1.15 Tensor Calculus 2: Tensor Functions

1.15.1 Vector-valued functions of a vector

Consider a vector-valued function of a vector

\[ \mathbf{a} = \mathbf{a}(\mathbf{b}), \quad a_i = a_i(b_j) \]

This is a function of three independent variables \( b_1, b_2, b_3 \), and there are nine partial derivatives \( \partial a_i / \partial b_j \). The partial derivative of the vector \( \mathbf{a} \) with respect to \( \mathbf{b} \) is defined to be a second-order tensor with these partial derivatives as its components:

\[ \frac{\partial \mathbf{a}(\mathbf{b})}{\partial b_j} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{(1.15.1)} \]

It follows from this that

\[ \frac{\partial \mathbf{a}}{\partial \mathbf{b}} = \left( \frac{\partial \mathbf{b}}{\partial \mathbf{a}} \right)^{-1} \quad \text{or} \quad \frac{\partial \mathbf{a}}{\partial \mathbf{b}} \frac{\partial \mathbf{b}}{\partial \mathbf{a}} = \mathbf{I}, \quad \frac{\partial a_i}{\partial b_m} \frac{\partial b_m}{\partial a_j} = \delta_{ij} \quad \text{(1.15.2)} \]

To show this, with \( a_i = a_i(b_j), b_i = b_i(a_j) \), note that the differential can be written as

\[ da_i = \frac{\partial a_i}{\partial b_j} db_j = \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} da_i = da_i \left( \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} \right) + da_2 \left( \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_2} \right) + da_3 \left( \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_3} \right) \]

Since \( da_1, da_2, da_3 \) are independent, one may set \( da_2 = da_3 = 0 \), so that

\[ \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} = 1 \]

Similarly, the terms inside the other brackets are zero and, in this way, one finds Eqn. 1.15.2.

1.15.2 Scalar-valued functions of a tensor

Consider a scalar valued function of a (second-order) tensor

\[ \phi = \phi(\mathbf{T}), \quad \mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]

This is a function of nine independent variables, \( \phi = \phi(T_{ij}) \), so there are nine different partial derivatives:
The partial derivative of $\phi$ with respect to $T$ is defined to be a second-order tensor with these partial derivatives as its components:

$$\frac{\partial \phi}{\partial T_{ij}} = e_i \otimes e_j$$

Partial Derivative with respect to a Tensor \hspace{1cm} (1.15.3)

The quantity $\partial \phi(T)/\partial T$ is also called the gradient of $\phi$ with respect to $T$.

Thus differentiation with respect to a second-order tensor raises the order by 2. This agrees with the idea of the gradient of a scalar field where differentiation with respect to a vector raises the order by 1.

### Derivatives of the Trace and Invariants

Consider now the trace: the derivative of $\text{tr} A$, with respect to $A$ can be evaluated as follows:

$$\frac{\partial}{\partial A} \text{tr} A = \frac{\partial A_{11}}{\partial A} + \frac{\partial A_{22}}{\partial A} + \frac{\partial A_{33}}{\partial A}$$

$$= \frac{\partial A_{11}}{\partial A} e_i \otimes e_j + \frac{\partial A_{22}}{\partial A} e_i \otimes e_j + \frac{\partial A_{33}}{\partial A} e_i \otimes e_j$$

$$= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

$$= I$$

Similarly, one finds that \hspace{1cm} \{ \text{Problem 1} \}

$$\frac{\partial (\text{tr} A)^2}{\partial A} = 2(\text{tr} A)I \hspace{1cm} \frac{\partial (\text{tr} A)^3}{\partial A} = 3(\text{tr} A)^2 I$$

Derivatives of Trace Functions \hspace{1cm} (1.15.5)

From these and 1.11.17, one can evaluate the derivatives of the invariants \hspace{1cm} \{ \text{Problem 2} \}:

$$\frac{\partial \text{I}_A}{\partial A} = I \hspace{1cm} \frac{\partial \text{II}_A}{\partial A} = I_A I - A^T$$

$$\frac{\partial \text{III}_A}{\partial A} = (A^T)^2 - I_A A^T + \text{II}_A I = \text{III}_A A^{-T}$$

Derivatives of the Invariants \hspace{1cm} (1.15.6)
Derivative of the Determinant

An important relation is

\( \frac{\partial}{\partial \mathbf{A}} (\det \mathbf{A}) = (\det \mathbf{A}) \mathbf{A}^{-T} \) \hspace{1cm} (1.15.7)

which follows directly from 1.15.6c.

Other Relations

The total differential can be written as

\( d\phi = \frac{\partial \phi}{\partial T_{11}} dT_{11} + \frac{\partial \phi}{\partial T_{12}} dT_{12} + \frac{\partial \phi}{\partial T_{13}} dT_{13} + \cdots \)

\( \equiv \frac{\partial \phi}{\partial \mathbf{T}} : d\mathbf{T} \) \hspace{1cm} (1.15.8)

This total differential gives an approximation to the total increment in \( \phi \) when the increments of the independent variables \( T_{11}, \cdots \) are small.

The second partial derivative is defined similarly:

\( \frac{\partial^2 \phi}{\partial T_{pq} \partial T_{rs}} = \frac{\partial^2 \phi}{\partial T_{pq}} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_p \otimes \mathbf{e}_q \), \hspace{1cm} (1.15.9)

the result being in this case a fourth-order tensor.

Consider a scalar-valued function of a tensor, \( \phi(\mathbf{A}) \), but now suppose that the components of \( \mathbf{A} \) depend upon some scalar parameter \( t \): \( \phi = \phi(\mathbf{A}(t)) \). By means of the chain rule of differentiation,

\( \dot{\phi} = \frac{\partial \phi}{\partial A_{ij}} \frac{dA_{ij}}{dt} \)

which in symbolic notation reads (see Eqn. 1.10.10e)

\( \frac{d\phi}{dt} = \frac{\partial \phi}{\partial \mathbf{A}} : \frac{d\mathbf{A}}{dt} = \text{tr} \left[ \left( \frac{\partial \phi}{\partial \mathbf{A}} \right)^T \frac{d\mathbf{A}}{dt} \right] \) \hspace{1cm} (1.15.11)

Identities for Scalar-valued functions of Symmetric Tensor Functions

Let \( \mathbf{C} \) be a symmetric tensor, \( \mathbf{C} = \mathbf{C}^T \). Then the partial derivative of \( \phi = \phi(\mathbf{C}(\mathbf{T})) \) with respect to \( \mathbf{T} \) can be written as \{ ▲ Problem 3 \}
(1) \[ \frac{\partial \phi}{\partial T} = 2T \frac{\partial \phi}{\partial C} \quad \text{for} \quad C = T^T T \]

(2) \[ \frac{\partial \phi}{\partial T} = 2 \frac{\partial \phi}{\partial C} C \quad \text{for} \quad C = T T^T \] (1.15.12)

(3) \[ \frac{\partial \phi}{\partial T} = 2T \frac{\partial \phi}{\partial C} = 2 \frac{\partial \phi}{\partial C} T = T \frac{\partial \phi}{\partial C} + \frac{\partial \phi}{\partial C} T \quad \text{for} \quad C = T T^T \quad \text{and symmetric} \ T \]

Scalar-valued functions of a Symmetric Tensor

Consider the expression

\[ B = \frac{\partial \phi(A)}{\partial A} \quad B_{ij} = \frac{\partial \phi(A_{ij})}{\partial A_{ij}} \] (1.15.13)

If \( A \) is a symmetric tensor, there are a number of ways to consider this expression: two possibilities are that \( \phi \) can be considered to be

(i) a symmetric function of the 9 variables \( A_{ij} \)

(ii) a function of 6 independent variables: \( \phi = \phi(A_{11}, A_{12}, A_{13}, A_{22}, A_{23}, A_{33}) \)

where

\[ A_{12} = \frac{1}{2} (A_{11} + A_{22}) = A_{12} = A_{21} \]
\[ A_{13} = \frac{1}{2} (A_{11} + A_{33}) = A_{13} = A_{31} \]
\[ A_{23} = \frac{1}{2} (A_{22} + A_{33}) = A_{23} = A_{32} \]

Looking at (i) and writing \( \phi = \phi(A_{11}, A_{12}, \overline{A}_{12}, \ldots, A_{21}, \overline{A}_{12}, \ldots) \), one has, for example,

\[ \frac{\partial \phi}{\partial A_{12}} = \frac{\partial \phi}{\partial A_{12}} \frac{\partial A_{12}}{\partial A_{12}} + \frac{\partial \phi}{\partial A_{21}} \frac{\partial A_{21}}{\partial A_{12}} = \frac{\partial \phi}{\partial A_{12}} + \frac{\partial \phi}{\partial A_{21}} = 2 \frac{\partial \phi}{\partial A_{12}} , \]

the last equality following from the fact that \( \phi \) is a symmetrical function of the \( A_{ij} \).

Thus, depending on how the scalar function is presented, one could write

(i) \[ B_{11} = \frac{\partial \phi}{\partial A_{11}} , \quad B_{12} = \frac{\partial \phi}{\partial A_{12}} , \quad B_{13} = \frac{\partial \phi}{\partial A_{13}} , \quad \text{etc.} \]

(ii) \[ B_{11} = \frac{\partial \phi}{\partial A_{11}} , \quad B_{12} = \frac{1}{2} \frac{\partial \phi}{\partial A_{12}} , \quad B_{13} = \frac{1}{2} \frac{\partial \phi}{\partial A_{13}} , \quad \text{etc.} \]
1.15.3 Tensor-valued functions of a tensor

The derivative of a (second-order) tensor $A$ with respect to another tensor $B$ is defined as

$$\frac{\partial A}{\partial B} = \frac{\partial A_{ij}}{\partial B_{pq}} e_i \otimes e_j \otimes e_p \otimes e_q$$

(1.15.14)

and forms therefore a fourth-order tensor. The total differential $dA$ can in this case be written as

$$dA = \frac{\partial A}{\partial B} : dB$$

(1.15.15)

Consider now

$$\frac{\partial A}{\partial A} = \frac{\partial A_{ij}}{\partial A_{kl}} e_i \otimes e_j \otimes e_k \otimes e_l$$

The components of the tensor are independent, so

$$\frac{\partial A_{11}}{\partial A_{11}} = 1, \quad \frac{\partial A_{12}}{\partial A_{12}} = 0, \quad \cdots \text{ etc.} \quad \frac{\partial A_{mn}}{\partial A_{pq}} = \delta_{mp} \delta_{nq}$$

(1.15.16)

and so

$$\frac{\partial A}{\partial A} = e_i \otimes e_j \otimes e_i \otimes e_j = I,$$

(1.15.17)

the fourth-order identity tensor of Eqn. 1.12.4.

Example

Consider the scalar-valued function $\phi$ of the tensor $A$ and vector $v$ (the “dot” can be omitted from the following and similar expressions),

$$\phi(A, v) = v \cdot Av$$

The gradient of $\phi$ with respect to $v$ is

$$\frac{\partial \phi}{\partial v} = \frac{\partial v}{\partial v} \cdot Av + v \cdot \frac{\partial A}{\partial v} v = Av + vA = (A + A^T)v$$

On the other hand, the gradient of $\phi$ with respect to $A$ is

$$\frac{\partial \phi}{\partial A} = v \cdot \frac{\partial A}{\partial A} v = v \cdot Iv = v \otimes v$$
Consider now the derivative of the inverse, $\frac{\partial A^{-1}}{\partial A}$. One can differentiate $A^{-1}A = 0$ using the product rule to arrive at

$$ \frac{\partial A^{-1}}{\partial A} A = -A^{-1} \frac{\partial A}{\partial A} $$

One needs to be careful with derivatives because of the position of the indices in (1.15.14); it looks like a post-operation of both sides with the inverse leads to

$$ \frac{\partial A^{-1}}{\partial A} = -A^{-1} \left( \frac{\partial A}{\partial A} \right) A^{-1} = -A^{-1} A^{-1} e_i \otimes e_j \otimes e_k \otimes e_l. $$

However, this is not correct (unless $A$ is symmetric). Using the index notation (there is no clear symbolic notation), one has

$$ \frac{\partial A^{-1}}{\partial A} A^{-1} = -A^{-1} \frac{\partial A}{\partial A} A^{-1} $$

Using the index notation (there is no clear symbolic notation), one has

$$ \frac{\partial A^{-1}}{\partial A} A^{-1} = -A^{-1} \frac{\partial A}{\partial A} A^{-1} $$

and

$$ \frac{\partial A^{-1}}{\partial A} A^{-1} = -A^{-1} \frac{\partial A}{\partial A} A^{-1} $$

1.15.4 The Directional Derivative

The directional derivative was introduced in §1.6.11. The ideas introduced there can be extended to tensors. For example, the directional derivative of the trace of a tensor $A$, in the direction of a tensor $T$, is

$$ \frac{\partial \text{tr} A[T]}{\partial T} = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \text{tr}(A + \varepsilon T) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\text{tr} A + \varepsilon \text{tr} T) = \text{tr} T $$

(1.15.19)

As a further example, consider the scalar function $\phi(A) = u \cdot Av$, where $u$ and $v$ are constant vectors. Then

$$ \frac{\partial \phi(A, u, v)}{\partial A} = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [u \cdot (A + \varepsilon T)v] = u \cdot Tv $$

(1.15.20)

Also, the gradient of $\phi$ with respect to $A$ is

$$ \frac{\partial \phi}{\partial A} = \frac{\partial}{\partial A} (u \cdot Av) = u \otimes v $$

(1.15.21)
and it can be seen that this is an example of the more general relation

\[ \partial_\lambda \phi[T] = \frac{\partial \phi}{\partial A} : T \]  

(1.15.22)

which is analogous to 1.6.41. Indeed,

\[ \partial_\lambda \phi[w] = \frac{\partial \phi}{\partial x} \cdot w \]

\[ \partial_\lambda \phi[T] = \frac{\partial \phi}{\partial A} : T \]  

(1.15.23)

\[ \partial_\lambda v[w] = \frac{\partial v}{\partial u} w \]

**Example (the Directional Derivative of the Determinant)**

It was shown in §1.6.11 that the directional derivative of the determinant of the $2 \times 2$ matrix $A$, in the direction of a second matrix $T$, is

\[ \partial_\lambda (\det A)[T] = A_{11}T_{22} + A_{22}T_{11} - A_{12}T_{21} - A_{21}T_{12} \]

This can be seen to be equal to $\det A\left(A^{-T} : T\right)$, which will now be proved more generally for tensors $A$ and $T$:

\[ \partial_\lambda (\det A)[T] = \frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} \det(A + \varepsilon T) \]

\[ = \frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} \det[A(I + \varepsilon A^{-T}T)] \]

\[ = \det A \frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} \det(I + \varepsilon A^{-T}T) \]

The last line here follows from (1.10.16). Now the characteristic equation for a tensor $B$ is given by (1.11.4, 1.11.5),

\[ (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0 = \det(B - \lambda I) \]

where $\lambda_i$ are the three eigenvalues of $B$. Thus, setting $\lambda = -1$ and $B = \varepsilon A^{-T}T$, 

\[
\partial_A (\det A[T]) = \det A \frac{d}{de} \left|_{e=0} \right. (1 + \lambda_1 A^{-T}) (1 + \lambda_2 A^{-T}) (1 + \lambda_3 A^{-T}) \\
= \det A \frac{d}{de} \left|_{e=0} \right. (1 + \varepsilon \lambda_1 A^{-T}) (1 + \varepsilon \lambda_2 A^{-T}) (1 + \varepsilon \lambda_3 A^{-T}) \\
= \det A \left( \lambda_1 A^{-T} + \lambda_2 A^{-T} + \lambda_3 A^{-T} \right) \\
= \det A \, \text{tr}(A^{-T}T)
\]

and, from (1.10.10e),

\[
\partial_A (\det A[T]) = \det A \left( A^{-T} : T \right) \tag{1.15.24}
\]

**Example (the Directional Derivative of a vector function)**

Consider the \( n \) homogeneous algebraic equations \( f(x) = 0 \):

\[
f_1(x_1, x_2, \cdots, x_n) = 0 \\
f_2(x_1, x_2, \cdots, x_n) = 0 \\
\vdots \\
f_n(x_1, x_2, \cdots, x_n) = 0
\]

The directional derivative of \( f \) in the direction of some vector \( u \) is

\[
\partial_x f(x)[u] = \frac{d}{de} \left|_{e=0} \right. f(z(e)) \quad (z = x + \varepsilon u) \\
= \left( \frac{\partial f(z)}{\partial z} \frac{dz}{de} \right) \left|_{e=0} \right. \\
= Ku
\]

where \( K \), called the tangent matrix of the system, is

\[
K = \frac{\partial f}{\partial x} = \left[ \begin{array}{cccc}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n}
\end{array} \right], \quad \partial_x f[u] = (\text{grad} f) u
\]

which can be compared to (1.15.23c).

**Properties of the Directional Derivative**

The directional derivative is a linear operator and so one can apply the usual product rule. For example, consider the directional derivative of \( A^{-1} \) in the direction of \( T \):
\[ \partial_A \left( A^{-1} \right) \hat{T} = \frac{d}{d\epsilon} \left( A + \epsilon \hat{T} \right)^{-1} \]

To evaluate this, note that \( \partial_A \left( A^{-1} A \right) \hat{T} = \partial_A \left( I \right) \hat{T} = 0 \), since \( I \) is independent of \( A \). The product rule then gives \( \partial_A \left( A^{-1} \right) \hat{T} A = -A^{-1} \partial_A \left( A \right) \hat{T} A^{-1} \), so that

\[ \partial_A \left( A^{-1} \right) \hat{T} = -A^{-1} \partial_A A \hat{T} A^{-1} = -A^{-1} T A^{-1} \quad \text{(1.15.26)} \]

Another important property of the directional derivative is the chain rule, which can be applied when the function is of the form \( f(x) = \hat{f}(B(x)) \). To derive this rule, consider (see §1.6.11)

\[ f(x + u) \approx f(x) + \partial_x f[u], \quad \text{(1.15.27)} \]

where terms of order \( o(u) \) have been neglected, i.e.

\[ \lim_{|u| \to 0} \frac{o(u)}{|u|} = 0. \]

The left-hand side of the previous expression can also be written as

\[ \hat{f}(B(x + u)) \approx \hat{f}(B(x) + \partial_x B[u]) \]

\[ \approx \hat{f}(B(x)) + \partial_u \hat{f}(B) \partial_x B[u] \]

Comparing these expressions, one arrives at the chain rule,

\[ \partial_x f[u] = \partial_u \hat{f}(B) \partial_x B[u] \quad \text{Chain Rule} \quad \text{(1.15.28)} \]

As an application of this rule, consider the directional derivative of \( \det A^{-1} \) in the direction \( \hat{T} \); here, \( f \) is \( \det A^{-1} \) and \( \hat{f} = \hat{f}(B(A)) \). Let \( B = A^{-1} \) and \( \hat{f} = \det B \). Then, from Eqns. 1.15.24, 1.15.25, 1.10.3h, \( f \),

\[ \partial_A \left( \det A^{-1} \right) \hat{T} = \partial_B \left( \det B \right) \partial_A A^{-1} \left[ \hat{T} \right] \]

\[ = \left( \det B \right) \left( B^{-T} : \left( -A^{-1} T A^{-1} \right) \right) \]

\[ = - \det A^{-1} \left( A^{-T} : \left( A^{-1} T A^{-1} \right) \right) \]

\[ = - \det A^{-1} \left( A^{-T} : T \right) \quad \text{(1.15.29)} \]

### 1.15.5 Formal Treatment of Tensor Calculus

Following on from §1.6.12 and §1.14.6, a scalar function \( f : V^2 \to R \) is differentiable at \( A \in V^2 \) if there exists a second order tensor \( Df(A) \in V^2 \) such that
\[ f(A + H) = f(A) + Df(A) : H + o(\|H\|) \quad \text{for all} \quad H \in V^2 \] (1.15.30)

In that case, the tensor \( Df(A) \) is called the derivative of \( f \) at \( A \). It follows from this that \( Df(A) \) is that tensor for which

\[ \partial_A f(B) = Df(A) : B = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(A + \varepsilon B) \quad \text{for all} \quad B \in V^2 \] (1.15.31)

For example, from 1.15.24,

\[ \partial_A (\det A[T]) = \det A (A^{-T} : T) = (\det A A^{-T}) : T \] (1.15.32)

from which it follows, from 1.15.31, that

\[ \frac{\partial}{\partial A} \det A = \det A A^{-T} \] (1.15.33)

which is 1.15.7.

Similarly, a tensor-valued function \( T : V^2 \rightarrow V^2 \) is differentiable at \( A \in V^2 \) if there exists a fourth order tensor \( DT(A) \in V^4 \) such that

\[ T(A + H) = T(A) + DT(A)H + o(\|H\|) \quad \text{for all} \quad H \in V^2 \] (1.15.34)

In that case, the tensor \( DT(A) \) is called the derivative of \( T \) at \( A \). It follows from this that \( DT(A) \) is that tensor for which

\[ \partial_A T[B] = DT(A) : B = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} T(A + \varepsilon B) \quad \text{for all} \quad B \in V^2 \] (1.15.35)

### 1.15.6 Problems

1. Evaluate the derivatives (use the chain rule for the last two of these)

\[ \frac{\partial (\text{tr}A^2)}{\partial A}, \quad \frac{\partial (\text{tr}A^3)}{\partial A}, \quad \frac{\partial ((\text{tr}A)^2)}{\partial A}, \quad \frac{\partial ((\text{tr}A)^3)}{\partial A} \]

2. Derive the derivatives of the invariants, Eqn. 1.15.5. [Hint: use the Cayley-Hamilton theorem, Eqn. 1.11.15, to express the derivative of the third invariant in terms of the third invariant.]

3. (a) Consider the scalar valued function \( \phi = \phi(C(F)) \), where \( C = F^T F \). Use the chain rule

\[ \frac{\partial \phi}{\partial F} = \frac{\partial \phi}{\partial C_{mn}} \frac{\partial C_{mn}}{\partial F_{ij}} e_i \otimes e_j \]

to show that
\[ \frac{\partial \phi}{\partial F} = 2F \frac{\partial \phi}{\partial C}, \quad \frac{\partial \phi}{\partial F_{ij}} = 2F_{ik} \frac{\partial \phi}{\partial C_{kj}} \]

(b) Show also that
\[ \frac{\partial \phi}{\partial U} = 2U \frac{\partial \phi}{\partial C} = 2 \frac{\partial \phi}{\partial C} U \]

for \( C = UU \) with \( U \) symmetric.

[Hint: for (a), use the index notation: first evaluate \( \partial C_{mn} / \partial F_{ij} \) using the product rule, then evaluate \( \partial \phi / \partial F_{ij} \) using the fact that \( C \) is symmetric.]

4. Show that
   
   (a) \[ \frac{\partial A^{-1}}{\partial A} : B = -A^{-1}BA^{-1}, \]
   
   (b) \[ \frac{\partial A^{-1}}{\partial A} : A \otimes A^{-1} = -A^{-1} \otimes A^{-1} \]

5. Show that
\[ \frac{\partial A^T}{\partial A} : B = B^T \]

6. By writing the norm of a tensor \(|A|\), 1.10.14, where \( A \) is symmetric, in terms of the trace (see 1.10.10), show that
\[ \frac{\partial |A|}{\partial A} = \frac{A}{|A|} \]

7. Evaluate
   
   (i) \[ \partial_A \left( A^2 \right)^T \]
   
   (ii) \[ \partial_A \left( \text{tr} A^2 \right) \] (see 1.10.10e)

8. Derive 1.15.29 by using the definition of the directional derivative and the relation 1.15.7, \( \partial (\det A) / \partial A = (\det A)A^{-T} \).