1.18 Curvilinear Coordinates: Tensor Calculus

1.18.1 Differentiation of the Base Vectors

Differentiation in curvilinear coordinates is more involved than that in Cartesian coordinates because the base vectors are no longer constant and their derivatives need to be taken into account, for example the partial derivative of a vector with respect to the Cartesian coordinates is

$$\frac{\partial \mathbf{v}}{\partial x_j} = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i \qquad \text{but}^1 \qquad \frac{\partial \mathbf{v}}{\partial \Theta^j} = \frac{\partial v^i}{\partial \Theta^j}\mathbf{g}_i + v^i \frac{\partial \mathbf{g}_i}{\partial \Theta^j}$$

The Christoffel Symbols of the Second Kind

First, from Eqn. 1.16.19 – and using the inverse relation,

$$\frac{\partial \mathbf{g}_i}{\partial \Theta^j} = \frac{\partial}{\partial \Theta^j}\left(\frac{\partial x^m}{\partial \Theta^i}\right)\mathbf{e}_m = \frac{\partial^2 x^m}{\partial \Theta^i \Theta^j}\frac{\partial \Theta^k}{\partial x^m}\mathbf{g}_k \qquad (1.18.1)$$

this can be written as

$$\boxed{\frac{\partial \mathbf{g}_i}{\partial \Theta^j} = \Gamma_{ij}^k \mathbf{g}_k} \quad \textbf{Partial Derivatives of Covariant Base Vectors} \quad (1.18.2)$$

where

$$\Gamma_{ij}^k = \frac{\partial^2 x^m}{\partial \Theta^i \Theta^j}\frac{\partial \Theta^k}{\partial x^m}, \qquad (1.18.3)$$

and Γ_{ij}^k is called the **Christoffel symbol of the second kind**; it can be seen to be equivalent to the *k*th contravariant component of the vector $\partial \mathbf{g}_i / \partial \Theta^j$. One then has {▲Problem 1}

$$\boxed{\Gamma_{ij}^k = \Gamma_{ji}^k = \frac{\partial \mathbf{g}_i}{\partial \Theta^j}\cdot \mathbf{g}^k = \frac{\partial \mathbf{g}_j}{\partial \Theta^i}\cdot \mathbf{g}^k} \quad \textbf{Christoffel Symbols of the 2}^{\textbf{nd}}\textbf{ kind} \quad (1.18.4)$$

and the symmetry in the indices *i* and *j* is evident[2]. Looking now at the derivatives of the contravariant base vectors $\mathbf{g}^i$: differentiating the relation $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k$ leads to

$$-\frac{\partial \mathbf{g}^k}{\partial \Theta^j}\cdot \mathbf{g}_i = \frac{\partial \mathbf{g}_i}{\partial \Theta^j}\cdot \mathbf{g}^k = \Gamma_{ij}^m \mathbf{g}_m \cdot \mathbf{g}^k = \Gamma_{ij}^k$$

---

[1] of course, one could express the $\mathbf{g}_i$ in terms of the $\mathbf{e}_i$, and use only the first of these expressions

[2] note that, in non-Euclidean space, this symmetry in the indices is not necessarily valid

and so

$$\boxed{\frac{\partial \mathbf{g}^i}{\partial \Theta^j} = -\Gamma^i_{jk}\mathbf{g}^k} \quad \textbf{Partial Derivatives of Contravariant Base Vectors} \quad (1.18.5)$$

Transformation formulae for the Christoffel Symbols

The Christoffel symbols are not the components of a (third order) tensor. This follows from the fact that these components do not transform according to the tensor transformation rules given in §1.17. In fact,

$$\overline{\Gamma}^k_{ij} = \frac{\partial \Theta^p}{\partial \overline{\Theta}^i}\frac{\partial \Theta^q}{\partial \overline{\Theta}^j}\frac{\partial \overline{\Theta}^k}{\partial \Theta^r}\Gamma^r_{pq} + \frac{\partial^2 \Theta^s}{\partial \overline{\Theta}^i \overline{\Theta}^j}\frac{\partial \overline{\Theta}^k}{\partial \Theta^s}$$

The Christoffel Symbols of the First Kind

The Christoffel symbols of the second kind relate derivatives of covariant (contravariant) base vectors to the covariant (contravariant) base vectors. A second set of symbols can be introduced relating the base vectors to the derivatives of the reciprocal base vectors, called the **Christoffel symbols of the first kind**:

$$\boxed{\Gamma_{ijk} = \Gamma_{jik} = \frac{\partial \mathbf{g}_i}{\partial \Theta^j}\cdot\mathbf{g}_k = \frac{\partial \mathbf{g}_j}{\partial \Theta^i}\cdot\mathbf{g}_k} \quad \textbf{Christoffel Symbols of the 1}^{\textbf{st}}\textbf{ kind} \quad (1.18.6)$$

so that the partial derivatives of the covariant base vectors can be written in the alternative form

$$\frac{\partial \mathbf{g}_i}{\partial \Theta^j} = \Gamma_{ijk}\mathbf{g}^k, \qquad (1.18.7)$$

and it also follows from Eqn. 1.18.2 that

$$\Gamma_{ijk} = \Gamma^m_{ij} g_{mk}, \quad \Gamma^k_{ij} = \Gamma_{ijm} g^{mk} \qquad (1.18.8)$$

showing that the index *k* here can be raised or lowered using the metric coefficients as for a third order tensor (but the first two indexes, *i* and *j*, cannot and, as stated, the Christoffel symbols are not the components of a third order tensor).

Example: Newton's Second Law

The position vector can be expressed in terms of curvilinear coordinates, $\mathbf{x} = \mathbf{x}(\Theta^i)$. The velocity is then

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial \Theta^i}\frac{d\Theta^i}{dt} = \frac{d\Theta^i}{dt}\mathbf{g}_i$$

and the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\Theta^i}{dt^2}\mathbf{g}_i + \frac{d\Theta^j}{dt}\frac{\partial \mathbf{g}_j}{\partial \Theta^k}\frac{d\Theta^k}{dt} = \left(\frac{d^2\Theta^i}{dt^2} + \Gamma^i_{jk}\frac{d\Theta^j}{dt}\frac{d\Theta^k}{dt}\right)\mathbf{g}_i$$

Equating the contravariant components of Newton's second law $\mathbf{f} = m\mathbf{a}$ then gives the general curvilinear expression

$$f^i = m\left(\ddot{\Theta}^i + \Gamma^i_{jk}\dot{\Theta}^j\dot{\Theta}^k\right)$$

■

Partial Differentiation of the Metric Coefficients

The metric coefficients can be differentiated with the aid of the Christoffel symbols of the first kind {▲Problem 3}:

$$\frac{\partial g_{ij}}{\partial \Theta^k} = \Gamma_{ikj} + \Gamma_{jki} \qquad (1.18.9)$$

Using the symmetry of the metric coefficients and the Christoffel symbols, this equation can be written in a number of different ways:

$$g_{ij,k} = \Gamma_{kij} + \Gamma_{jki}, \qquad g_{jk,i} = \Gamma_{ijk} + \Gamma_{kij}, \qquad g_{ki,j} = \Gamma_{jki} + \Gamma_{ijk}$$

Subtracting the first of these from the sum of the second and third then leads to the useful relations (using also 1.18.8)

$$\begin{aligned} \Gamma_{ijk} &= \frac{1}{2}\left(g_{jk,i} + g_{ki,j} - g_{ij,k}\right) \\ \Gamma^k_{ij} &= \frac{1}{2}g^{mk}\left(g_{jm,i} + g_{mi,j} - g_{ij,m}\right) \end{aligned} \qquad (1.18.10)$$

which show that the Christoffel symbols depend on the metric coefficients only.

Alternatively, one can write the derivatives of the metric coefficients in the form (the first of these is 1.18.9)

$$\begin{aligned} g_{ij,k} &= \Gamma_{ikj} + \Gamma_{jki} \\ g^{ij}{}_{,k} &= -g^{im}\Gamma^j_{km} - g^{jm}\Gamma^i_{km} \end{aligned} \qquad (1.18.11)$$

Also, directly from 1.15.7, one has the relations

$$\frac{\partial g}{\partial g_{ij}} = g g^{ij}, \qquad \frac{\partial g}{\partial g^{ij}} = g g_{ij} \qquad (1.18.12)$$

and from these follow other useful relations, for example {▲Problem 4}

$$\Gamma^i_{ij} = \frac{\partial \log\left(\sqrt{g}\right)}{\partial \Theta^j} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \Theta^j} = J^{-1} \frac{\partial J}{\partial \Theta^j} \tag{1.18.13}$$

and

$$\begin{aligned} \frac{\partial e_{ijk}}{\partial \Theta^m} &= \varepsilon_{ijk} \frac{\partial \sqrt{g}}{\partial \Theta^m} = \varepsilon_{ijk} \sqrt{g}\Gamma^n_{mn} = e_{ijk}\Gamma^n_{mn} \\ \frac{\partial e^{ijk}}{\partial \Theta^m} &= \varepsilon^{ijk} \frac{\partial \left(1/\sqrt{g}\right)}{\partial \Theta^m} = -\varepsilon^{ijk} \frac{1}{\sqrt{g}} \Gamma^n_{mn} = -e^{ijk}\Gamma^n_{mn} \end{aligned} \tag{1.18.14}$$

1.18.2 Partial Differentiation of Tensors

The Partial Derivative of a Vector

The derivative of a vector in curvilinear coordinates can be written as

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial \Theta^j} &= \frac{\partial v^i}{\partial \Theta^j}\mathbf{g}_i + v^i \frac{\partial \mathbf{g}_i}{\partial \Theta^j} & \qquad \frac{\partial \mathbf{v}}{\partial \Theta^j} &= \frac{\partial v_i}{\partial \Theta^j}\mathbf{g}^i + v_i \frac{\partial \mathbf{g}^i}{\partial \Theta^j} \\ &= \frac{\partial v^i}{\partial \Theta^j}\mathbf{g}_i + v^i \Gamma^k_{ij}\mathbf{g}_k \qquad \text{or} & &= \frac{\partial v_i}{\partial \Theta^j}\mathbf{g}^i - v_i \Gamma^i_{jk}\mathbf{g}^k \\ &\equiv v^i\Big|_j \mathbf{g}_i & &\equiv v_i\Big|_j \mathbf{g}^i \end{aligned} \tag{1.18.15}$$

where

$$\boxed{\begin{aligned} v^i\,|_j &= v^i{}_{,j} + \Gamma^i_{kj} v^k \\ v_i\,|_j &= v_{i,j} - \Gamma^k_{ij} v_k \end{aligned}} \quad \textbf{Covariant Derivative of Vector Components} \tag{1.18.16}$$

The first term here is the ordinary partial derivative of the vector components. The second term enters the expression due to the fact that the curvilinear base vectors are changing. The complete quantity is defined to be the **covariant derivative** of the vector components. The covariant derivative reduces to the ordinary partial derivative in the case of rectangular Cartesian coordinates.

The $v_i\,|_j$ is the *i*th component of the *j* – derivative of **v**. The $v_i\,|_j$ are also the components of a second order covariant tensor, transforming under a change of coordinate system according to the tensor transformation rule 1.17.4 (see the gradient of a vector below).

The covariant derivative of vector components is given by 1.18.10. In the same way, the covariant derivative of a *vector* is defined to be the complete expression in 1.18.9, $\mathbf{v}_{,j}$, with $\mathbf{v}_{,j} = v^i|_j \, \mathbf{g}_i$.

The Partial Derivative of a Tensor

The rules for covariant differentiation of vectors can be extended to higher order tensors. The various partial derivatives of a second-order tensor

$$\mathbf{A} = A^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = A_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = A_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j$$

are indicated using the following notation:

$$\frac{\partial \mathbf{A}}{\partial \Theta^k} = A^{ij}|_k \, \mathbf{g}_i \otimes \mathbf{g}_j = A_{ij}|_k \, \mathbf{g}^i \otimes \mathbf{g}^j = A_i^{\cdot j}|_k \, \mathbf{g}^i \otimes \mathbf{g}_j = A_{\cdot j}^{i}|_k \, \mathbf{g}_i \otimes \mathbf{g}^j \qquad (1.18.17)$$

Thus, for example,

$$\begin{aligned} \mathbf{A}_{,k} &= A_{ij,k}\mathbf{g}^i \otimes \mathbf{g}^j + A_{ij}\mathbf{g}^i{}_{,k} \otimes \mathbf{g}^j + A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j{}_{,k} \\ &= A_{ij,k}\mathbf{g}^i \otimes \mathbf{g}^j - A_{ij}\Gamma^i_{mk}\mathbf{g}^m \otimes \mathbf{g}^j - A_{ij}\mathbf{g}^i \otimes \Gamma^j_{km}\mathbf{g}^m \\ &= \left[A_{ij,k} - \Gamma^m_{ik}A_{mj} - \Gamma^m_{jk}A_{im}\right]\mathbf{g}^i \otimes \mathbf{g}^j \end{aligned}$$

and, in summary,

$$\boxed{\begin{aligned} A_{ij}|_k &= A_{ij,k} - \Gamma^m_{ik}A_{mj} - \Gamma^m_{jk}A_{im} \\ A^{ij}|_k &= A^{ij}{}_{,k} + \Gamma^i_{mk}A^{mj} + \Gamma^j_{mk}A^{im} \\ A_i^{\cdot j}|_k &= A_i^{\cdot j}{}_{,k} + \Gamma^j_{mk}A_i^{\cdot m} - \Gamma^m_{ik}A_m^{\cdot j} \\ A_{\cdot j}^{i}|_k &= A_{\cdot j,k}^{i} + \Gamma^i_{mk}A_{\cdot j}^{m} - \Gamma^m_{jk}A_{\cdot m}^{i} \end{aligned}} \qquad (1.18.18)$$

Covariant Derivative of Tensor Components

The covariant derivative formulas can be remembered as follows: the formula contains the usual partial derivative plus

- for each contravariant index a term containing a Christoffel symbol in which that index has been inserted on the upper level, multiplied by the tensor component with that index replaced by a dummy summation index which also appears in the Christoffel symbol
- for each covariant index a term prefixed by a minus sign and containing a Christoffel symbol in which that index has been inserted on the lower level, multiplied by the tensor with that index replaced by a dummy which also appears in the Christoffel symbol.
- the remaining symbol in all of the Christoffel symbols is the index of the variable with respect to which the covariant derivative is taken.

For example,

$$A_{\cdot jk}^{i}|_l = A_{\cdot jk,l}^{i} + \Gamma^i_{ml}A_{\cdot jk}^{m} - \Gamma^m_{jl}A_{\cdot mk}^{i} - \Gamma^m_{kl}A_{\cdot jm}^{i}$$

Note that the covariant derivative of a product obeys the same rules as the ordinary differentiation, e.g.

$$\left(u_i A^{jk}\right)|_m = u_{i|}\,|_m\; A^{jk} + u_i A^{jk}\;|_m$$

Covariantly Constant Coefficients

It can be shown that the metric coefficients are **covariantly constant**[3] {▲Problem 5},

$$g_{ij}|_k = g^{ij}|_k = 0\,,$$

This implies that the metric (identity) tensor **I** is constant, $\mathbf{I}_{,k} = 0$ (see Eqn. 1.16.32) – although its components g_{ij} are not constant. Similarly, the components of the permutation tensor, are covariantly constant

$$e_{ijk}|_m = e^{ijk}|_m = 0\,.$$

In fact, specialising the identity tensor **I** and the permutation tensor **E** to Cartesian coordinates, one has $g_{ij} = g^{ij} \to \delta_{ij}$, $e_{ijk} = e^{ijk} \to \varepsilon_{ijk}$, which are clearly constant. Specialising the derivatives, $g_{ij}\,|_k \to \delta_{ij,k}$, $e_{ijk}\,|_m \to \varepsilon_{ijk,m}$, and these are clearly zero. From §1.17, since if the components of a tensor vanish in one coordinate system, they vanish in all coordinate systems, the curvilinear coordinate versions vanish also, as stated above.

The above implies that any time any of these factors appears in a covariant derivative, they may be extracted, as in $\left(g_{ij}u^i\right)|_k = \left(g_{ij}\right)u^i\,|_k$.

The Riemann-Christoffel Curvature Tensor

Higher-order covariant derivatives are defined by repeated application of the first-order derivative. This is straight-forward but can lead to algebraically lengthy expressions. For example, to evaluate $v_i|_{mn}$, first write the first covariant derivative in the form of a second order covariant tensor **B**,

$$v_i\,|_m = v_{i,m} - \Gamma^k_{im} v_k \equiv B_{im}$$

so that

$$\begin{aligned} v_i\,|_{mn} &= B_{im}\,|_n \\ &= B_{im,n} - \Gamma^k_{in} B_{km} - \Gamma^k_{mn} B_{ik} \\ &= \left(v_{i,m} - \Gamma^k_{im} v_k\right)_{,n} - \Gamma^k_{in}\left(v_{k,m} - \Gamma^l_{km} v_l\right) - \Gamma^k_{mn}\left(v_{i,k} - \Gamma^l_{ik} v_l\right) \end{aligned} \tag{1.18.19}$$

3

The covariant derivative $v_i|_{nm}$ is obtained by interchaning *m* and *n* in this expression. Now investigate the difference

$$v_i\,|_{mn} - v_i\,|_{nm} = \left(v_{i,m} - \Gamma^k_{im} v_k\right)_{,n} - \left(v_{i,n} - \Gamma^k_{in} v_k\right)_{,m} - \Gamma^k_{in}\left(v_{k,m} - \Gamma^l_{km} v_l\right) + \Gamma^k_{im}\left(v_{k,n} - \Gamma^l_{kn} v_l\right)$$
$$- \Gamma^k_{mn}\left(v_{i,k} - \Gamma^l_{ik} v_l\right) + \Gamma^k_{nm}\left(v_{i,k} - \Gamma^l_{ik} v_l\right)$$

The last two terms cancel here because of the symmetry of the Christoffel symbol, leaving

$$v_i\,|_{mn} - v_i\,|_{nm} = v_{i,mn} - \Gamma^k_{im,n} v_k - \Gamma^k_{im} v_{k,n} - v_{i,nm} + \Gamma^k_{in,m} v_k + \Gamma^k_{in} v_{k,m}$$
$$- \Gamma^k_{in}\left(v_{k,m} - \Gamma^l_{km} v_l\right) + \Gamma^k_{im}\left(v_{k,n} - \Gamma^l_{kn} v_l\right)$$

The order on the ordinary partial differentiation is interchangeable and so the second order partial derivative terms cancel,

$$v_i\,|_{mn} - v_i\,|_{nm} = v_{i,mn} - \Gamma^k_{im,n} v_k - \Gamma^k_{im} v_{k,n} - v_{i,nm} + \Gamma^k_{in,m} v_k + \Gamma^k_{in} v_{k,m}$$
$$- \Gamma^k_{in}\left(v_{k,m} - \Gamma^l_{km} v_l\right) + \Gamma^k_{im}\left(v_{k,n} - \Gamma^l_{kn} v_l\right)$$

After further cancellation one arrives at

$$v_i|_{mn} - v_i|_{nm} = R^{j}_{\cdot imn} v_j \qquad (1.18.20)$$

where $\mathbf{R}$ is the fourth-order **Riemann-Christoffel curvature tensor**, with (mixed) components

$$R^{j}_{\cdot imn} = \Gamma^j_{in,m} - \Gamma^j_{im,n} + \Gamma^k_{in}\Gamma^j_{km} - \Gamma^k_{im}\Gamma^j_{kn} \qquad (1.18.21)$$

Since the Christoffel symbols vanish in a Cartesian coordinate system, then so does $R^{j}_{\cdot imn}$. Again, any tensor that vanishes in one coordinate system must be zero in all coordinate systems, and so $R^{j}_{\cdot imn} = 0$, implying that the order of covariant differentiation is immaterial, $v_i|_{mn} = v_i|_{nm}$.

From 1.18.10, it follows that

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$
$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

The latter of these known as the **Bianchi identities**. In fact, only six components of the Riemann-Christoffel tensor are independent; the expression $R^{j}_{\cdot imn} = 0$ then represents 6 equations in the 6 independent components g_{ij}.

This analysis is for a Euclidean space – the usual three-dimensional space in which quantities can be expressed in terms of a Cartesian reference system – such a space is called a **flat space**. These ideas can be extended to other, curved spaces, so-called **Riemannian spaces** (**Riemannian manifolds**), for which the Riemann-Christoffel tensor is non-zero (see §1.19).

1.18.3 Differential Operators and Tensors

In this section, the concepts of the gradient, divergence and curl from §1.6 and §1.14 are generalized to the case of curvilinear components.

Space Curves and the Gradient

Consider first a scalar function $f(\mathbf{x})$, where $\mathbf{x} = x^i \mathbf{e}_i$ is the position vector, with $x^i = x^i(\Theta^j)$. Let the curvilinear coordinates depend on some parameter s, $\Theta^j = \Theta^j(s)$, so that $\mathbf{x}(s)$ traces out a space curve C.

For example, the cylindrical coordinates $\Theta^j = \Theta^j(s)$, with $r = a$, $\theta = s/c$, $z = sb/c$, $0 \le s \le 2\pi c$, generate a helix.

From §1.6.2, a tangent to C is

$$\boldsymbol{\tau} = \frac{d\mathbf{x}}{ds} = \frac{\partial \mathbf{x}}{\partial \Theta^i}\frac{d\Theta^i}{ds} = \tau^i \mathbf{g}_i$$

so that $d\Theta^i / ds$ are the contravariant components of $\boldsymbol{\tau}$. Thus

$$\frac{df}{ds} = \frac{\partial f}{\partial \Theta^i}\tau^i = \left(\frac{\partial f}{\partial \Theta^i}\mathbf{g}^i\right)\cdot\left(\tau^j \mathbf{g}_j\right).$$

For Cartesian coordinates, $df/ds = \nabla f \cdot \boldsymbol{\tau}$ (see the discussion on normals to surfaces in §1.6.4). For curvilinear coordinates, therefore, the Nabla operator of 1.6.11 now reads

$$\nabla = \mathbf{g}^i \frac{\partial}{\partial \Theta^i} \qquad (1.18.22)$$

so that again the directional derivative is

$$\frac{df}{ds} = \nabla f \cdot \boldsymbol{\tau}$$

The Gradient of a Scalar

In general then, the gradient of a scalar valued function Φ is defined to be

$$\boxed{\nabla \Phi \equiv \operatorname{grad}\Phi = \frac{\partial \Phi}{\partial \Theta^i}\mathbf{g}^i} \quad \textbf{Gradient of a Scalar} \qquad (1.18.23)$$

and, with $d\mathbf{x} = dx^i \mathbf{e}_i = d\Theta^i \mathbf{g}_i$, one has

$$d\Phi \equiv \frac{\partial \Phi}{\partial \Theta^i} d\Theta^i = \nabla\Phi \cdot d\mathbf{x} \tag{1.18.24}$$

The Gradient of a Vector

Analogous to Eqn. 1.14.3, the gradient of a vector is defined to be the tensor product of the derivative $\partial \mathbf{u} / \Theta^j$ with the *contravariant* base vector $\mathbf{g}^j$:

$$\boxed{\begin{aligned} \text{grad}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \Theta^j} \otimes \mathbf{g}^j &= u_i \,|_j\, \mathbf{g}^i \otimes \mathbf{g}^j \\ &= u^i \,|_j\, \mathbf{g}_i \otimes \mathbf{g}^j \end{aligned}} \quad \textbf{Gradient of a Vector} \tag{1.18.25}$$

Note that

$$\nabla \otimes \mathbf{u} = \mathbf{g}^i \frac{\partial}{\partial \Theta^i} \otimes \mathbf{u} = \mathbf{g}^i \otimes \frac{\partial \mathbf{u}}{\partial \Theta^i} = u_j \,|_i\, \mathbf{g}^i \otimes \mathbf{g}^j = u^j \,|_i\, \mathbf{g}^i \otimes \mathbf{g}_j$$

so that again one arrives at Eqn. 1.14.7, $(\nabla \otimes \mathbf{u})^{\mathrm{T}} = \text{grad}\mathbf{u}$.

Again, one has for a space curve parameterised by s,

$$\frac{d\mathbf{u}}{ds} = \frac{\partial \mathbf{u}}{\partial \Theta^i} \tau^i = \frac{\partial \mathbf{u}}{\partial \Theta^i} \left(\boldsymbol{\tau} \cdot \mathbf{g}^i\right) = \left(\mathbf{g}^i \otimes \frac{\partial \mathbf{u}}{\partial \Theta^i}\right)^{\mathrm{T}} \cdot \boldsymbol{\tau} = \text{grad}\mathbf{u} \cdot \boldsymbol{\tau}$$

Similarly, from 1.18.18, the gradient of a second-order tensor is

$$\boxed{\begin{aligned} \text{grad}\mathbf{A} = \frac{\partial \mathbf{A}}{\partial \Theta^k} \otimes \mathbf{g}^k &= A^{ij} \,|_k\, \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= A_{ij} \,|_k\, \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \\ &= A_i^{\cdot j} \,|_k\, \mathbf{g}^i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= A_{\cdot j}^{i} \,|_k\, \mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \end{aligned}} \quad \textbf{Gradient of a Tensor} \tag{1.18.26}$$

The Divergence

From 1.14.9, the divergence of a vector is {▲Problem 6}

$$\boxed{\text{div}\mathbf{u} = \text{grad}\,\mathbf{u} : \mathbf{I} = u^i \,|_i \qquad \left(= \frac{\partial \mathbf{u}}{\partial \Theta^j} \cdot \mathbf{g}^j \right)} \quad \textbf{Divergence of a Vector} \tag{1.18.27}$$

This is equivalent to the divergence operation involving the Nabla operator, $\text{div}\mathbf{u} = \nabla \cdot \mathbf{u}$. An alternative expression can be obtained from 1.18.13 {▲Problem 7},

$$\operatorname{div}\mathbf{u} = u^i\,|_i = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}u^i\right)}{\partial\Theta^i} = J^{-1}\frac{\partial\left(Ju^i\right)}{\partial\Theta^i}$$

Similarly, using 1.14.12, the divergence of a second-order tensor is

$$\begin{aligned}\operatorname{div}\mathbf{A} &= \operatorname{grad}\mathbf{A} : \mathbf{I} = A^{ij}\,|_j\ \mathbf{g}_i \qquad \left(= \frac{\partial\mathbf{A}}{\partial\Theta^j}\mathbf{g}^j\right) \\ &= A_i^{\cdot j}\,|_j\ \mathbf{g}^i\end{aligned}$$

Divergence of a Tensor (1.18.28)

Here, one has the alternative definition,

$$\nabla\cdot\mathbf{A} = \mathbf{g}^i\frac{\partial}{\partial\Theta^i}\cdot\mathbf{A} = \mathbf{g}^i\cdot\frac{\partial\mathbf{A}}{\partial\Theta^i} = A^{ji}\,|_j\ \mathbf{g}_i = \cdots$$

so that again one arrives at Eqn. 1.14.14, $\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A}^{\mathrm{T}}$.

The Curl

The curl of a vector is defined by {▲Problem 8}

$$\operatorname{curl}\mathbf{u} = \nabla\times\mathbf{u} = \mathbf{g}^k\times\frac{\partial\mathbf{u}}{\partial\Theta^k} = e^{ijk}u_j\,|_i\ \mathbf{g}_k = e^{ijk}\frac{\partial u_j}{\partial\Theta^i}\mathbf{g}_k$$

Curl of a Vector (1.18.29)

the last equality following from the fact that all the Christoffel symbols cancel out.

Covariant derivatives as Tensor Components

Equation 1.18.25 shows clearly that the covariant derivatives of vector components are themselves the components of second order tensors. It follows that they can be manipulated as other tensors, for example,

$$g^{im}u_m\,|_j = u^i\,|_j$$

and it is also helpful to introduce the following notation:

$$u_i\,|^j = u_i\,|_m\ g^{mj}, \quad u^i\,|^j = u^i\,|_m\ g^{mj}.$$

The divergence and curl can then be written as {▲Problem 10}

$$\begin{aligned}\operatorname{div}\mathbf{u} &= u^i\,|_i = u_i\,|^i \\ \operatorname{curl}\mathbf{u} &= e^{ijk}u_j\,|_i\ \mathbf{g}_k = e_{ijk}u^j\,|^i\ \mathbf{g}^k\end{aligned}.$$

Generalising Tensor Calculus from Cartesian to Curvilinear Coordinates

It was seen in §1.16.7 how formulae could be generalised from the Cartesian system to the corresponding formulae in curvilinear coordinates. In addition, formulae for the gradient, divergence and curl of tensor fields may be generalised to curvilinear components simply by replacing the partial derivatives with the covariant derivatives. Thus:

		Cartesian	Curvilinear
Gradient	Of a scalar field	$\text{grad}\phi,\ \nabla\phi = \partial\phi/\partial x_i$	$\phi_{,i} = \phi\,|_i \equiv \partial\phi/\partial\Theta^i$
	of a vector field	$\text{grad}\mathbf{u} = \partial u_i / \partial x_j$	$u^i|_j$
	of a tensor field	$\text{grad}\mathbf{T} = \partial T_{ij} / \partial x_k$	$T^{ij}|_k$
Divergence	of a vector field	$\text{div}\mathbf{u},\ \nabla\cdot\mathbf{u} = \partial u_i / \partial x_i$	$u^i|_i$
	of a tensor field	$\text{div}\mathbf{T} = \partial T_{ij} / \partial x_j$	$T^{ij}|_j$
Curl	of a vector field	$\text{curl}\mathbf{u},\ \nabla\times\mathbf{u} = \varepsilon_{ijk}\partial u_j / \partial x_i$	$e^{ijk}u_j\,|_i$

Table 1.18.1: generalising formulae from Cartesian to General Curvilinear Coordinates

All the tensor identities derived for Cartesian bases (§1.6.9, §1.14.3) hold also for curvilinear coordinates, for example {▲Problem 11}

$$\text{grad}(\alpha\mathbf{v}) = \alpha\,\text{grad}\mathbf{v} + \mathbf{v}\otimes\text{grad}\alpha$$
$$\text{div}(\mathbf{vA}) = \mathbf{v}\cdot\text{div}\mathbf{A} + \mathbf{A} : \text{grad}\mathbf{v}$$

1.18.4 Partial Derivatives with respect to a Tensor

The notion of differentiation of one tensor with respect to another can be generalised from the Cartesian differentiation discussed in §1.15. For example:

$$\frac{\partial\Phi}{\partial\mathbf{A}} = \frac{\partial\Phi}{\partial A_{ij}}\mathbf{g}_i\otimes\mathbf{g}_j = \frac{\partial\Phi}{\partial A_i^{\cdot j}}\mathbf{g}_i\otimes\mathbf{g}^j = \cdots$$
$$\frac{\partial\mathbf{B}}{\partial\mathbf{A}} = \frac{\partial B_{ij}}{\partial A_{mn}}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}_m\otimes\mathbf{g}_n = \cdots$$
$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial\mathbf{A}} &= \frac{\partial A_{ij}}{\partial A_{mn}}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}_m\otimes\mathbf{g}_n = \delta_i^m\delta_j^n\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}_m\otimes\mathbf{g}_n \\ &= \mathbf{g}^m\otimes\mathbf{g}^n\otimes\mathbf{g}_m\otimes\mathbf{g}_n\end{aligned}$$

1.18.5 Orthogonal Curvilinear Coordinates

This section is based on the groundwork carried out in §1.16.9. In orthogonal curvilinear systems, it is best to write all equations in terms of the covariant base vectors, or in terms of the corresponding physical components, using the identities (see Eqn. 1.16.45)

$$\mathbf{g}^i = \frac{1}{h_i^2}\mathbf{g}_i = \frac{1}{h_i}\hat{\mathbf{g}}_i \qquad \text{(no sum)} \tag{1.18.30}$$

The Gradient of a Scalar Field

From the definition 1.18.23 for the gradient of a scalar field, and Eqn. 1.18.30, one has for an orthogonal curvilinear coordinate system,

$$\begin{aligned}\nabla\Phi &= \frac{1}{h_1^2}\frac{\partial\Phi}{\partial\Theta^1}\mathbf{g}_1 + \frac{1}{h_2^2}\frac{\partial\Phi}{\partial\Theta^2}\mathbf{g}_2 + \frac{1}{h_3^2}\frac{\partial\Phi}{\partial\Theta^3}\mathbf{g}_3 \\ &= \frac{1}{h_1}\frac{\partial\Phi}{\partial\Theta^1}\hat{\mathbf{g}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial\Theta^2}\hat{\mathbf{g}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial\Theta^3}\hat{\mathbf{g}}_3\end{aligned} \tag{1.18.31}$$

The Christoffel Symbols

The Christoffel symbols simplify considerably in orthogonal coordinate systems. First, from the definition 1.18.4,

$$\Gamma_{ij}^k = \frac{1}{h_k^2}\frac{\partial\mathbf{g}_i}{\partial\Theta^j}\cdot\mathbf{g}_k \tag{1.18.32}$$

Note that the introduction of the scale factors h into this and the following equations disrupts the summation and index notation convention used hitherto. To remain consistent, one should use the metric coefficients and leave this equation in the form

$$\Gamma_{ij}^k = \frac{\partial\mathbf{g}_i}{\partial\Theta^j}\cdot g^{km}\mathbf{g}_m$$

Now

$$\frac{\partial}{\partial\Theta^j}\left(\mathbf{g}_i\cdot\mathbf{g}_i\right) = 2\left(\frac{\partial\mathbf{g}_i}{\partial\Theta^j}\cdot\mathbf{g}_i\right) = 2h_i^2\Gamma_{ij}^i$$

and $\mathbf{g}_i\cdot\mathbf{g}_i = h_i^2$ so, in terms of the derivatives of the scale factors,

$$\Gamma_{ij}^i = \Gamma_{ij}^k\Big|_{k=i} = \frac{1}{h_i}\frac{\partial h_i}{\partial\Theta^j} \qquad \text{(no sum)} \tag{1.18.33}$$

Similarly, it can be shown that {▲Problem 14}

$$h_k^2\Gamma_{ij}^k = -h_i^2\Gamma_{jk}^i = -h_j^2\Gamma_{ki}^j = h_i^2\Gamma_{jk}^i \text{ when } i\neq j\neq k \tag{1.18.34}$$

so that the Christoffel symbols are zero when the indices are distinct, so that there are only 21 non-zero symbols of the 27. Further, {▲Problem 15}

$$\Gamma_{ii}^{k} = \Gamma_{ij}^{k}\Big|_{i=j} = -\frac{h_i}{h_k^2}\frac{\partial h_i}{\partial \Theta^k}, \quad i \neq k \quad \text{(no sum)} \tag{1.18.35}$$

From the symmetry condition (see Eqn. 1.18.4), only 15 of the 21 non-zero symbols are distinct:

$$\begin{gathered}
\Gamma_{11}^{1}, \Gamma_{12}^{1} = \Gamma_{21}^{1}, \Gamma_{13}^{1} = \Gamma_{31}^{1}, \Gamma_{22}^{1}, \Gamma_{33}^{1} \\
\Gamma_{11}^{2}, \Gamma_{12}^{2} = \Gamma_{21}^{2}, \Gamma_{22}^{2}, \Gamma_{23}^{2} = \Gamma_{32}^{2}, \Gamma_{33}^{2} \\
\Gamma_{11}^{3}, \Gamma_{13}^{3} = \Gamma_{31}^{3}, \Gamma_{22}^{3}, \Gamma_{23}^{3} = \Gamma_{32}^{3}, \Gamma_{33}^{3}
\end{gathered}$$

Note also that these are related to each other through the relation between (1.18.33, 1.18.35), i.e.

$$\Gamma_{ii}^{k} = -\frac{h_i^2}{h_k^2}\Gamma_{ik}^{i}, \quad i \neq k \quad \text{(no sum)}$$

so that

$$\begin{aligned}
&\Gamma_{11}^{1}, \quad \Gamma_{22}^{2}, \quad \Gamma_{33}^{3} \\
&\Gamma_{12}^{1} = \Gamma_{21}^{1} = -\frac{h_2^2}{h_1^2}\Gamma_{11}^{2}, \quad \Gamma_{13}^{1} = \Gamma_{31}^{1} = -\frac{h_3^2}{h_1^2}\Gamma_{11}^{3}, \quad \Gamma_{12}^{2} = \Gamma_{21}^{2} = -\frac{h_1^2}{h_2^2}\Gamma_{22}^{1} \\
&\Gamma_{23}^{2} = \Gamma_{32}^{2} = -\frac{h_3^2}{h_2^2}\Gamma_{22}^{3}, \quad \Gamma_{13}^{3} = \Gamma_{31}^{3} = -\frac{h_1^2}{h_3^2}\Gamma_{33}^{1}, \quad \Gamma_{23}^{3} = \Gamma_{32}^{3} = -\frac{h_2^2}{h_3^2}\Gamma_{33}^{2}
\end{aligned} \tag{1.18.36}$$

The Gradient of a Vector

From the definition 1.18.25, the gradient of a vector is

$$\operatorname{grad}\mathbf{v} = v^i\Big|_j \mathbf{g}_i \otimes \mathbf{g}^j = \frac{1}{h_j^2} v^i\Big|_j \mathbf{g}_i \otimes \mathbf{g}_j \quad \text{(no sum over } h_j\text{)} \tag{1.18.37}$$

In terms of physical components,

$$\begin{aligned}
\operatorname{grad}\mathbf{v} &= \frac{1}{h_j^2}\left(\frac{\partial v^i}{\partial \Theta^j} + v^k \Gamma_{kj}^{i}\right)\mathbf{g}_i \otimes \mathbf{g}_j \\
&= \frac{1}{h_j}\left(\frac{\partial v^{\langle i\rangle}}{\partial \Theta^j} - \Gamma_{ij}^{i} v^{\langle i\rangle} + \frac{h_i}{h_k}\Gamma_{kj}^{i} v^{\langle k\rangle}\right)\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j
\end{aligned} \tag{1.18.38}$$

The Divergence of a Vector

From the definition 1.18.27, the divergence of a vector is $\operatorname{div}\mathbf{v} = v^i\big|_i$ or {▲Problem 16}

$$\operatorname{div}\mathbf{v} = \frac{\partial v_i}{\partial \Theta^i} + v_k \Gamma^i_{ki} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(v^{\langle 1\rangle} h_2 h_3\right)}{\partial \Theta^1} + \frac{\partial\left(v^{\langle 2\rangle} h_1 h_3\right)}{\partial \Theta^2} + \frac{\partial\left(v^{\langle 3\rangle} h_1 h_2\right)}{\partial \Theta^3}\right] \qquad (1.18.39)$$

The Curl of a Vector

From §1.16.10, the permutation symbol in orthogonal curvilinear coordinates reduceS to

$$e^{ijk} = \frac{1}{h_1 h_2 h_3}\varepsilon^{ijk} \qquad (1.18.40)$$

where $\varepsilon^{ijk} = \varepsilon_{ijk}$ is the Cartesian permutation symbol. From the definition 1.18.29, the curl of a vector is then

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= \frac{1}{h_1 h_2 h_3}\varepsilon_{ijk}\frac{\partial v_j}{\partial \Theta^i}\mathbf{g}_k = \frac{1}{h_1 h_2 h_3}\left\{\left[\frac{\partial v_2}{\partial \Theta^1} - \frac{\partial v_1}{\partial \Theta^2}\right]\mathbf{g}_3 + \cdots\right\} \\ &= \frac{1}{h_1 h_2 h_3}\left\{\left[\frac{\partial\left(v^2 h_2^2\right)}{\partial \Theta^1} - \frac{\partial\left(v^1 h_1^2\right)}{\partial \Theta^2}\right]\mathbf{g}_3 + \cdots\right\} \\ &= \frac{1}{h_1 h_2 h_3}\left\{\left[\frac{\partial\left(v^{\langle 2\rangle} h_2\right)}{\partial \Theta^1} - \frac{\partial\left(v^{\langle 1\rangle} h_1\right)}{\partial \Theta^2}\right] h_3 \hat{\mathbf{g}}_3 + \cdots\right\}\end{aligned} \qquad (1.18.41)$$

or

$$\operatorname{curl}\mathbf{v} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{g}}_1 & h_2\hat{\mathbf{g}}_2 & h_3\hat{\mathbf{g}}_3 \\ \dfrac{\partial}{\partial \Theta^1} & \dfrac{\partial}{\partial \Theta^2} & \dfrac{\partial}{\partial \Theta^3} \\ h_1 v^{\langle 1\rangle} & h_2 v^{\langle 2\rangle} & h_3 v^{\langle 3\rangle}\end{vmatrix} \qquad (1.18.42)$$

The Laplacian

From the above results, the Laplacian is given by

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \Theta^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial \Theta^1}\right) + \frac{\partial}{\partial \Theta^2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \Phi}{\partial \Theta^2}\right) + \frac{\partial}{\partial \Theta^3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial \Theta^3}\right)\right]$$

Divergence of a Tensor

From the definition 1.18.28, and using 1.16.59, 1.16.62 {▲Problem 17}

$$\begin{aligned}\operatorname{div}\mathbf{A} = A_{i}^{\cdot j}\,|_{j}\ \mathbf{g}^{i} &= \left\{\frac{\partial A_{i}^{\cdot j}}{\partial\Theta^{j}} + \Gamma_{mj}^{j}A_{i}^{\cdot m} - \Gamma_{ij}^{m}A_{m}^{\cdot j}\right\}\mathbf{g}^{i} \\ &= \left\{\frac{1}{h_i}\frac{\partial}{\partial\Theta^{j}}\left(\frac{h_i}{h_j}A^{\langle ij\rangle}\right) + \frac{1}{h_m}\Gamma_{mj}^{j}A^{\langle im\rangle} - \frac{h_m}{h_i h_j}\Gamma_{ij}^{m}A^{\langle mj\rangle}\right\}\hat{\mathbf{g}}_i\end{aligned} \qquad (1.18.43)$$

Examples

1. Cylindrical Coordinates

Gradient of a Scalar Field:

$$\nabla\Phi = \frac{\partial\Phi}{\partial\Theta^1}\hat{\mathbf{g}}_1 + \frac{1}{\Theta^2}\frac{\partial\Phi}{\partial\Theta_2}\hat{\mathbf{g}}_2 + \frac{\partial\Phi}{\partial\Theta^3}\hat{\mathbf{g}}_3$$

Christoffel symbols:

With $h_1 = 1, h_2 = \Theta^1, h_3 = 1$, there are two distinct non-zero symbols:

$$\Gamma_{22}^{1} = -\Theta^1$$

$$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{\Theta^1}$$

Derivatives of the base vectors:

The non-zero derivatives are

$$\frac{\partial\mathbf{g}_1}{\partial\Theta^2} = \frac{\partial\mathbf{g}_2}{\partial\Theta^1} = \frac{1}{\Theta^1}\mathbf{g}_2, \quad \frac{\partial\mathbf{g}_2}{\partial\Theta^2} = -\Theta^1\mathbf{g}_1$$

and in terms of physical components, the non-zero derivatives are

$$\frac{\partial\hat{\mathbf{g}}_1}{\partial\Theta^2} = \hat{\mathbf{g}}_2, \quad \frac{\partial\hat{\mathbf{g}}_2}{\partial\Theta^2} = -\hat{\mathbf{g}}_1$$

which agree with 1.6.32.

The Divergence (see 1.6.33), Curl (see 1.6.34) and Gradient {▲Problem 18} (see 1.14.18) of a vector:

$$\operatorname{div}\mathbf{v} = \left[\frac{\partial v_{\langle 1\rangle}}{\partial\Theta^1} + \frac{v_{\langle 1\rangle}}{\Theta^1} + \frac{1}{\Theta^1}\frac{\partial v_{\langle 2\rangle}}{\partial\Theta^2} + \frac{\partial v_{\langle 3\rangle}}{\partial\Theta^3}\right]$$

$$\operatorname{curl}\mathbf{v} = \frac{1}{\Theta^1}\begin{vmatrix}\hat{\mathbf{g}}_1 & \Theta^1\hat{\mathbf{g}}_2 & \hat{\mathbf{g}}_3 \\ \dfrac{\partial}{\partial\Theta^1} & \dfrac{\partial}{\partial\Theta^2} & \dfrac{\partial}{\partial\Theta^3} \\ v^{\langle 1\rangle} & \Theta^1 v^{\langle 2\rangle} & v^{\langle 3\rangle}\end{vmatrix}$$

$$\begin{aligned}\operatorname{grad}\mathbf{v} = & \frac{\partial v_{\langle 1\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_1 + \frac{\partial v_{\langle 2\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_1 + \frac{\partial v_{\langle 3\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_1 \\ & + \frac{1}{\Theta^1}\left(\frac{\partial v_{\langle 1\rangle}}{\partial\Theta^2} - v_{\langle 2\rangle}\right)\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_2 + \frac{1}{\Theta^1}\left(\frac{\partial v_{\langle 2\rangle}}{\partial\Theta^2} + v_{\langle 1\rangle}\right)\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_2 \\ & + \frac{1}{\Theta^1}\left(\frac{\partial v_{\langle 3\rangle}}{\partial\Theta^2}\right)\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_2 + \frac{\partial v_{\langle 1\rangle}}{\partial\Theta^3}\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_3 + \frac{\partial v_{\langle 2\rangle}}{\partial\Theta^3}\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_3 + \frac{\partial v_{\langle 3\rangle}}{\partial\Theta^3}\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_3\end{aligned}$$

The Divergence of a tensor {▲Problem 19} (see 1.14.19):

$$\begin{aligned}\operatorname{div}\mathbf{A} = & \left\{\frac{\partial A^{\langle 11\rangle}}{\partial\Theta^1} + \frac{1}{\Theta^1}\frac{\partial A^{\langle 12\rangle}}{\partial\Theta^2} + \frac{\partial A^{\langle 13\rangle}}{\partial\Theta^3} + \frac{A^{\langle 11\rangle} - A^{\langle 22\rangle}}{\Theta^1}\right\}\hat{\mathbf{g}}_1 \\ & + \left\{\frac{\partial A^{\langle 21\rangle}}{\partial\Theta^1} + \frac{1}{\Theta^1}\frac{\partial A^{\langle 22\rangle}}{\partial\Theta^2} + \frac{\partial A^{\langle 23\rangle}}{\partial\Theta^3} + \frac{A^{\langle 21\rangle} + A^{\langle 12\rangle}}{\Theta^1}\right\}\hat{\mathbf{g}}_2 \\ & + \left\{\frac{\partial A^{\langle 31\rangle}}{\partial\Theta^1} + \frac{A^{\langle 31\rangle}}{\Theta^1} + \frac{1}{\Theta^1}\frac{\partial A^{\langle 32\rangle}}{\partial\Theta^2} + \frac{\partial A^{\langle 33\rangle}}{\partial\Theta^3}\right\}\hat{\mathbf{g}}_3\end{aligned}$$

2. Spherical Coordinates

Gradient of a Scalar Field:

$$\nabla\Phi = \frac{\partial\Phi}{\partial\Theta^1}\hat{\mathbf{g}}_1 + \frac{1}{\Theta^1}\frac{\partial\Phi}{\partial\Theta^2}\hat{\mathbf{g}}_2 + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial\Phi}{\partial\Theta^3}\hat{\mathbf{g}}_3$$

Christoffel symbols:

With $h_1 = 1, h_2 = \Theta^1, h_3 = \Theta^1\sin\Theta^2$, there are six distinct non-zero symbols:

$$\Gamma^1_{22} = -\Theta^1, \Gamma^1_{33} = -\Theta^1\sin^2\Theta^2$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\Theta^1}, \Gamma^2_{33} = -\sin\Theta^2\cos\Theta^2$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{\Theta^1},\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\Theta^2$$

Derivatives of the base vectors:

The non-zero derivatives are

$$\frac{\partial\mathbf{g}_1}{\partial\Theta^2} = \frac{\partial\mathbf{g}_2}{\partial\Theta^1} = \frac{1}{\Theta^1}\mathbf{g}_2, \quad \frac{\partial\mathbf{g}_1}{\partial\Theta^3} = \frac{\partial\mathbf{g}_3}{\partial\Theta^1} = \frac{1}{\Theta^1}\mathbf{g}_3, \quad \frac{\partial\mathbf{g}_2}{\partial\Theta^2} = -\Theta^1\mathbf{g}_1$$

$$\frac{\partial\mathbf{g}_2}{\partial\Theta_3} = \frac{\partial\mathbf{g}_3}{\partial\Theta_2} = \cot\Theta^2\mathbf{g}_3, \quad \frac{\partial\mathbf{g}_3}{\partial\Theta_3} = -\Theta^1\sin^2\Theta^2\mathbf{g}_1 - \sin\Theta^2\cos\Theta^2\mathbf{g}_2$$

and in terms of physical components, the non-zero derivatives are

$$\frac{\partial\hat{\mathbf{g}}_1}{\partial\Theta^2} = \hat{\mathbf{g}}_2, \quad \frac{\partial\hat{\mathbf{g}}_1}{\partial\Theta^3} = \sin\Theta^2\hat{\mathbf{g}}_3, \quad \frac{\partial\hat{\mathbf{g}}_2}{\partial\Theta^2} = -\hat{\mathbf{g}}_1$$

$$\frac{\partial\hat{\mathbf{g}}_2}{\partial\Theta^3} = \cos\Theta^2\hat{\mathbf{g}}_3, \quad \frac{\partial\hat{\mathbf{g}}_3}{\partial\Theta^3} = -\sin\Theta^2\hat{\mathbf{g}}_1 - \cos\Theta^2\hat{\mathbf{g}}_2$$

which agree with 1.6.37.

The Divergence (see 1.6.38), Curl and Gradient of a Vector:

$$\operatorname{div}\mathbf{v} = \frac{1}{\left(\Theta^1\right)^2}\frac{\partial\left(\left(\Theta^1\right)^2 v_{\langle 1\rangle}\right)}{\partial\Theta^1} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial\left(\sin\Theta^2 v_{\langle 2\rangle}\right)}{\partial\Theta^2} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial v_{\langle 3\rangle}}{\partial\Theta^3}$$

$$\operatorname{curl}\mathbf{v} = \frac{1}{\left(\Theta^1\right)^2\sin\Theta^2}\begin{vmatrix} \hat{\mathbf{g}}_1 & \Theta^1\hat{\mathbf{g}}_2 & \Theta^1\sin\Theta^2\hat{\mathbf{g}}_3 \\ \dfrac{\partial}{\partial\Theta^1} & \dfrac{\partial}{\partial\Theta^2} & \dfrac{\partial}{\partial\Theta^3} \\ v^{\langle 1\rangle} & \Theta^1 v^{\langle 2\rangle} & \Theta^1\sin\Theta^2 v^{\langle 3\rangle}\end{vmatrix}$$

$$\begin{aligned}\operatorname{grad}\mathbf{v} = {} & \frac{\partial v_{\langle 1\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_1 + \left(\frac{1}{\Theta^1}\frac{\partial v_{\langle 1\rangle}}{\partial\Theta^2} - \frac{v_{\langle 2\rangle}}{\Theta^1}\right)\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_2 \\ & + \left(\frac{1}{\Theta^1\sin\Theta^2}\frac{\partial v_{\langle 1\rangle}}{\partial\Theta^3} - \frac{v_{\langle 3\rangle}}{\Theta^1}\right)\hat{\mathbf{g}}_1\otimes\hat{\mathbf{g}}_3 + \frac{\partial v_{\langle 2\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_1 \\ & + \left(\frac{1}{\Theta^1}\frac{\partial v_{\langle 2\rangle}}{\partial\Theta^2} + \frac{v_{\langle 1\rangle}}{\Theta^1}\right)\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_2 + \left(\frac{1}{\Theta^1\sin\Theta^2}\frac{\partial v_{\langle 2\rangle}}{\partial\Theta^3} - \cot\Theta^2\frac{v_{\langle 3\rangle}}{\Theta^1}\right)\hat{\mathbf{g}}_2\otimes\hat{\mathbf{g}}_3 \\ & + \frac{\partial v_{\langle 3\rangle}}{\partial\Theta^1}\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_1 + \frac{1}{\Theta^1}\frac{\partial v_{\langle 3\rangle}}{\partial\Theta^2}\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_2 \\ & + \left(\frac{1}{\Theta^1\sin\Theta^2}\frac{\partial v_{\langle 3\rangle}}{\partial\Theta^3} + \frac{v_{\langle 1\rangle}}{\Theta^1} + \cot\Theta^2\frac{v_{\langle 2\rangle}}{\Theta^1}\right)\hat{\mathbf{g}}_3\otimes\hat{\mathbf{g}}_3\end{aligned}$$

The Divergence of a tensor {▲Problem 20}

$$\begin{aligned}\operatorname{div}\mathbf{A} = {} & \left\{\frac{\partial A^{\langle 11\rangle}}{\partial\Theta^1} + \frac{1}{\Theta^1}\frac{\partial A^{\langle 12\rangle}}{\partial\Theta^2} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial A^{\langle 13\rangle}}{\partial\Theta^3} + \frac{2A^{\langle 11\rangle} + \cot\Theta^2 A^{\langle 12\rangle} - A^{\langle 22\rangle} - A^{\langle 33\rangle}}{\Theta^1}\right\}\hat{\mathbf{g}}_1 \\ & + \left\{\frac{\partial A^{\langle 21\rangle}}{\partial\Theta^1} + \frac{1}{\Theta^1}\frac{\partial A^{\langle 22\rangle}}{\partial\Theta^2} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial A^{\langle 23\rangle}}{\partial\Theta^3} + \frac{A^{\langle 12\rangle} + 2A^{\langle 21\rangle} + \cot\Theta^2\left(A^{\langle 22\rangle} - A^{\langle 33\rangle}\right)}{\Theta^1}\right\}\hat{\mathbf{g}}_2 \\ & + \left\{\frac{\partial A^{\langle 31\rangle}}{\partial\Theta^1} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial A^{\langle 32\rangle}}{\partial\Theta^2} + \frac{1}{\Theta^1\sin\Theta^2}\frac{\partial A^{\langle 33\rangle}}{\partial\Theta^3} + \frac{A^{\langle 13\rangle} + 2A^{\langle 31\rangle} + \cot\Theta^2\left(A^{\langle 23\rangle} + A^{\langle 32\rangle}\right)}{\Theta^1}\right\}\hat{\mathbf{g}}_3\end{aligned}$$

1.18.6 Problems

1 Show that the Christoffel symbol of the second kind is symmetric, i.e. $\Gamma_{ij}^k = \Gamma_{ji}^k$, and that it is explicitly given by $\Gamma_{ij}^k = \dfrac{\partial\mathbf{g}_i}{\partial\Theta^j}\cdot\mathbf{g}^k$.

2 Consider the scalar-valued function $\Phi = (\mathbf{Au})\cdot\mathbf{v} = A_{ij}u^i v^j$. By taking the gradient of this function, and using the relation for the covariant derivative of **A**, i.e. $A_{ij}|_k = A_{ij,k} - \Gamma_{ik}^m A_{mj} - \Gamma_{jk}^m A_{im}$, show that

$$\frac{\partial\left(A_{ij}u^i v^j\right)}{\partial\Theta^k} = \left(A_{ij}u^i v^j\right)|_k,$$

i.e. the partial derivative and covariant derivative are equivalent for a scalar-valued function.

3 Prove 1.18.9:

(i) $\dfrac{\partial g_{ij}}{\partial \Theta^k} = \Gamma_{ikj} + \Gamma_{jki}$, (ii) $\dfrac{\partial g^{ij}}{\partial \Theta^k} = -g^{im}\Gamma^j_{km} - g^{jm}\Gamma^i_{km}$

[Hint: for (ii), first differentiate Eqn. 1.16.10, $g^{ij} g_{kj} = \delta^i_k$.]

4 Derive 1.18.13, relating the Christoffel symbols to the partial derivatives of $\sqrt{g}$ and $\log\left(\sqrt{g}\right)$. [Hint: begin by using the chain rule $\dfrac{\partial g}{\partial \Theta^j} = \dfrac{\partial g}{\partial g_{mn}} \dfrac{\partial g_{mn}}{\partial \Theta^j}$.]

5 Use the definition of the covariant derivative of second order tensor components, Eqn. 1.18.18, to show that (i) $g_{ij}|_k = 0$ and (ii) $g^{ij}|_k = 0$.

6 Use the definition of the gradient of a vector, 1.18.25, to show that $\text{div}\mathbf{u} = \text{grad}\,\mathbf{u} : \mathbf{I} = u^i\,|_i$.

7 Derive the expression $\text{div}\mathbf{u} = \left(1/\sqrt{g}\right)\partial\left(\sqrt{g}u^i\right)/\partial\Theta^i$

8 Use 1.16.54 to show that $\mathbf{g}^k \times \left(\partial \mathbf{u}/\partial\Theta^k\right) = e^{ijk} u_j\,|_i\;\mathbf{g}_k$.

9 Use the relation $\varepsilon^{ijk}\varepsilon_{imn} = \delta^j_m\delta^k_n - \delta^k_m\delta^j_n$ (see Eqn. 1.3.19) to show that

$$\text{curl}(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} \mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\ \dfrac{\partial}{\partial\Theta^1} + \Gamma^k_{1k} & \dfrac{\partial}{\partial\Theta^2} + \Gamma^k_{2k} & \dfrac{\partial}{\partial\Theta^3} + \Gamma^k_{3k} \\ \left(u^2v^3 - u^3v^2\right) & -\left(u^1v^3 - u^3v^1\right) & \left(u^1v^2 - u^2v^1\right) \end{vmatrix}.$$

10 Show that (i) $u^i\,|_i = u_i\,|^i$, (ii) $e^{ijk}u_j\,|_i\;\mathbf{g}_k = e_{ijk}u^j\,|^i\;\mathbf{g}^k$

11 Show that

(i) $\text{grad}(\alpha\mathbf{v}) = \alpha\text{grad}\mathbf{v} + \mathbf{v}\otimes\text{grad}\alpha$, (ii) $\text{div}(\mathbf{v}\mathbf{A}) = \mathbf{v}\cdot\text{div}\mathbf{A} + \mathbf{A} : \text{grad}\mathbf{v}$

[Hint: you might want to use the relation $\mathbf{a}\mathbf{T}\cdot\mathbf{b} = \mathbf{T} : (\mathbf{a}\otimes\mathbf{b})$ for the second of these.]

12 Derive the relation $\partial(\text{tr}\mathbf{A})/\partial\mathbf{A} = \mathbf{I}$ in curvilinear coordinates.

13 Consider a (two dimensional) curvilinear coordinate system with covariant base vectors $\mathbf{g}_1 = \Theta^2\mathbf{e}_1 - 2\mathbf{e}_2$, $\mathbf{g}_2 = \Theta^1\mathbf{e}_1$.

(a) Evaluate the transformation equations $x^i = x^i(\Theta^j)$ and the Jacobian *J*.

(b) Evaluate the inverse transformation equations $\Theta^i = \Theta^i\left(x^j\right)$ and the contravariant base vectors $\mathbf{g}^i$.

(c) Evaluate the metric coefficients g_{ij}, g^{ij} and the function *g*:

(d) Evaluate the Christoffel symbols (only 2 are non-zero

(e) Consider the scalar field $\Phi = \Theta^1 + \Theta^2$. Evaluate $\text{grad}\Phi$.

(f) Consider the vector fields $\mathbf{u} = \mathbf{g}_1 + \Theta^2\mathbf{g}_2$, $\mathbf{v} = -\left(\Theta^1\right)^2\mathbf{g}_1 + 2\mathbf{g}_2$:

(i) Evaluate the covariant components of the vectors **u** and **v**

(ii) Evaluate $\text{div}\mathbf{u}$, $\text{div}\mathbf{v}$

(iii)Evaluate $\text{curl}\mathbf{u}$, $\text{curl}\mathbf{v}$

(iv)Evaluate $\text{grad}\mathbf{u}$, $\text{grad}\mathbf{v}$

(g) Verify the vector identities

$$\begin{aligned}\operatorname{div}(\Phi\mathbf{u}) &= \Phi\operatorname{div}\mathbf{u} + \operatorname{grad}\Phi\cdot\mathbf{u}\\ \operatorname{curl}(\Phi\mathbf{u}) &= \Phi\operatorname{curl}\mathbf{u} + \operatorname{grad}\Phi\times\mathbf{u}\\ \operatorname{div}(\mathbf{u}\times\mathbf{v}) &= \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}\\ \operatorname{curl}(\operatorname{grad}\Phi) &= \mathbf{o}\\ \operatorname{div}(\operatorname{curl}\mathbf{u}) &= 0\end{aligned}$$

(h) Verify the identities

$$\begin{aligned}\operatorname{grad}(\Phi\mathbf{v}) &= \Phi\operatorname{grad}\mathbf{v} + \mathbf{v}\otimes\operatorname{grad}\Phi\\ \operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= (\operatorname{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u}\\ \operatorname{div}(\mathbf{u}\otimes\mathbf{v}) &= (\operatorname{grad}\mathbf{u})\mathbf{v} + (\operatorname{div}\mathbf{v})\mathbf{u}\\ \operatorname{curl}(\mathbf{u}\times\mathbf{v}) &= \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} + (\operatorname{grad}\mathbf{u})\mathbf{v} - (\operatorname{grad}\mathbf{v})\mathbf{u}\end{aligned}$$

(i) Consider the tensor field

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -\Theta^2 \end{bmatrix}(\mathbf{g}^i \otimes \mathbf{g}^j)$$

Evaluate all contravariant and mixed components of the tensor **A**

14 Use the fact that $\mathbf{g}_k \cdot \mathbf{g}_i = 0$, $k \neq i$ to show that $h_k^2 \Gamma_{ij}^k = -h_i^2 \Gamma_{jk}^i{}_{ij}^{k}$. Then permutate the indices to show that $h_k^2 \Gamma_{ij}^k = -h_i^2 \Gamma_{jk}^i = -h_j^2 \Gamma_{ki}^j = h_i^2 \Gamma_{jk}^i$ when $i \neq j \neq k$.

15 Use the relation

$$\frac{\partial}{\partial \Theta^i}(\mathbf{g}_i \cdot \mathbf{g}_j) = 0, \quad i \neq j$$

to derive $\Gamma_{ii}^j = -\dfrac{h_i^2}{h_j^2}\Gamma_{ij}^i$.

16 Derive the expression 1.18.39 for the divergence of a vector field **v**.

17 Derive 1.18.43 for the divergence of a tensor in orthogonal coordinate systems.

18 Use the expression 1.18.38 to derive the expression for the gradient of a vector field in cylindrical coordinates.

19 Use the expression 1.18.43 to derive the expression for the divergence of a tensor field in cylindrical coordinates.

20 Use the expression 1.18.43 to derive the expression for the divergence of a tensor field in spherical coordinates.