{\mathrm{T}}$, and rotates if $\lambda_1 \neq \lambda_2$.

It will be shown below that, for any deformation, there are always three mutually orthogonal directions along which material undergoes a pure stretch. These directions, the coordinate axes in this example, are called the **principal axes** of the material and the associated stretches are called the **principal stretches**.

[3] with the material and spatial basis vectors coincident

The Case of F Real and Symmetric

Consider now another special deformation, where **F** is a real symmetric tensor, in which case the eigenvalues are real and the eigenvectors form an orthonormal basis (*cf.* §1.11.2)[4]. In any given coordinate system, **F** will in general result in the stretching of line elements and the changing of the angles between line elements. However, if one chooses a coordinate set to be the eigenvectors of **F**, then from Eqn. 1.11.11-12 one can write[5]

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad [\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.2.20}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of **F**. The eigenvalues are the principal stretches and the eigenvectors are the principal axes. This indicates that as long as **F** is real and symmetric, one can always find a coordinate system along whose axes the material undergoes a pure stretch, with no rotation. This topic will be discussed more fully in §2.2.5 below.

2.2.4 The Green-Lagrange and Euler-Almansi Strain Tensors

Whereas the left and right Cauchy-Green tensors give information about the change in angle between line elements and the stretch of line elements, the **Green-Lagrange strain** and the **Euler-Almansi strain** tensors directly give information about the change in the squared length of elements.

Specifically, when the Green-Lagrange strain **E** operates on a line element d**X**, it gives (half) the change in the squares of the undeformed and deformed lengths:

$$\begin{aligned} \frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} &= \frac{1}{2}\{d\mathbf{X}\mathbf{C}d\mathbf{X} - d\mathbf{X}\cdot d\mathbf{X}\} \\ &= \frac{1}{2}\{d\mathbf{X}(\mathbf{C}-\mathbf{I})d\mathbf{X}\} \\ &\equiv d\mathbf{X}\mathbf{E}d\mathbf{X} \end{aligned} \qquad \textbf{action of E} \tag{2.2.21}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F}-\mathbf{I}), \quad E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \qquad \textbf{Green-Lagrange Strain} \tag{2.2.22}$$

It is a symmetric positive definite material tensor. Similarly, the (symmetric spatial) Euler-Almansi strain tensor is defined through

[4] in fact, **F** in this case will have to be positive definite, with $\det \mathbf{F} > 0$ (see later in §2.2.8)

[5] $\hat{\mathbf{n}}_i$ are the eigenvectors for the basis $\mathbf{e}_i$, $\hat{\mathbf{N}}_I$ for the basis $\hat{\mathbf{E}}_i$, with $\hat{\mathbf{n}}_i$, $\hat{\mathbf{N}}_I$ coincident; when the bases are not coincident, the notion of rotating line elements becomes ambiguous – this topic will be examined later in the context of *objectivity*

$$\boxed{\frac{|d\mathbf{x}|^2 - |d\mathbf{X}|^2}{2} = d\mathbf{x}\,\mathbf{e}\,d\mathbf{x}} \qquad \textbf{action of e} \tag{2.2.23}$$

and

$$\boxed{\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right)} \qquad \textbf{Euler-Almansi Strain} \tag{2.2.24}$$

Physical Meaning of the Components of E

Take a line element in the 1-direction, $d\mathbf{X}_{(1)} = [dX_1,\ 0,\ 0]^{\mathrm{T}}$, so that $d\hat{\mathbf{X}}_{(1)} = [1,\ 0,\ 0]^{\mathrm{T}}$. The square of the stretch of this element is

$$\lambda_{(1)}^2 = d\hat{\mathbf{X}}_{(1)}\mathbf{C}d\hat{\mathbf{X}}_{(1)} = C_{11} \quad \rightarrow \quad E_{11} = \frac{1}{2}\left(C_{11} - 1\right) = \frac{1}{2}\left(\lambda_{(1)}^2 - 1\right)$$

The unit extension is $\left(|d\mathbf{x}| - |d\mathbf{X}|\right)/|d\mathbf{X}| = \lambda - 1$. Denoting the unit extension of $d\mathbf{X}_{(1)}$ by $\mathbf{E}_{(1)}$, one has

$$E_{11} = \mathbf{E}_{(1)} + \frac{1}{2}\mathbf{E}_{(1)}^2 \tag{2.2.25}$$

and similarly for the other diagonal elements E_{22}, E_{33}.

When the deformation is small, $\mathbf{E}_{(1)}^2$ is small in comparison to $\mathbf{E}_{(1)}$, so that $E_{11} \approx \mathbf{E}_{(1)}$. For small deformations then, the diagonal terms are equivalent to the unit extensions.

Let θ_{12} denote the angle between the deformed elements which were initially parallel to the X_1 and X_2 axes. Then

$$\begin{aligned}\cos\theta_{12} &= \frac{d\mathbf{x}_{(1)}}{|d\mathbf{x}_{(1)}|} \cdot \frac{d\mathbf{x}_{(2)}}{|d\mathbf{x}_{(2)}|} = \frac{|d\mathbf{X}_{(1)}||d\mathbf{X}_{(2)}|}{|d\mathbf{x}_{(1)}||d\mathbf{x}_{(2)}|}\left\{\frac{d\mathbf{X}_{(1)}}{|d\mathbf{X}_{(1)}|} \cdot \mathbf{C}\frac{d\mathbf{X}_{(2)}}{|d\mathbf{X}_{(2)}|}\right\} = \frac{C_{12}}{\lambda_{(1)}\lambda_{(2)}} \\ &= \frac{2E_{12}}{\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}}\end{aligned} \tag{2.2.26}$$

and similarly for the other off-diagonal elements. Note that if $\theta_{12} = \pi/2$, so that there is no angle change, then $E_{12} = 0$. Again, if the deformation is small, then E_{11}, E_{22} are small, and

$$\frac{\pi}{2} - \theta_{12} \approx \sin\left(\frac{\pi}{2} - \theta_{12}\right) = \cos\theta_{12} \approx 2E_{12} \tag{2.2.27}$$

In words: for small deformations, the component E_{12} gives half the change in the original right angle.

2.2.5 Stretch and Rotation Tensors

The deformation gradient can always be decomposed into the product of two tensors, a stretch tensor and a rotation tensor (in one of two different ways, material or spatial versions). This is known as the **polar decomposition**, and is discussed in §1.11.7. One has

$$\boxed{\mathbf{F} = \mathbf{R}\mathbf{U}} \qquad \textbf{Polar Decomposition (Material)} \qquad (2.2.28)$$

Here, **R** is a proper orthogonal tensor, i.e. $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ with $\det \mathbf{R} = 1$, called *the* **rotation tensor**. It is a measure of the local rotation at **X**.

The decomposition is not unique; it is made unique by choosing **U** to be a *symmetric* tensor, called the **right stretch tensor**. It is a measure of the local stretching (or contraction) of material at **X**. Consider a line element $d\mathbf{X}$. Then

$$\lambda d\hat{\mathbf{x}} = \mathbf{F}d\hat{\mathbf{X}} = \mathbf{R}\mathbf{U}d\hat{\mathbf{X}} \qquad (2.2.29)$$

and so {▲Problem 2}

$$\lambda^2 = d\hat{\mathbf{X}}\mathbf{U} \cdot \mathbf{U}d\hat{\mathbf{X}} \qquad (2.2.30)$$

Thus (this is a definition of **U**)

$$\boxed{\mathbf{U} = \sqrt{\mathbf{C}} \qquad (\mathbf{C} = \mathbf{U}\mathbf{U})} \qquad \textbf{The Right Stretch Tensor} \qquad (2.2.31)$$

From 2.2.30, the right Cauchy-Green strain **C** (and by consequence the Euler-Lagrange strain **E**) only give information about the stretch of line elements; it does not give information about the rotation that is experienced by a particle during motion. The deformation gradient **F**, however, contains information about both the stretch and rotation. It can also be seen from 2.2.30-1 that **U** is a material tensor.

Note that, since

$$d\mathbf{x} = \mathbf{R}(\mathbf{U}d\mathbf{X}),$$

the undeformed line element is *first* stretched by **U** and is *then* rotated by **R** into the deformed element $d\mathbf{x}$ (the element may also undergo a rigid body translation **c**), Fig. 2.2.4. **R** is a two-point tensor.

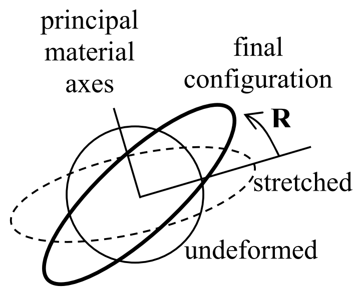

Figure 2.2.4: the polar decomposition

Evaluation of U

In order to evaluate **U**, it is necessary to evaluate $\sqrt{\mathbf{C}}$. To evaluate the square-root, **C** must first be obtained in relation to its principal axes, so that it is diagonal, and then the square root can be taken of the diagonal elements, since its eigenvalues will be positive (see §1.11.6). Then the tensor needs to be transformed back to the original coordinate system.

Example

Consider the motion

$$x_1 = 2X_1 - 2X_2, \quad x_2 = X_1 + X_2, \quad x_3 = X_3$$

The (homogeneous) deformation of a unit square in the $x_1 - x_2$ plane is as shown in Fig. 2.2.5.

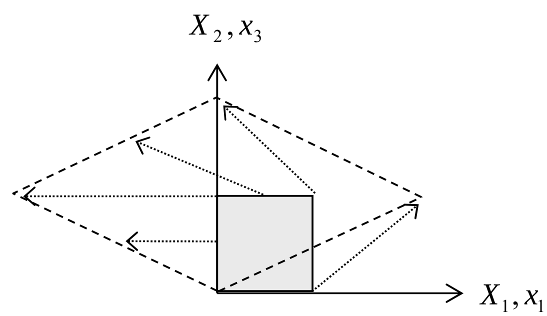

Figure 2.2.5: deformation of a square

One has

$$[\mathbf{F}] = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: (\mathbf{e}_i \otimes \mathbf{E}_j), \qquad [\mathbf{C}] = [\mathbf{F}^{\mathrm{T}}\mathbf{F}] = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: (\mathbf{E}_i \otimes \mathbf{E}_j)$$

Note that **F** is not symmetric, so that it might have only one real eigenvalue (in fact here it does have complex eigenvalues), and the eigenvectors may not be orthonormal. **C**, on the other hand, by its very definition, *is* symmetric; it is in fact positive definite and so has positive real eigenvalues forming an orthonormal set.

To determine the principal axes of **C**, it is necessary to evaluate the eigenvalues/eigenvectors of the tensor. The eigenvalues are the roots of the characteristic equation 1.11.5,

$$\alpha^3 - \mathrm{I}_{\mathbf{C}}\alpha^2 + \mathrm{II}_{\mathbf{C}}\alpha - \mathrm{III}_{\mathbf{C}} = 0$$

and the first, second and third invariants of the tensor are given by 1.11.6 so that $\alpha^3 - 11\alpha^2 + 26\alpha - 16 = 0$, with roots $\alpha = 8, 2, 1$. The three corresponding eigenvectors are found from 1.11.8,

$$\begin{array}{lcr} (C_{11} - \alpha)\hat{N}_1 + C_{12}\hat{N}_2 + C_{13}\hat{N}_3 = 0 & & (5-\alpha)\hat{N}_1 - 3\hat{N}_2 = 0 \\ C_{21}\hat{N}_1 + (C_{22} - \alpha)\hat{N}_2 + C_{23}\hat{N}_3 = 0 & \rightarrow & -3\hat{N}_1 + (5-\alpha)\hat{N}_2 = 0 \\ C_{31}\hat{N}_1 + C_{32}\hat{N}_2 + (C_{33} - \alpha)\hat{N}_3 = 0 & & (1-\alpha)\hat{N}_3 = 0 \end{array}$$

Thus (normalizing the eigenvectors so that they are unit vectors, and form a right-handed set, Fig. 2.2.6):

(i) for $\alpha = 8$, $-3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 - 3\hat{N}_2 = 0, -7\hat{N}_3 = 0$, $\qquad \hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 - \frac{1}{\sqrt{2}}\mathbf{E}_2$

(ii) for $\alpha = 2$, $3\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 3\hat{N}_2 = 0, -\hat{N}_3 = 0$, $\qquad \hat{\mathbf{N}}_2 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$

(iii) for $\alpha = 1$, $4\hat{N}_1 - 3\hat{N}_2 = 0, -3\hat{N}_1 + 4\hat{N}_2 = 0, 0\hat{N}_3 = 0$, $\qquad \hat{\mathbf{N}}_3 = \mathbf{E}_3$

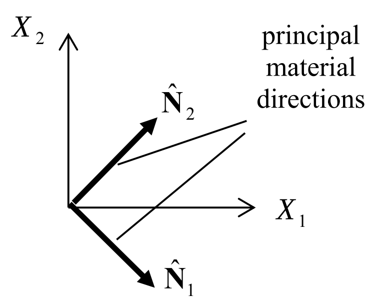

Figure 2.2.6: deformation of a square

Thus the right Cauchy-Green strain tensor **C**, with respect to coordinates with base vectors $\mathbf{E}_1' = \hat{\mathbf{N}}_1$, $\mathbf{E}_2' = \hat{\mathbf{N}}_2$ and $\mathbf{E}_3' = \hat{\mathbf{N}}_3$, that is, in terms of principal coordinates, is

$$[\mathbf{C}] = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

This result can be checked using the tensor transformation formulae 1.13.6, $[\mathbf{C}'] = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{C}][\mathbf{Q}]$, where Q is the transformation matrix of direction cosines (see also the example at the end of §1.5.2),

$$Q_{ij} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1' & \mathbf{e}_1 \cdot \mathbf{e}_2' & \mathbf{e}_1 \cdot \mathbf{e}_3' \\ \mathbf{e}_2 \cdot \mathbf{e}_1' & \mathbf{e}_2 \cdot \mathbf{e}_2' & \mathbf{e}_2 \cdot \mathbf{e}_3' \\ \mathbf{e}_3 \cdot \mathbf{e}_1' & \mathbf{e}_3 \cdot \mathbf{e}_2' & \mathbf{e}_3 \cdot \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2 & \hat{\mathbf{N}}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stretch tensor **U**, with respect to the principal directions is

$$[\mathbf{U}] = \left[\sqrt{\mathbf{C}}\right] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{basis}: \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_j$$

These eigenvalues of **U** (which are the square root of those of **C**) are the principal stretches and, as before, they are labeled $\lambda_1, \lambda_2, \lambda_3$.

In the original coordinate system, using the inverse tensor transformation rule 1.13.6, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$,

$$[\mathbf{U}] = \begin{bmatrix} 3/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 3/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: \mathbf{E}_i \otimes \mathbf{E}_j$$

so that

$$[\mathbf{R}] = \left[\mathbf{F}\mathbf{U}^{-1}\right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis}: \mathbf{e}_i \otimes \mathbf{E}_j$$

and it can be verified that **R** is a rotation tensor, i.e. is proper orthogonal.

Returning to the deformation of the unit square, the stretch and rotation are as illustrated in Fig. 2.2.7 – the action of **U** is indicated by the arrows, deforming the unit square to the dotted parallelogram, whereas **R** rotates the parallelogram through 45^{o} as a rigid body to its final position.

Note that the line elements along the diagonals (indicated by the heavy lines) lie along the principal directions of **U** and therefore undergo a pure stretch; the diagonal in the $\hat{\mathbf{N}}_1$ direction has stretched but has also moved with a rigid translation.

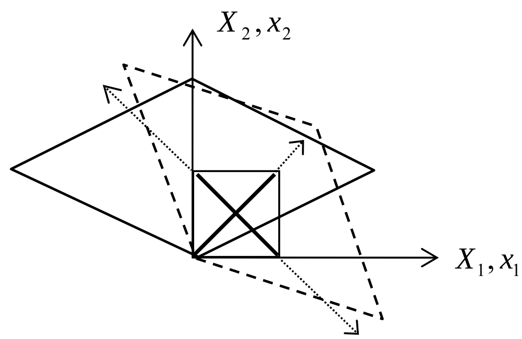

Figure 2.2.7: stretch and rotation of a square

■

Spatial Description

A polar decomposition can be made in the spatial description. In that case,

$$\boxed{\mathbf{F} = \mathbf{vR}} \qquad \textbf{Polar Decomposition (Spatial)} \tag{2.2.32}$$

Here **v** is a symmetric, positive definite second order tensor called the **left stretch tensor**, and $\mathbf{vv} = \mathbf{b}$, where **b** is the left Cauchy-Green tensor. **R** is the same rotation tensor as appears in the material description. Thus an elemental sphere can be regarded as first stretching into an ellipsoid, whose axes are the principal material axes (the principal axes of **U**), and then rotating; or first rotating, and then stretching into an ellipsoid whose axes are the **principal spatial axes** (the principal axes of **v**). The end result is the same.

The development in the spatial description is similar to that given above for the material description, and one finds by analogy with 2.2.30,

$$\lambda^{-2} = d\hat{\mathbf{x}}\mathbf{v}^{-1} \cdot \mathbf{v}^{-1} d\hat{\mathbf{x}} \tag{2.2.33}$$

In the above example, it turns out that **v** takes the simple diagonal form

$$[\mathbf{v}] = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{basis} : \mathbf{e}_i \otimes \mathbf{e}_j .$$

so the unit square rotates first and then undergoes a pure stretch along the coordinate axes, which are the principal spatial axes, and the sequence is now as shown in Fig. 2.2.9.

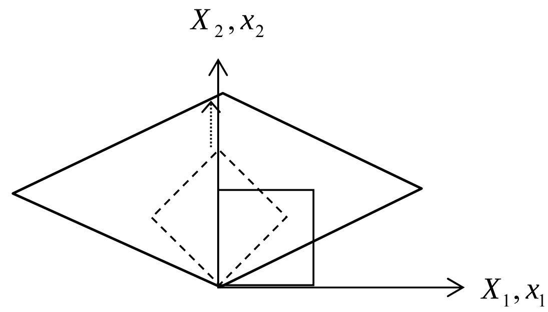

Figure 2.2.8: stretch and rotation of a square in spatial description

Relationship between the Material and Spatial Decompositions

Comparing the two decompositions, one sees that the material and spatial tensors involved are related through

$$\mathbf{v} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{b} = \mathbf{R}\mathbf{C}\mathbf{R}^{\mathrm{T}} \tag{2.2.34}$$

Further, suppose that $\mathbf{U}$ has an eigenvalue λ and an eigenvector $\hat{\mathbf{N}}$. Then $\mathbf{U}\hat{\mathbf{N}} = \lambda\hat{\mathbf{N}}$, so that $\mathbf{RUN} = \lambda\mathbf{RN}$. But $\mathbf{RU} = \mathbf{vR}$, so $\mathbf{v}\left(\mathbf{R}\hat{\mathbf{N}}\right) = \lambda\left(\mathbf{R}\hat{\mathbf{N}}\right)$. Thus $\mathbf{v}$ also has an eigenvalue λ, but an eigenvector $\hat{\mathbf{n}} = \mathbf{R}\hat{\mathbf{N}}$. From this, it is seen that the rotation tensor $\mathbf{R}$ maps the principal material axes into the principal spatial axes. It also follows that $\mathbf{R}$ and $\mathbf{F}$ can be written explicitly in terms of the material and spatial principal axes (compare the first of these with 1.10.25)[6]:

$$\mathbf{R} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i, \qquad \mathbf{F} = \mathbf{RU} = \mathbf{R}\sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}_i \tag{2.2.35}$$

and the deformation gradient acts on the principal axes base vectors according to {▲Problem 4}

$$\mathbf{F}\hat{\mathbf{N}}_i = \lambda_i \hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-\mathrm{T}}\hat{\mathbf{N}}_i = \frac{1}{\lambda_i}\hat{\mathbf{n}}_i, \qquad \mathbf{F}^{-1}\hat{\mathbf{n}}_i = \frac{1}{\lambda_i}\hat{\mathbf{N}}_i, \qquad \mathbf{F}^{\mathrm{T}}\hat{\mathbf{n}}_i = \lambda_i \hat{\mathbf{N}}_i \tag{2.2.36}$$

The representation of $\mathbf{F}$ and $\mathbf{R}$ in terms of both material and spatial principal base vectors in 2.3.35 highlights their two-point character.

Other Strain Measures

Some other useful measures of strain are

The **Hencky strain** measure: $\mathbf{H} \equiv \ln \mathbf{U}$ (material) or $\mathbf{h} = \ln \mathbf{v}$ (spatial)

[6] this is not a spectral decomposition of $\mathbf{F}$ (unless $\mathbf{F}$ happens to be symmetric, which it must be in order to have a spectral decomposition)

The **Biot strain** measure: $\overline{\mathbf{B}} = \mathbf{U} - \mathbf{I}$ (material) or $\overline{\mathbf{b}} = \mathbf{v} - \mathbf{I}$ (spatial)

The Hencky strain is evaluated by first evaluating **U** along the principal axes, so that the logarithm can be taken of the diagonal elements.

The material tensors **H**, $\overline{\mathbf{B}}$, **C**, **U** and **E** are coaxial tensors, with the same eigenvectors $\hat{\mathbf{N}}_i$. Similarly, the spatial tensors **h**, $\overline{\mathbf{b}}$, **b**, **v** and **e** are coaxial with the same eigenvectors $\hat{\mathbf{n}}_i$. From the definitions, the spectral decompositions of these tensors are

$$
\begin{aligned}
\mathbf{U} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{v} &= \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{C} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{b} &= \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{E} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(\lambda_i^2 - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{e} &= \sum_{i=1}^{3} \tfrac{1}{2}\left(1 - 1/\lambda_i^2\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\mathbf{H} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \mathbf{h} &= \sum_{i=1}^{3} \left(\ln \lambda_i\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \\
\overline{\mathbf{B}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i & \overline{\mathbf{b}} &= \sum_{i=1}^{3} \left(\lambda_i - 1\right) \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i
\end{aligned}
\tag{2.2.37}
$$

Deformation of a Circular Material Element

A circular material element will deform into an ellipse, as indicated in Figs. 2.2.2 and 2.2.4. This can be shown as follows. With respect to the principal axes, an undeformed line element $d\mathbf{X} = dX_1\mathbf{N}_1 + dX_2\mathbf{N}_2$ has magnitude squared $\left(dX_1\right)^2 + \left(dX_2\right)^2 = c^2$, where c is the radius of the circle, Fig. 2.2.9. The deformed element is $d\mathbf{x} = \mathbf{U}d\mathbf{X}$, or $d\mathbf{x} = \lambda_1 dX_1\mathbf{N}_1 + \lambda_2 dX_2\mathbf{N}_2 \equiv dx_1\mathbf{n}_1 + dx_2\mathbf{n}_2$. Thus $dx_1/\lambda_1 = dX_1$, $dx_2/\lambda_2 = dX_2$, which leads to the standard equation of an ellipse with major and minor axes $\lambda_1 c, \lambda_2 c$: $\left(dx_1/\lambda_1 c\right)^2 + \left(dx_2/\lambda_2 c\right)^2 = 1$.

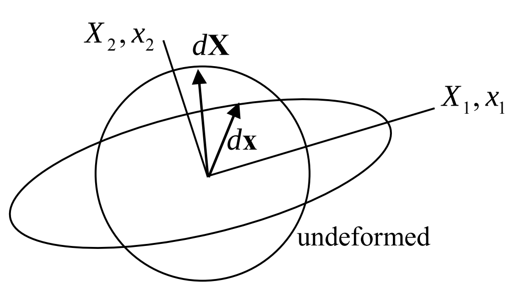

Figure 2.2.9: a circular element deforming into an ellipse

2.2.6 Some Simple Deformations

In this section, some elementary deformations are considered.

Pure Stretch

This deformation has already been seen, but now it can be viewed as a special case of the polar decomposition. The motion is

$$\boxed{x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3} \qquad \textbf{Pure Stretch} \qquad (2.2.38)$$

and the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Here, $\mathbf{R} = \mathbf{I}$ and there is no rotation. $\mathbf{U} = \mathbf{F}$ and the principal material axes are coincident with the material coordinate axes. $\lambda_1, \lambda_2, \lambda_3$, the eigenvalues of **U**, are the principal stretches.

Stretch with rotation

Consider the motion

$$x_1 = X_1 - kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & -k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sec\theta & 0 & 0 \\ 0 & \sec\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $k = \tan\theta$. This decomposition shows that the deformation consists of material stretching by $\sec\theta (= \sqrt{1+k^2})$, the principal stretches, along each of the axes, followed by a rigid body rotation through an angle θ about the $X_3 = 0$ axis, Fig. 2.2.10. The deformation is relatively simple because the principal material axes are aligned with the material coordinate axes (so that **U** is diagonal). The deformation of the unit square is as shown in Fig. 2.2.10.

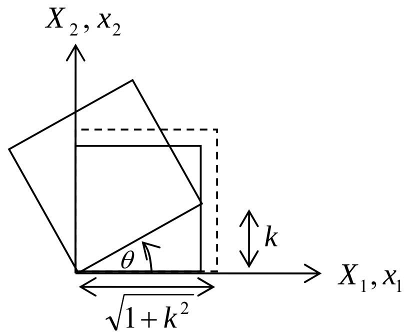

Figure 2.2.10: stretch with rotation

Pure Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = kX_1 + X_2, \quad x_3 = X_3} \qquad \textbf{Pure Shear} \qquad (2.2.39)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where, since **F** is symmetric, there is no rotation, and $\mathbf{F} = \mathbf{U}$. Since the rotation is zero, one can work directly with **U** and not have to consider **C**. The eigenvalues of **U**, the principal stretches, are $1+k, 1-k, 1$, with corresponding principal directions $\hat{\mathbf{N}}_1 = \frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$, $\hat{\mathbf{N}}_2 = -\frac{1}{\sqrt{2}}\mathbf{E}_1 + \frac{1}{\sqrt{2}}\mathbf{E}_2$ and $\hat{\mathbf{N}}_3 = \mathbf{E}_3$.

The deformation of the unit square is as shown in Fig. 2.2.11. The diagonal indicated by the heavy line stretches by an amount $1+k$ whereas the other diagonal contracts by an amount $1-k$. An element of material along the diagonal will undergo a pure stretch as indicated by the stretching of the dotted box.

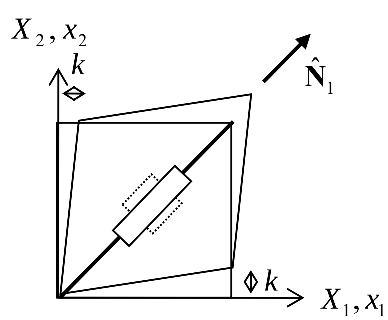

Figure 2.2.11: pure shear

Simple Shear

Consider the motion

$$\boxed{x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3} \quad \textbf{Simple Shear} \qquad (2.2.40)$$

so that

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The invariants of **C** are $\mathrm{I}_\mathbf{C} = 3 + k^2$, $\mathrm{II}_\mathbf{C} = 3 + k^2$, $\mathrm{III}_\mathbf{C} = 1$ and the characteristic equation is $\lambda^3 + (3 + k^2)\lambda(1 - \lambda) - 1 = 0$, so the principal values of **C** are $\lambda = 1 + \frac{1}{2}k^2 \pm \frac{1}{2}k\sqrt{4 + k^2}, 1$. The principal values of **U** are the (positive) square-roots of these: $\lambda = \frac{1}{2}\sqrt{4 + k^2} \pm \frac{1}{2}k, 1$. These can be written as $\lambda = \sec\theta \pm \tan\theta, 1$ by letting $\tan\theta = \frac{1}{2}k$. The corresponding eigenvectors of **C** are

$$\hat{\mathbf{N}}_1 = \frac{k}{\frac{1}{2}k^2 + \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = \frac{k}{\frac{1}{2}k^2 - \frac{1}{2}k\sqrt{4+k^2}}\mathbf{E}_1 + \mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

or, normalizing so that they are of unit size, and writing in terms of θ,

$$\hat{\mathbf{N}}_1 = \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = -\sqrt{\frac{1+\sin\theta}{2}}\mathbf{E}_1 + \sqrt{\frac{1-\sin\theta}{2}}\mathbf{E}_2, \quad \hat{\mathbf{N}}_3 = \mathbf{E}_3$$

The transformation matrix of direction cosines is then

$$[\mathbf{Q}] = \begin{bmatrix} \sqrt{(1-\sin\theta)/2} & -\sqrt{(1+\sin\theta)/2} & 0 \\ \sqrt{(1+\sin\theta)/2} & \sqrt{(1-\sin\theta)/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that, using the inverse transformation formula, $[\mathbf{U}] = [\mathbf{Q}][\mathbf{U}'][\mathbf{Q}]^{\mathrm{T}}$, one obtains **U** in terms of the original coordinates, and hence

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & (1+\sin^2\theta)/\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The deformation of the unit square is shown in Fig. 2.2.12 (for $k = 0.2, \theta = 5.71^{\circ}$). The square first undergoes a pure stretch/contraction ($\hat{\mathbf{N}}_1$ is in this case at 47.86° to the X_1

axis, with the diagonal of the square becoming the diagonal of the parallelogram, at 45.5° to the X_1 axis), and is then brought to its final position by a negative (clockwise) rotation of θ.

For this deformation, $\det \mathbf{F} = 1$ and, as will be shown below, this means that the simple shear deformation is volume-preserving.

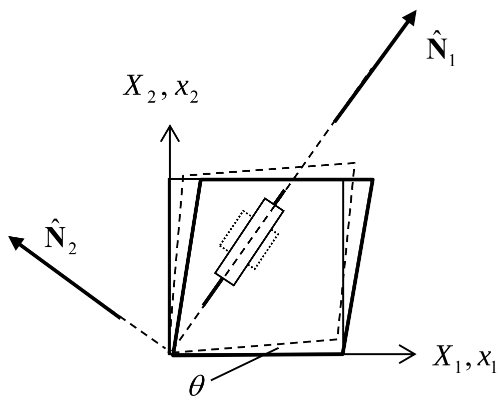

Figure 2.2.12: simple shear

2.2.7 Displacement & Displacement Gradients

The displacement of a material particle[7] is the movement it undergoes in the transition from the reference configuration to the current configuration. Thus, Fig. 2.2.13,[8]

$$\boxed{\mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}} \qquad \textbf{Displacement (Material Description)} \qquad (2.2.41)$$

$$\boxed{\mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t)} \qquad \textbf{Displacement (Spatial Description)} \qquad (2.2.42)$$

Note that **U** and **u** have the same values, they just have different arguments.

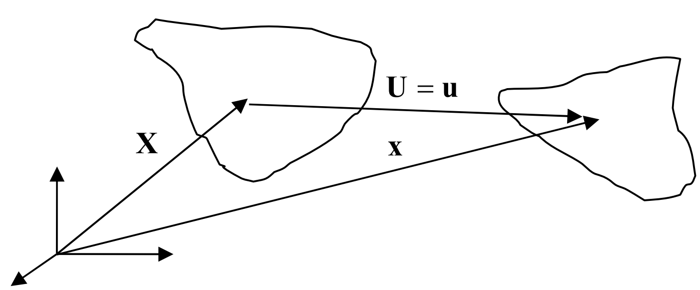

Figure 2.2.13: the displacement

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[7] In solid mechanics, the motion and deformation are often described in terms of the displacement **u**. In fluid mechanics, however, the primary field quantity describing the kinematic properties is the velocity **v** (and the acceleration $\mathbf{a} = \dot{\mathbf{v}}$) – see later.

[8] The material displacement **U** here is not to be confused with the right stretch tensor discussed earlier.

Displacement Gradients

The displacement gradient in the material and spatial descriptions, $\partial \mathbf{U}(\mathbf{X},t)/\partial \mathbf{X}$ and $\partial \mathbf{u}(\mathbf{x},t)/\partial \mathbf{x}$, are related to the deformation gradient and the inverse deformation gradient through

$$
\begin{aligned}
\mathrm{Grad}\mathbf{U} &= \frac{\partial \mathbf{U}}{\partial \mathbf{X}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial \mathbf{X}} = \mathbf{F}-\mathbf{I} & \qquad \frac{\partial U_i}{\partial X_j} &= \frac{\partial x_i}{\partial X_j} - \delta_{ij} \\
\mathrm{grad}\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial(\mathbf{x}-\mathbf{X})}{\partial \mathbf{x}} = \mathbf{I}-\mathbf{F}^{-1} & \qquad \frac{\partial u_i}{\partial x_j} &= \delta_{ij} - \frac{\partial X_i}{\partial x_j}
\end{aligned}
\tag{2.2.43}
$$

and it is clear that the displacement gradients are related through (see Eqn. 2.2.8)

$$
\mathrm{grad}\mathbf{u} = \mathrm{Grad}\,\mathbf{U}\,\mathbf{F}^{-1} \tag{2.2.44}
$$

The deformation can now be written in terms of either the material or spatial displacement gradients:

$$
\begin{aligned}
d\mathbf{x} &= d\mathbf{X} + d\mathbf{U}(\mathbf{X}) = d\mathbf{X} + \mathrm{Grad}\mathbf{U}\,d\mathbf{X} \\
d\mathbf{x} &= d\mathbf{X} + d\mathbf{u}(\mathbf{x}) = d\mathbf{X} + \mathrm{grad}\mathbf{u}\,d\mathbf{x}
\end{aligned}
\tag{2.2.45}
$$

Example

Consider again the extension of the bar shown in Fig. 2.1.5. The displacement is

$$
\mathbf{U}(\mathbf{X}) = \left(t + 3X_1 t\right)\mathbf{E}_1, \qquad \mathbf{u}(\mathbf{x}) = \left(\frac{t+3x_1 t}{1+3t}\right)\mathbf{e}_1
$$

and the displacement gradients are

$$
\mathrm{Grad}\mathbf{U} = 3t\mathbf{E}_1, \qquad \mathrm{grad}\mathbf{u} = \left(\frac{3t}{1+3t}\right)\mathbf{e}_1
$$

The displacement is plotted in Fig. 2.2.14 for $t = 1$. The two gradients $\partial U_1 / \partial X_1$ and $\partial u_1 / \partial x_1$ have different values (see the horizontal axes on Fig. 2.2.14). In this example, $\partial U_1 / \partial X_1 > \partial u_1 / \partial x_1$ – the change in displacement is not as large when "seen" from the spatial coordinates.

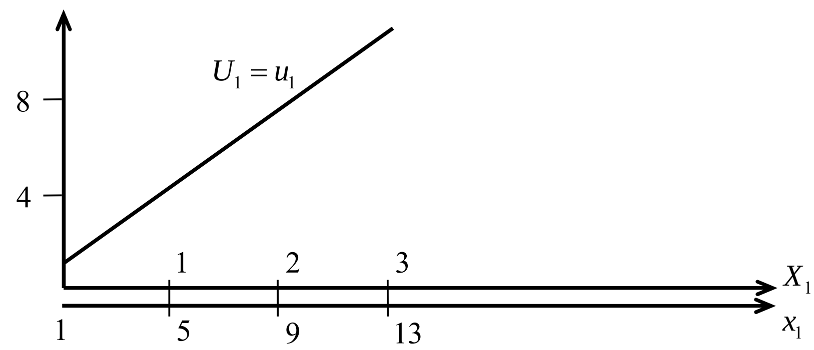

Figure 2.1.14: displacement and displacement gradient

■

Strains in terms of Displacement Gradients

The strains can be written in terms of the displacement gradients. Using 1.10.3b,

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) \\ &= \frac{1}{2}\left((\mathrm{Grad}\mathbf{U} + \mathbf{I})^{\mathrm{T}}(\mathrm{Grad}\mathbf{U} + \mathbf{I}) - \mathbf{I}\right) \\ &= \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + (\mathrm{Grad}\mathbf{U})^{\mathrm{T}} + (\mathrm{Grad}\mathbf{U})^{\mathrm{T}}\mathrm{Grad}\mathbf{U}\right), \quad E_{IJ} = \frac{1}{2}\left\{\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_K}{\partial X_I}\frac{\partial U_K}{\partial X_J}\right\}\end{aligned} \tag{2.2.46a}$$

$$\begin{aligned}\mathbf{e} &= \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right) \\ &= \frac{1}{2}\left(\mathbf{I} - (\mathbf{I} - \mathrm{grad}\mathbf{u})^{\mathrm{T}}(\mathbf{I} - \mathrm{grad}\mathbf{u})\right) \\ &= \frac{1}{2}\left(\mathrm{grad}\mathbf{u} + (\mathrm{grad}\mathbf{u})^{\mathrm{T}} - (\mathrm{grad}\mathbf{u})^{\mathrm{T}}\mathrm{grad}\mathbf{u}\right), \quad e_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right\}\end{aligned} \tag{2.2.46b}$$

Small Strain

If the displacement gradients are small, then the quadratic terms, their products, are small relative to the gradients themselves, and may be neglected. With this assumption, the Green-Lagrange strain **E** (and the Euler-Almansi strain) reduces to the **small-strain tensor**,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + (\mathrm{Grad}\mathbf{U})^{\mathrm{T}}\right), \quad \varepsilon_{IJ} = \frac{1}{2}\left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I}\right) \tag{2.2.47}$$

Since in this case the displacement gradients are small, it does not matter whether one refers the strains to the reference or current configurations – the error is of the same order as the quadratic terms already neglected[9], so the small strain tensor can equally well be written as

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathrm{grad}\mathbf{u} + (\mathrm{grad}\mathbf{u})^{\mathrm{T}}\right), \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \textbf{Small Strain Tensor} \qquad (2.2.48)$$

2.2.8 The Deformation of Area and Volume Elements

Line elements transform between the reference and current configurations through the deformation gradient. Here, the transformation of area and volume elements is examined.

The Jacobian Determinant

The **Jacobian determinant** of the deformation is defined as the determinant of the deformation gradient,

$$\boxed{J(\mathbf{X},t) = \det\mathbf{F}} \qquad \det\mathbf{F} = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} \qquad \textbf{The Jacobian Determinant} \qquad (2.2.49)$$

Equivalently, it can be considered to be the Jacobian of the transformation from material to spatial coordinates (see Appendix 1.B.2).

From Eqn. 1.3.17, the Jacobian can also be written in the form of the triple scalar product

$$J = \frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3} \right) \qquad (2.2.50)$$

Consider now a volume element in the reference configuration, a parallelepiped bounded by the three line-elements $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$. The volume of the parallelepiped[10] is given by the triple scalar product (Eqns. 1.1.4):

$$dV = d\mathbf{X}^{(1)} \cdot \left(d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)} \right) \qquad (2.2.51)$$

After deformation, the volume element is bounded by the three vectors $d\mathbf{x}^{(i)}$, so that the volume of the deformed element is, using 1.10.16f,

[9] although large rigid body rotations must not be allowed – see §2.7 .
[10] the vectors should form a right-handed set so that the volume is positive.

$$\begin{aligned} dv &= d\mathbf{x}^{(1)} \cdot \left(d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)} \right) \\ &= \mathbf{F}d\mathbf{X}^{(1)} \cdot \left(\mathbf{F}d\mathbf{X}^{(2)} \times \mathbf{F}d\mathbf{X}^{(3)} \right) \\ &= \det \mathbf{F} \left(d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} \times d\mathbf{X}^{(3)} \right) \\ &= \det \mathbf{F}\; dV \end{aligned} \tag{2.2.52}$$

Thus the scalar J is a measure of how the volume of a material element has changed with the deformation and for this reason is often called the **volume ratio**.

$$\boxed{dv = J\, dV} \qquad \textbf{Volume Ratio} \tag{2.2.53}$$

Since volumes cannot be negative, one must insist on physical grounds that $J > 0$. Also, since **F** has an inverse, $J \neq 0$. Thus one has the restriction

$$J > 0 \tag{2.2.54}$$

Note that a rigid body rotation does not alter the volume, so the volume change is completely characterised by the stretching tensor **U**. Three line elements lying along the principal directions of **U** form an element with volume dV, and then undergo pure stretch into new line elements defining an element of volume $dv = \lambda_1\lambda_2\lambda_3 dV$, where λ_i are the principal stretches, Fig. 2.2.15. The unit change in volume is therefore also

$$\frac{dv - dV}{dV} = \lambda_1\lambda_2\lambda_3 - 1 \tag{2.2.55}$$

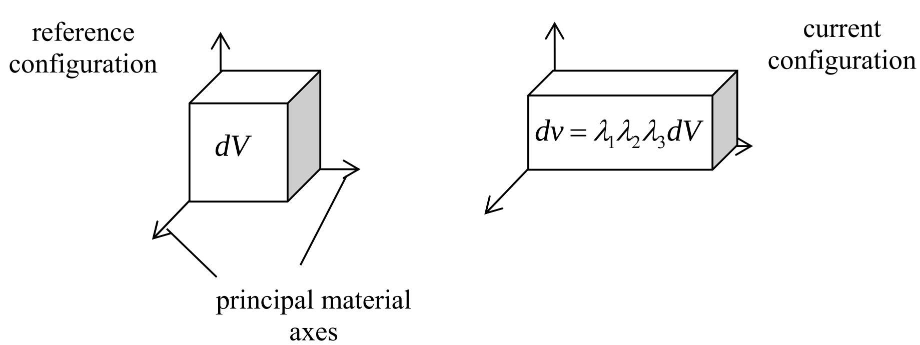

Figure 2.2.15: change in volume

For example, the volume change for pure shear is $-k^2$ (volume decreasing) and, for simple shear, is zero (*cf*. Eqn. 2.2.40 *et seq*., $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)(1) - 1 = 0$).

An **incompressible** material is one for which the volume change is zero, i.e. the deformation is isochoric. For such a material, $J = 1$, and the three principal stretches are not independent, but are constrained by

$$\boxed{\lambda_1\lambda_2\lambda_3 = 1} \qquad \textbf{Incompressibility Constraint} \tag{2.2.56}$$

Nanson's Formula

Consider an area element in the reference configuration, with area dS, unit normal $\hat{\mathbf{N}}$, and bounded by the vectors $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$, Fig. 2.2.16. Then

$$\hat{\mathbf{N}}dS = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \tag{2.2.57}$$

The volume of the element bounded by the vectors $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and some arbitrary line element $d\mathbf{X}$ is $dV = \hat{\mathbf{N}}dS \cdot d\mathbf{X}$. The area element is now deformed into an element of area ds with normal $\hat{\mathbf{n}}$ and bounded by the line elements $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$. The volume of the new element bounded by the area element and $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ is then

$$dv = \hat{\mathbf{n}}ds \cdot d\mathbf{x} = \hat{\mathbf{n}}ds \cdot \mathbf{F}d\mathbf{X} \equiv J\hat{\mathbf{N}}dS \cdot d\mathbf{X} \tag{2.2.58}$$

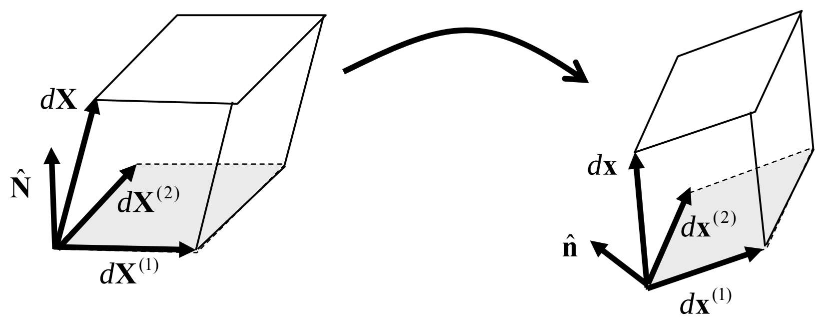

Figure 2.2.16: change of surface area

Thus, since $d\mathbf{X}$ is arbitrary, and using 1.10.3d,

$$\boxed{\hat{\mathbf{n}}ds = J\,\mathbf{F}^{-\mathrm{T}}\hat{\mathbf{N}}dS} \qquad \textbf{Nanson's Formula} \tag{2.2.59}$$

Nanson's formula shows how the vector element of area $\hat{\mathbf{n}}ds$ in the current configuration is related to the vector element of area $\hat{\mathbf{N}}dS$ in the reference configuration.

2.2.9 Inextensibility and Orientation Constraints

A constraint on the principal stretches was introduced for an incompressible material, 2.2.56. Other constraints arise in practice. For example, consider a material which is inextensible in a certain direction, defined by a unit vector $\hat{\mathbf{A}}$ in the reference configuration. It follows that $\left|\mathbf{F}\hat{\mathbf{A}}\right| = 1$ and the constraint can be expressed as 2.2.17,

$$\boxed{\hat{\mathbf{A}}\mathbf{C}\hat{\mathbf{A}} = 1} \qquad \textbf{Inextensibility Constraint} \tag{2.2.60}$$

If there are two such directions in a plane, defined by $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, making angles θ and ϕ respectively with the principal material axes $\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2$, then

$$1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and $(\lambda_1^2 - \lambda_2^2)\cos^2\theta = 1 - \lambda_2^2 = (\lambda_1^2 - \lambda_2^2)\cos^2\phi$. It follows that $\phi = \theta$, $\phi = \theta + \pi$, $\theta + \phi = \pi$ or $\theta + \phi = 2\pi$ (or $\lambda_1 = \lambda_2 = 1$, i.e. no deformation).

Similarly, one can have orientation constraints. For example, suppose that the direction associated with the vector $\hat{\mathbf{A}}$ maintains that direction. Then

$$\boxed{\mathbf{F}\hat{\mathbf{A}} = \mu\hat{\mathbf{A}}} \qquad \textbf{Orientation Constraint} \qquad (2.2.61)$$

for some scalar $\mu > 0$.

2.2.10 Problems

1. In equations 2.2.8, one has from the chain rule
$$\mathrm{grad}\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \frac{\partial\phi}{\partial X_m}\frac{\partial X_m}{\partial x_i}\mathbf{e}_i = \left(\frac{\partial\phi}{\partial X_j}\mathbf{E}_j\right)\left(\frac{\partial X_m}{\partial x_i}\mathbf{E}_m \otimes \mathbf{e}_i\right) = \mathrm{Grad}\phi\,\mathbf{F}^{-1}$$
Derive the other two relations.
2. Take the dot product $(\lambda d\hat{\mathbf{x}})\cdot(\lambda d\hat{\mathbf{x}})$ in Eqn. 2.2.29. Then use $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$, and 1.10.3e to show that
$$\lambda^2 = \frac{d\mathbf{X}}{|d\mathbf{X}|}\mathbf{U}\cdot\mathbf{U}\frac{d\mathbf{X}}{|d\mathbf{X}|}$$
3. For the deformation
$$x_1 = X_1 + 2X_3, \quad x_2 = X_2 - 2X_3, \quad x_3 = -2X_1 + 2X_2 + X_3$$
   (a) Determine the Deformation Gradient and the Right Cauchy-Green tensors
   (b) Consider the two line elements $d\mathbf{X}^{(1)} = \mathbf{e}_1$, $d\mathbf{X}^{(2)} = \mathbf{e}_2$ (emanating from (0,0,0)). Use the Right Cauchy Green tensor to determine whether these elements in the current configuration ($d\mathbf{x}^{(1)}$, $d\mathbf{x}^{(2)}$) are perpendicular.
   (c) Use the right Cauchy Green tensor to evaluate the stretch of the line element $d\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$, and hence determine whether the element contracts, stretches, or stays the same length after deformation.
   (d) Determine the Green-Lagrange and Eulerian strain tensors
   (e) Decompose the deformation into a stretching and rotation (check that $\mathbf{U}$ is symmetric and $\mathbf{R}$ is orthogonal). What are the principal stretches?
4. Derive Equations 2.2.36.
5. For the deformation
$$x_1 = X_1, \quad x_2 = X_2 + X_3, \quad x_3 = aX_2 + X_3$$

(a) Determine the displacement vector in both the material and spatial forms
(b) Determine the displaced location of the particles in the undeformed state which originally comprise
   (i) the plane circular surface $X_1 = 0, \quad X_2^2 + X_3^2 = 1/(1-a^2)$
   (ii) the infinitesimal cube with edges along the coordinate axes of length $dX_i = \varepsilon$

   Sketch the displaced configurations if $a = 1/2$

6. For the deformation

$$x_1 = X_1 + aX_2, \quad x_2 = X_2 + aX_3, \quad x_3 = aX_1 + X_3$$

(a) Determine the displacement vector in both the material and spatial forms
(b) Calculate the full material (Green-Lagrange) strain tensor and the full spatial strain tensor
(c) Calculate the infinitesimal strain tensor as derived from the material and spatial tensors, and compare them for the case of very small *a*.

7. In the example given above on the polar decomposition, §2.2.5, check that the relations $\mathbf{C}\mathbf{n}_i = \lambda\mathbf{n}_i$, $i = 1,2,3$ are satisfied (with respect to the original axes). Check also that the relations $\mathbf{C}\mathbf{n}'_i = \lambda\mathbf{n}'_i$, $i = 1,2,3$ are satisfied (here, the eigenvectors are the unit vectors in the second coordinate system, the principal directions of **C**, and **C** is with respect to these axes, i.e. it is diagonal).