1.19 Curvilinear Coordinates: Curved Geometries

In this section is examined the special case of a two-dimensional curved surface.

1.19.1 Monoclinic Coordinate Systems

Base Vectors

A curved surface can be defined using two covariant base vectors $\mathbf{a}_1, \mathbf{a}_2$, with the third base vector, $\mathbf{a}_3$, everywhere of unit size and normal to the other two, Fig. 1.19.1 These base vectors form a **monoclinic** reference frame, that is, only one of the angles between the base vectors is not necessarily a right angle.

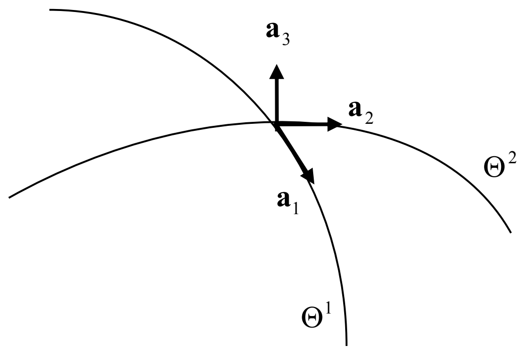

Figure 1.19.1: Geometry of the Curved Surface

In what follows, in the index notation, Greek letters such as α, β take values 1 and 2; as before, Latin letters take values from 1..3.

Since $\mathbf{a}^3 = \mathbf{a}_3$ and

$$a_{\alpha 3} = \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0, \qquad a^{\alpha 3} = \mathbf{a}^\alpha \cdot \mathbf{a}^3 = 0 \qquad (1.19.1)$$

the determinant of metric coefficients is

$$J^2 = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \frac{1}{J^2} = \begin{vmatrix} g^{11} & g^{12} & 0 \\ g^{21} & g^{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad (1.19.2)$$

The Cross Product

Particularising the results of §1.16.10, define the surface permutation symbol to be the triple scalar product

$$e_{\alpha\beta} \equiv \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \times \mathbf{a}_3 = \varepsilon_{\alpha\beta}\sqrt{g}, \qquad e^{\alpha\beta} \equiv \mathbf{a}^\alpha \cdot \mathbf{a}^\beta \times \mathbf{a}^3 = \varepsilon^{\alpha\beta}\frac{1}{\sqrt{g}} \tag{1.19.3}$$

where $\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}$ is the Cartesian permutation symbol, $\varepsilon_{12} = +1$, $\varepsilon_{21} = -1$, and zero otherwise, with

$$e^{\alpha\beta}e_{\mu\eta} = \varepsilon^{\alpha\beta}\varepsilon_{\mu\eta}, \qquad e^{\alpha\beta}e_{\mu\eta} = \delta^\alpha_\mu\delta^\beta_\eta - \delta^\beta_\mu\delta^\alpha_\eta = e^{\beta\alpha}e_{\eta\mu} \tag{1.19.4}$$

From 1.19.3,

$$\begin{aligned} \mathbf{a}_\alpha \times \mathbf{a}_\beta &= e_{\alpha\beta}\mathbf{a}^3 \\ \mathbf{a}^\alpha \times \mathbf{a}^\beta &= e^{\alpha\beta}\mathbf{a}_3 \end{aligned} \tag{1.19.5}$$

and so

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{g}} \tag{1.19.6}$$

The cross product of surface vectors, that is, vectors with component in the normal ($\mathbf{g}_3$) direction zero, can be written as

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = e_{\alpha\beta}u^\alpha v^\beta \mathbf{a}^3 &= \sqrt{g}\begin{vmatrix} u^1 & u^2 \\ v^1 & v^2 \end{vmatrix}\mathbf{a}^3 \\ = e^{\alpha\beta}u_\alpha v_\beta \mathbf{a}_3 &= \frac{1}{\sqrt{g}}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{a}_3 \end{aligned} \tag{1.19.7}$$

The Metric and Surface elements

Considering a line element lying within the surface, so that $\Theta^3 = 0$, the metric for the surface is

$$(\Delta s)^2 = d\mathbf{s} \cdot d\mathbf{s} = \left(d\Theta^\alpha \mathbf{a}_\alpha\right) \cdot \left(d\Theta^\beta \mathbf{a}_\beta\right) = g_{\alpha\beta}d\Theta^\alpha d\Theta^\beta \tag{1.19.8}$$

which is in this context known as the **first fundamental form of the surface**.

Similarly, from 1.16.41, a surface element is given by

$$\Delta S = \sqrt{g}\,\Delta\Theta^1\Delta\Theta^2 \tag{1.5.9}$$

Christoffel Symbols

The Christoffel symbols can be simplified as follows. A differentiation of $\mathbf{a}_3 \cdot \mathbf{a}_3 = 1$ leads to

$$\mathbf{a}_{3,\alpha} \cdot \mathbf{a}_3 = -\mathbf{a}_{3,\alpha} \cdot \mathbf{a}_3 \tag{1.19.10}$$

so that, from Eqn 1.18.6,

$$\Gamma_{3\alpha 3} = \Gamma_{\alpha 33} = 0 \tag{1.19.11}$$

Further, since $\partial \mathbf{a}_3 / \partial \Theta^3 = 0$,

$$\Gamma_{33\alpha} = 0, \quad \Gamma_{333} = 0 \tag{1.19.12}$$

These last two equations imply that the Γ_{ijk} vanish whenever two or more of the subscripts are 3.

Next, differentiate 1.19.1 to get

$$\mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 = -\mathbf{a}_{3,\beta} \cdot \mathbf{a}_\alpha, \qquad \mathbf{a}^\alpha{}_{,\beta} \cdot \mathbf{a}^3 = -\mathbf{a}^3{}_{,\beta} \cdot \mathbf{a}^\alpha \tag{1.19.13}$$

and Eqns. 1.18.6 now lead to

$$\Gamma_{\alpha\beta 3} = \Gamma_{\beta\alpha 3} = -\Gamma_{3\beta\alpha} = -\Gamma_{\beta 3\alpha} \tag{1.19.14}$$

From 1.18.8, using 1.19.11,

$$\begin{aligned} \Gamma^3_{\alpha\beta} &= \Gamma_{\alpha\beta\gamma} g^{\gamma 3} + \Gamma_{\alpha\beta 3} g^{33} = \Gamma_{\alpha\beta 3} \\ \Gamma^3_{3\alpha} &= \Gamma_{3\alpha\beta} g^{\beta 3} + \Gamma_{3\alpha 3} g^{33} = \Gamma_{3\alpha 3} = 0 \end{aligned} \tag{1.19.15}$$

and, similarly {▲Problem 1}

$$\Gamma^3_{\alpha 3} = \Gamma^\alpha_{33} = \Gamma^3_{33} = 0 \tag{1.19.16}$$

1.19.2 The Curvature Tensor

In this section is introduced a tensor which, with the metric coefficients, completely describes the surface.

First, although the base vector $\mathbf{a}_3$ maintains unit length, its direction changes as a function of the coordinates Θ^1, Θ^2, and its derivative is, from 1.18.2 or 1.18.5 (and using 1.19.15)

$$\frac{\partial \mathbf{a}_3}{\partial \Theta^\alpha} = \Gamma^k_{3\alpha} \mathbf{a}_k = \Gamma^\beta_{3\alpha} \mathbf{a}_\beta, \qquad \frac{\partial \mathbf{a}^3}{\partial \Theta^\alpha} = -\Gamma^3_{\alpha k} \mathbf{a}^k = -\Gamma^3_{\alpha\beta} \mathbf{a}^\beta \tag{1.19.17}$$

Define now the **curvature tensor K** to have the covariant components $K_{\alpha\beta}$, through

$$\frac{\partial \mathbf{a}_3}{\partial \Theta^\alpha} = -K_{\alpha\beta}\mathbf{a}^\beta \tag{1.19.18}$$

and it follows from 1.19.13, 1.19.15a and 1.19.14,

$$K_{\alpha\beta} = \Gamma^3_{\alpha\beta} = \Gamma_{\alpha\beta 3} = -\Gamma_{3\beta\alpha} \tag{1.19.19}$$

and, since these Christoffel symbols are symmetric in the α, β, *the curvature tensor is symmetric*.

The mixed and contravariant components of the curvature tensor follows from 1.16.58-9:

$$\begin{gathered} K_\alpha^\beta = g^{\gamma\beta}K_{\alpha\gamma} = g_{\alpha\gamma}K^{\gamma\beta}, \quad K^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\lambda}K_{\gamma\lambda} \\ \frac{\partial \mathbf{a}_3}{\partial \Theta^\alpha} \equiv -K_{\alpha\beta}\mathbf{a}^\beta = -K_{\alpha\beta}g^{\gamma\beta}\mathbf{a}_\gamma = -K_\alpha^\gamma \mathbf{a}_\gamma \end{gathered} \tag{1.19.20}$$

and the "dot" is not necessary in the mixed notation because of the symmetry property. From these and 1.18.8, it follows that

$$K_\beta^\alpha = g^{\gamma\alpha}K_{\gamma\beta} = -g^{\gamma\alpha}\Gamma_{3\beta\gamma} = -\Gamma^\alpha_{3\beta} = -\Gamma^\alpha_{\beta 3} \tag{1.19.21}$$

Also,

$$\begin{aligned} d\mathbf{a}_3 \cdot d\mathbf{s} &= \left(\mathbf{a}_{3,\alpha} d\Theta^\alpha\right)\cdot\left(d\Theta^\beta \mathbf{a}_\beta\right) \\ &= \left(-K_{\alpha\gamma} d\Theta^\alpha \mathbf{a}^\gamma\right)\cdot\left(d\Theta^\beta \mathbf{a}_\beta\right) \\ &= -K_{\alpha\beta} d\Theta^\alpha d\Theta^\beta \end{aligned} \tag{1.19.22}$$

which is known as the **second fundamental form of the surface**.

From 1.19.19 and the definitions of the Christoffel symbols, 1.18.4, 1.18.6, the curvature can be expressed as

$$K_{\alpha\beta} = \frac{\partial \mathbf{a}_\alpha}{\partial \Theta^\beta}\cdot \mathbf{a}_3 = -\frac{\partial \mathbf{a}_3}{\partial \Theta^\beta}\cdot \mathbf{a}_\alpha \tag{1.19.23}$$

showing that the curvature is a measure of the change of the base vector $\mathbf{a}_\alpha$ along the Θ^β curve, in the direction of the normal vector; alternatively, the rate of change of the normal vector along Θ^β, in the direction $-\mathbf{a}_\alpha$. Looking at this in more detail, consider now the change in the normal vector $\mathbf{a}_3$ in the Θ^1 direction, Fig. 1.19.2. Then

$$d\mathbf{a}_3 = \mathbf{a}_{3,1}d\Theta^1 = -K_1^\gamma d\Theta^1 \mathbf{a}_\gamma \tag{1.19.24}$$

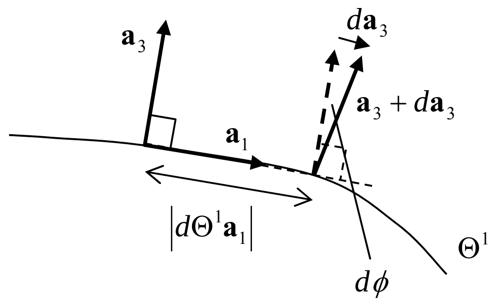

Figure 1.19.2: Curvature of the Surface

Taking the case of $K_1^1 \neq 0, K_1^2 = 0$, one has $d\mathbf{a}_3 = -K_1^1 d\Theta^1 \mathbf{a}_1$. From Fig. 1.19.2, and since the normal vector is of unit length, the magnitude $|d\mathbf{a}_3|$ equals $d\phi$, the small angle through which the normal vector rotates as one travels along the Θ^1 coordinate curve. The **curvature** of the surface is defined to be the rate of change of the angle ϕ:[1]

$$\frac{d\phi}{ds} = \frac{\left|-K_1^1 d\Theta^1 \mathbf{a}_1\right|}{\left|d\Theta^1 \mathbf{a}_1\right|} = \left|K_1^1\right| \tag{1.19.25}$$

and so the mixed component K_1^1 is the curvature in the Θ^1 direction. Similarly, K_2^2 is the curvature in the Θ^2 direction.

Assume now that $K_1^1 = 0, K_1^2 \neq 0$. Eqn. 1.19.24 now reads $d\mathbf{a}_3 = -K_1^2 d\Theta^1 \mathbf{a}_2$ and, referring Fig. 1.19.3, the **twist** of the surface with respect to the coordinates is

$$\frac{d\varphi}{ds} = \frac{\left|-K_1^2 d\Theta^1 \mathbf{a}_2\right|}{\left|d\Theta^1 \mathbf{a}_1\right|} = \left|K_1^2\right| \frac{|\mathbf{a}_2|}{|\mathbf{a}_1|} \tag{1.5.26}$$

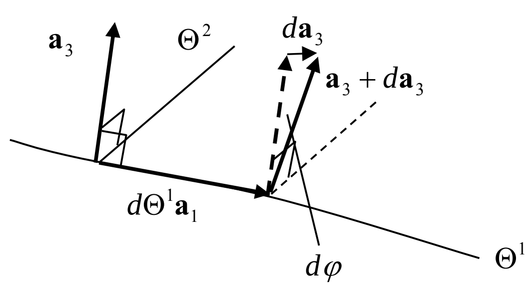

Figure 1.19.3: Twisting over the Surface

When $|\mathbf{a}_1| = |\mathbf{a}_2|$, $\left|K_1^2\right|$ is the twist; when they are not equal, $\left|K_1^2\right|$ is closely related to the twist.

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[1] this is essentially the same definition as for the space curve of §1.6.2; there, the angle $\phi = \kappa \Delta s$

Two important quantities are often used to describe the curvature of a surface. These are the first and the third principal scalar invariants:

$$\begin{aligned} \mathrm{I}_{\mathbf{K}} &= K^i_{\cdot i} = K^1_1 + K^2_2 \\ \mathrm{III}_{\mathbf{K}} &= \det K^i_{\cdot j} = \begin{vmatrix} K^1_1 & K^1_2 \\ K^2_1 & K^2_2 \end{vmatrix} = K^1_1 K^2_2 - K^1_2 K^2_1 = \varepsilon_{\alpha\beta} K^\alpha_1 K^\beta_2 \end{aligned} \tag{1.19.27}$$

The first invariant is twice the **mean curvature** K_M whilst the third invariant is called the **Gaussian curvature** (or **Total curvature**) K_G of the surface.

Example (Curvature of a Sphere)

The surface of a sphere of radius a can be described by the coordinates (Θ^1, Θ^2), Fig. 1.19.4, where

$$x^1 = a \sin\Theta^1 \cos\Theta^2, \quad x^2 = a \sin\Theta^1 \sin\Theta^2, \quad x^3 = a \cos\Theta^1$$

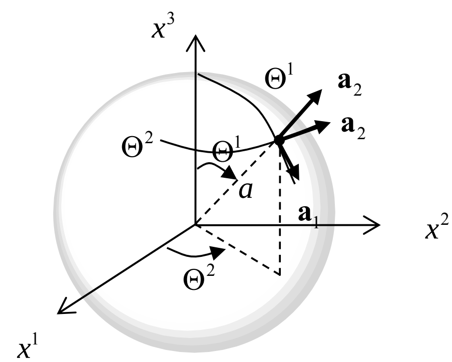

Figure 1.19.4: a spherical surface

Then, from the definitions 1.16.19, 1.16.27-28, 1.16.34, {▲Problem 2}

$$\begin{aligned} \mathbf{a}_1 &= +a\cos\Theta^1 \cos\Theta^2 \mathbf{e}_1 + a\cos\Theta^1 \sin\Theta^2 \mathbf{e}_2 - a\sin\Theta^1 \mathbf{e}_3 \\ \mathbf{a}_2 &= -a\sin\Theta^1 \sin\Theta^2 \mathbf{e}_1 + a\sin\Theta^1 \cos\Theta^2 \mathbf{e}_2 \\ \mathbf{a}^1 &= \frac{1}{a^2}\mathbf{a}_1 \\ \mathbf{a}^2 &= \frac{1}{a^2 \sin^2\Theta^1}\mathbf{a}_2 \\ g_{\alpha\beta} &= \begin{vmatrix} a^2 & 0 \\ 0 & a^2\sin^2\Theta^1 \end{vmatrix}, \qquad g = a^4 \sin^2\Theta^1 \end{aligned} \tag{1.19.28}$$

From 1.19.6,

$$\mathbf{a}_3 = \sin\Theta^1 \cos\Theta^2 \mathbf{e}_1 + \sin\Theta^1 \sin\Theta^2 \mathbf{e}_2 + \cos\Theta^1 \mathbf{e}_3 \tag{1.19.29}$$

and this is clearly an orthogonal coordinate system with scale factors

$$h_1 = a, \quad h_2 = a\sin\Theta^1, \quad h_3 = 1 \tag{1.19.30}$$

The surface Christoffel symbols are, from 1.18.33, 1.18.36,

$$\Gamma_{11}^1 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{11}^2 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\cos\Theta^1}{\sin^2\Theta^1}, \quad \Gamma_{22}^1 = -\sin\Theta^1\cos\Theta^1 \tag{1.19.31}$$

Using the definitions 1.18.4, {▲Problem 3}

$$\begin{aligned} &\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{a}, \quad \Gamma_{23}^1 = \Gamma_{32}^1 = 0 \\ &\Gamma_{13}^2 = \Gamma_{31}^2 = 0, \quad \Gamma_{23}^2 = \Gamma_{32}^2 = \frac{1}{a} \\ &\Gamma_{11}^3 = -a, \quad \Gamma_{12}^3 = \Gamma_{21}^3 = 0, \quad \Gamma_{22}^3 = -a\sin^2\Theta^1 \end{aligned} \tag{1.19.32}$$

with the remaining symbols $\Gamma_{\alpha 3}^3 = \Gamma_{3\alpha}^3 = \Gamma_{33}^\alpha = \Gamma_{33}^3 = 0$.

The components of the curvature tensor are then, from 1.19.21, 1.19.19,

$$\left[K_\beta^\alpha\right] = \begin{bmatrix} -\frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}, \qquad \left[K_{\alpha\beta}\right] = \begin{bmatrix} -a & 0 \\ 0 & -a\sin^2\Theta^1 \end{bmatrix} \tag{1.19.33}$$

The mean and Gaussian curvature of a sphere are then

$$\begin{aligned} K_M &= -\frac{2}{a} \\ K_G &= \frac{1}{a^2} \end{aligned} \tag{1.19.34}$$

The principal curvatures are evidently K_1^1 and K_2^2. As expected, they are simply the reciprocal of the radius of curvature a.

■

1.19.3 Covariant Derivatives

Vectors

Consider a vector **v**, which is not necessarily a surface vector, that is, it might have a normal component $v_3 = v^3$. The covariant derivative is

$$
\begin{aligned}
v^{\alpha}\,|_{\beta} &= v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\gamma\beta}v^{\gamma} + \Gamma^{\alpha}_{3\beta}v^{3} & v_{\alpha}\,|_{\beta} &= v_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}v_{\gamma} - \Gamma^{3}_{\alpha\beta}v_{3} \\
v^{\alpha}\,|_{3} &= v^{\alpha}{}_{,3} + \Gamma^{\alpha}_{\gamma 3}v^{\gamma} + \Gamma^{\alpha}_{33}v^{3} & v_{\alpha}\,|_{3} &= v_{\alpha,3} - \Gamma^{\gamma}_{\alpha 3}v_{\gamma} - \Gamma^{3}_{\alpha 3}v_{3} \\
&= v^{\alpha}{}_{,3} + \Gamma^{\alpha}_{\gamma 3}v^{\gamma} \quad , & &= v_{\alpha,3} - \Gamma^{\gamma}_{\alpha 3}v_{\gamma} \\
v^{3}\,|_{\alpha} &= v^{3}{}_{,\alpha} + \Gamma^{3}_{\gamma\alpha}v^{\gamma} + \Gamma^{3}_{3\alpha}v^{3} & v_{3}\,|_{\alpha} &= v_{3,\alpha} - \Gamma^{\gamma}_{3\alpha}v_{\gamma} - \Gamma^{3}_{3\alpha}v_{3} \\
&= v^{3}{}_{,\alpha} + \Gamma^{3}_{\gamma\alpha}v^{\gamma} & &= v_{3,\alpha} - \Gamma^{\gamma}_{3\alpha}v_{\gamma}
\end{aligned}
\qquad (1.19.35)
$$

Define now a two-dimensional analogue of the three-dimensional covariant derivative through

$$
\begin{aligned}
v^{\alpha}\,\|_{\beta} &= v^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\gamma\beta}v^{\gamma} \\
v_{\alpha}\,\|_{\beta} &= v_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}v_{\gamma}
\end{aligned}
\qquad (1.19.36)
$$

so that, using 1.19.19, 1.19.21, the covariant derivative can be expressed as

$$
\begin{aligned}
v^{\alpha}\,|_{\beta} &= v^{\alpha}\,\|_{\beta} - K^{\alpha}_{\beta}v^{3} \\
v_{\alpha}\,|_{\beta} &= v_{\alpha}\,\|_{\beta} - K_{\alpha\beta}v_{3}
\end{aligned}
\qquad (1.19.37)
$$

In the special case when the vector is a plane vector, then $v_3 = v^3 = 0$, and there is no difference between the three-dimensional and two-dimensional covariant derivatives. In the general case, the covariant derivatives can now be expressed as

$$
\begin{aligned}
\mathbf{v}_{,\beta} &= v^{i}\,|_{\beta}\,\mathbf{a}_{i} \\
&= \left(v^{\alpha}\,\|_{\beta} - K^{\alpha}_{\beta}v^{3}\right)\mathbf{a}_{\alpha} + v^{3}\,|_{\beta}\,\mathbf{a}_{3} \\
\mathbf{v}_{,\beta} &= v_{i}\,|_{\beta}\,\mathbf{a}^{i} \\
&= \left(v_{\alpha}\,\|_{\beta} - K_{\alpha\beta}v_{3}\right)\mathbf{a}^{\alpha} + v_{3}\,|_{\beta}\,\mathbf{a}^{3}
\end{aligned}
\qquad (1.19.38)
$$

From 1.18.25, the gradient of a surface vector is (using 1.19.21)

$$
\operatorname{grad}\mathbf{v} = \left(v_{\alpha}\,\|_{\beta} - K_{\alpha\beta}v_{3}\right)\mathbf{a}^{\alpha}\otimes\mathbf{a}^{\beta} + K^{\gamma}_{\alpha}v_{\gamma}\mathbf{a}^{\alpha}\otimes\mathbf{a}^{3} \qquad (1.19.39)
$$

Tensors

The covariant derivatives of second order tensor components are given by 1.18.18. For example,

$$
\begin{aligned}
A^{ij}\,|_{\gamma} &= A^{ij}{}_{,\gamma} + \Gamma^{i}_{m\gamma}A^{mj} + \Gamma^{j}_{m\gamma}A^{im} \\
&= A^{ij}{}_{,\gamma} + \Gamma^{i}_{\lambda\gamma}A^{\lambda j} + \Gamma^{i}_{3\gamma}A^{3j} + \Gamma^{j}_{\lambda\gamma}A^{i\lambda} + \Gamma^{j}_{3\gamma}A^{i3}
\end{aligned}
\qquad (1.19.40)
$$

Here, only surface tensors will be examined, that is, all components with an index 3 are zero. The two dimensional (plane) covariant derivative is

$$A^{\alpha\beta}\|_\gamma \equiv A^{\alpha\beta}{}_{,\gamma} + \Gamma^{\alpha}_{\lambda\gamma}A^{\lambda\beta} + \Gamma^{\beta}_{\lambda\gamma}A^{\alpha\lambda} \tag{1.19.41}$$

Although $A^{\alpha 3} = A^{3\alpha} = 0$ for plane tensors, one still has non-zero

$$\begin{aligned} A^{\alpha 3}|_\gamma &= A^{\alpha 3}{}_{,\gamma} + \Gamma^{\alpha}_{\lambda\gamma}A^{\lambda 3} + \Gamma^{3}_{\lambda\gamma}A^{\alpha\lambda} \\ &= \Gamma^{3}_{\lambda\gamma}A^{\alpha\lambda} \\ &= K_{\lambda\gamma}A^{\alpha\lambda} \\ A^{3\beta}|_\gamma &= A^{3\beta}{}_{,\gamma} + \Gamma^{3}_{\lambda\gamma}A^{\lambda\beta} + \Gamma^{\beta}_{\lambda\gamma}A^{3\lambda} \\ &= \Gamma^{3}_{\lambda\gamma}A^{\lambda\beta} \\ &= K_{\lambda\gamma}A^{\lambda\beta} \end{aligned} \tag{1.19.42}$$

with $A^{33}|_\gamma = 0$.

From 1.18.28, the divergence of a surface tensor is

$$\operatorname{div}\mathbf{A} = A^{\alpha\beta}\|_\beta \,\mathbf{a}_\alpha + K_{\beta\gamma}A^{\beta\gamma}\mathbf{a}_3 \tag{1.19.43}$$

1.19.4 The Gauss-Codazzi Equations

Some useful equations can be derived by considering the second derivatives of the base vectors. First, from 1.18.2,

$$\begin{aligned} \mathbf{a}_{\alpha,\beta} &= \Gamma^{\lambda}_{\alpha\beta}\mathbf{a}_\lambda + \Gamma^{3}_{\alpha\beta}\mathbf{a}_3 \\ &= \Gamma^{\lambda}_{\alpha\beta}\mathbf{a}_\lambda + K_{\alpha\beta}\mathbf{a}_3 \end{aligned} \tag{1.19.44}$$

A second derivative is

$$\mathbf{a}_{\alpha,\beta\gamma} = \Gamma^{\lambda}_{\alpha\beta,\gamma}\mathbf{a}_\lambda + \Gamma^{\lambda}_{\alpha\beta}\mathbf{a}_{\lambda,\gamma} + K_{\alpha\beta,\gamma}\mathbf{a}_3 + K_{\alpha\beta}\mathbf{a}_{3,\gamma} \tag{1.19.45}$$

Eliminating the base vectors derivatives using 1.19.44 and 1.19.20b leads to {▲Problem 4}

$$\mathbf{a}_{\alpha,\beta\gamma} = \left(\Gamma^{\lambda}_{\alpha\beta,\gamma} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\lambda}_{\eta\gamma} - K_{\alpha\beta}K^{\lambda}_{\gamma}\right)\mathbf{a}_\lambda + \left(\Gamma^{\lambda}_{\alpha\beta}K_{\lambda\gamma} + K_{\alpha\beta,\gamma}\right)\mathbf{a}_3 \tag{1.19.46}$$

This equals the partial derivative $\mathbf{a}_{\alpha,\gamma\beta}$. Comparison of the coefficient of $\mathbf{a}_3$ for these alternative expressions for the second partial derivative leads to

$$K_{\alpha\beta,\gamma} - \Gamma^{\lambda}_{\alpha\gamma}K_{\lambda\beta} = K_{\alpha\gamma,\beta} - \Gamma^{\lambda}_{\alpha\beta}K_{\lambda\gamma} \tag{1.19.47}$$

From Eqn. 1.18.18,

$$K_{\alpha\beta}\,\|_{\gamma} = K_{\alpha\beta,\gamma} - \Gamma^{\lambda}_{\alpha\gamma} K_{\lambda\beta} - \Gamma^{\lambda}_{\beta\gamma} K_{\alpha\lambda} \tag{1.19.48}$$

and so

$$K_{\alpha\beta}\,\|_{\gamma} = K_{\alpha\gamma}\,\|_{\beta} \tag{1.19.49}$$

These are the **Codazzi equations**, in which there are only two independent non-trivial relations:

$$K_{11}\,\|_{2} = K_{12}\,\|_{1}, \qquad K_{22}\,\|_{1} = K_{12}\,\|_{2} \tag{1.19.50}$$

Raising indices using the metric coefficients leads to the similar equations

$$K^{\alpha}_{\beta}\,\|_{\gamma} = K^{\alpha}_{\gamma}\,\|_{\beta} \tag{1.19.51}$$

The Riemann-Christoffel Curvature Tensor

Comparing the coefficients of $\mathbf{a}_{\lambda}$ in 1.19.46 and the similar expression for the second partial derivative shows that

$$\Gamma^{\lambda}_{\alpha\gamma,\beta} - \Gamma^{\lambda}_{\alpha\beta,\gamma} + \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\lambda}_{\eta\beta} - \Gamma^{\eta}_{\alpha\beta}\Gamma^{\lambda}_{\eta\gamma} = K_{\alpha\gamma}K^{\lambda}_{\beta} - K_{\alpha\beta}K^{\lambda}_{\gamma} \tag{1.19.52}$$

The terms on the left are the two-dimensional Riemann-Christoffel, Eqn. 1.18.21, and so

$$R^{\lambda}_{\cdot\alpha\beta\gamma} = K_{\alpha\gamma}K^{\lambda}_{\beta} - K_{\alpha\beta}K^{\lambda}_{\gamma} \tag{1.19.53}$$

Further,

$$R_{\lambda\alpha\beta\gamma} = g_{\lambda\eta}R^{\eta}_{\cdot\alpha\beta\gamma} = K_{\alpha\gamma}g_{\lambda\eta}K^{\eta}_{\beta} - K_{\alpha\beta}g_{\lambda\eta}K^{\eta}_{\gamma} = K_{\alpha\gamma}K_{\beta\lambda} - K_{\alpha\beta}K_{\gamma\lambda} \tag{1.19.54}$$

These are the **Gauss equations**. From 1.18.21 *et seq.*, only 4 of the Riemann-Christoffel symbols are non-zero, and they are related through

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121} \tag{1.19.55}$$

so that there is in fact only one independent non-trivial Gauss relation. Further,

$$\begin{aligned} R_{\lambda\alpha\beta\gamma} &= K_{\alpha\gamma}K_{\beta\lambda} - K_{\alpha\beta}K_{\gamma\lambda} \\ &= K^{\mu}_{\alpha}K^{\eta}_{\lambda}\left(g_{\gamma\mu}g_{\beta\eta} - g_{\beta\mu}g_{\gamma\eta}\right) \\ &= K^{\mu}_{\alpha}K^{\eta}_{\lambda}\left(\delta^{\nu}_{\mu}\delta^{\rho}_{\eta} - \delta^{\rho}_{\mu}\delta^{\nu}_{\eta}\right)g_{\beta\rho}g_{\gamma\nu} \end{aligned} \tag{1.19.56}$$

Using 1.19.4b, 1.19.3,

$$R_{\lambda\alpha\beta\gamma} = K_\alpha^\mu K_\lambda^\eta e^{\rho\nu} e_{\eta\mu} g_{\beta\rho} g_{\gamma\nu}$$
$$= K_\alpha^\mu K_\lambda^\eta e_{\beta\gamma} e_{\eta\mu} \tag{1.19.57}$$
$$= g\varepsilon_{\beta\gamma}\varepsilon_{\eta\mu} K_\alpha^\mu K_\lambda^\eta$$

and so the Gauss relation can be expressed succinctly as

$$K_G = \frac{R_{1212}}{g} \tag{1.19.58}$$

where K_G is the Gaussian curvature, 1.19.27b. Thus the Riemann-Christoffel tensor is zero if and only if the Gaussian curvature is zero, and in this case only can the order of the two covariant differentiations be interchanged.

The Gauss-Codazzi equations, 1.19.50 and 1.19.58, are equivalent to a set of two first order and one second order differential equations that must be satisfied by the three independent metric coefficients $g_{\alpha\beta}$ and the three independent curvature tensor coefficients $K_{\alpha\beta}$.

Intrinsic Surface Properties

An **intrinsic** property of a surface is any quantity that remains unchanged when the surface is bent into another shape without stretching or shrinking. Some examples of intrinsic properties are the length of a curve on the surface, surface area, the components of the surface metric tensor $g_{\alpha\beta}$ (and hence the components of the Riemann-Christoffel tensor) and the Guassian curvature (which follows from the Gauss equation 1.19.58).

A **developable surface** is one which can be obtained by bending a plane, for example a piece of paper. Examples of developable surfaces are the cylindrical surface and the surface of a cone. Since the Riemann-Christoffel tensor and hence the Gaussian curvature vanish for the plane, they vanish for all developable surfaces.

1.19.5 Geodesics

The Geodesic Curvature and Normal Curvature

Consider a curve C lying on the surface, with arc length s measured from some fixed point. As for the space curve, §1.6.2, one can define the unit tangent vector $\boldsymbol{\tau}$, principal normal $\mathbf{v}$ and binormal vector $\mathbf{b}$ (Eqn. 1.6.3 *et seq.*):

$$\boldsymbol{\tau} = \frac{d\mathbf{x}}{ds} = \frac{d\Theta^\alpha}{ds}\mathbf{a}_\alpha, \qquad \mathbf{v} = \frac{1}{\kappa}\frac{d\boldsymbol{\tau}}{ds}, \qquad \mathbf{b} = \boldsymbol{\tau}\times\mathbf{v} \tag{1.19.59}$$

so that the curve passes along the intersection of the osculating plane containing $\boldsymbol{\tau}$ and $\mathbf{v}$ (see Fig. 1.6.3), and the surface These vectors form an orthonormal set but, although $\mathbf{v}$ is normal to the tangent, it is not necessarily normal to the surface, as illustrated in Fig.

1.19.5. For this reason, form the new orthonormal triad $(\boldsymbol{\tau}, \boldsymbol{\tau}_2, \mathbf{a}_3)$, so that the unit vector $\boldsymbol{\tau}_2$ lies in the plane tangent to the surface. From 1.19.59, 1.19.3,

$$\boldsymbol{\tau}_2 = \mathbf{a}_3 \times \boldsymbol{\tau} = \frac{d\Theta^\alpha}{ds}\mathbf{a}_3 \times \mathbf{a}_\alpha = e_{\alpha\beta}\frac{d\Theta^\alpha}{ds}\mathbf{a}^\beta \qquad (1.19.60)$$

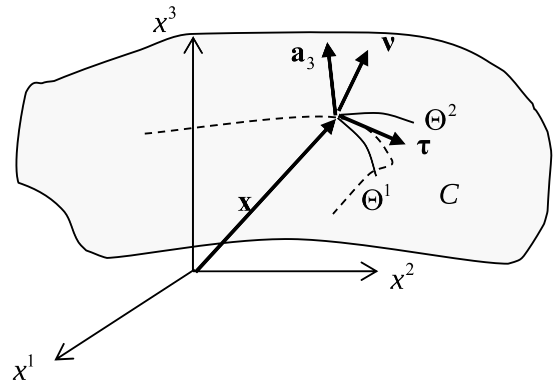

Figure 1.19.5: a curve lying on a surface

Next, the vector $d\boldsymbol{\tau}/ds$ will be decomposed into components along $\boldsymbol{\tau}_2$ and the normal $\mathbf{a}_3$. First, differentiate 1.19.59a and use 1.19.44b to get {▲Problem 5}

$$\frac{d\boldsymbol{\tau}}{ds} = \left(\frac{d^2\Theta^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{d\Theta^\alpha}{ds}\frac{d\Theta^\beta}{ds}\right)\mathbf{a}_\gamma + K_{\alpha\beta}\frac{d\Theta^\alpha}{ds}\frac{d\Theta^\beta}{ds}\mathbf{a}_3 \qquad (1.19.61)$$

Then

$$\frac{d\boldsymbol{\tau}}{ds} = \kappa_g \boldsymbol{\tau}_2 + \kappa_n \mathbf{a}_3 \qquad (1.19.62)$$

where

$$\begin{aligned}\kappa_g &= e_{\lambda\gamma}\frac{d\Theta^\lambda}{ds}\left(\frac{d^2\Theta^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{d\Theta^\alpha}{ds}\frac{d\Theta^\beta}{ds}\right) \\ \kappa_n &= K_{\alpha\beta}\frac{d\Theta^\alpha}{ds}\frac{d\Theta^\beta}{ds}\end{aligned} \qquad (1.19.63)$$

These are formulae for the **geodesic curvature** κ_g and the **normal curvature** κ_n. Many different curves with representations $\Theta^\alpha(s)$ can pass through a certain point with a given tangent vector $\boldsymbol{\tau}$. Form 1.19.59, these will all have the same value of $d\Theta^\alpha / ds$ and so, from 1.19.63, these curves will have the same normal curvature but, in general, different geodesic curvatures.

A curve passing through a **normal section**, that is, along the intersection of a plane containing $\boldsymbol{\tau}$ and $\mathbf{a}_3$, and the surface, will have zero geodesic curvature.

The normal curvature can be expressed as

$$\kappa_n = \boldsymbol{\tau}\,\mathbf{K}\,\boldsymbol{\tau} \tag{1.19.64}$$

If the tangent is along an eigenvector of **K**, then κ_n is an eigenvalue, and hence a maximum or minimum normal curvature. Surface curves with the property that an eigenvector of the curvature tensor is tangent to it at every point is called a **line of curvature**. A convenient coordinate system for a surface is one in which the coordinate curves are lines of curvature. Such a system, with Θ^1 containing the maximum values of κ_n, has at every point a curvature tensor of the form

$$\left[K_i^{\,j}\right] = \begin{bmatrix} K_1^1 & 0 \\ 0 & K_2^2 \end{bmatrix} = \begin{bmatrix} (\kappa_n)_{\max} & 0 \\ 0 & (\kappa_n)_{\min} \end{bmatrix} \tag{1.19.65}$$

This was the case with the spherical surface example discussed in §1.19.2.

The Geodesic

A **geodesic** is defined to be a curve which has zero geodesic curvature *at every point* along the curve. Form 1.19.63, parametric equations for the geodesics over a surface are

$$\frac{d^2\Theta^\gamma}{ds^2} + \Gamma_{\alpha\beta}^\gamma \frac{d\Theta^\alpha}{ds}\frac{d\Theta^\beta}{ds} = 0 \tag{1.19.64}$$

It can be proved that the geodesic is the curve of shortest distance joining two points on the surface. Thus the geodesic curvature is a measure of the deviance of the curve from the shortest-path curve.

The Geodesic Coordinate System

If the Gaussian curvature of a surface is not zero, then it is not possible to find a surface coordinate system for which the metric tensor components $g_{\alpha\beta}$ equal the Kronecker delta $\delta_{\alpha\beta}$ everywhere. Such a geometry is called **Riemannian**. However, it is always possible to construct a coordinate system in which $g_{\alpha\beta} = \delta_{\alpha\beta}$, and the derivatives of the metric coefficients are zero, *at a particular point* on the surface. This is the **geodesic coordinate system**.

1.19.6 Problems

1 Derive Eqns. 1.19.16, $\Gamma_{\alpha 3}^3 = \Gamma_{33}^\alpha = \Gamma_{33}^3 = 0$.

2 Derive the Cartesian components of the curvilinear base vectors for the spherical surface, Eqn. 1.19.28.

3 Derive the Christoffel symbols for the spherical surface, Eqn. 1.19.32.
4 Use Eqns. 1.19.44-5 and 1.19.20b to derive 1.19.46.
5 Use Eqns. 1.19.59a and 1.19.44b to derive 1.19.61.