1.15 Tensor Calculus 2: Tensor Functions

1.15.1 Vector-valued functions of a vector

Consider a vector-valued function of a vector

\[ \mathbf{a} = \mathbf{a}(\mathbf{b}), \quad a_i = a_i(b_j) \]

This is a function of three independent variables \( b_1, b_2, b_3 \), and there are nine partial derivatives \( \partial a_i / \partial b_j \). The partial derivative of the vector \( \mathbf{a} \) with respect to \( \mathbf{b} \) is defined to be a second-order tensor with these partial derivatives as its components:

\[ \frac{\partial \mathbf{a}(\mathbf{b})}{\partial \mathbf{b}} = \frac{\partial a_i}{\partial b_j} \mathbf{e}_i \otimes \mathbf{e}_j \quad (1.15.1) \]

It follows from this that

\[ \frac{\partial \mathbf{a}}{\partial \mathbf{b}} = \left( \frac{\partial \mathbf{b}}{\partial \mathbf{a}} \right)^{-1} \quad \text{or} \quad \frac{\partial \mathbf{a}}{\partial \mathbf{b}} \frac{\partial \mathbf{b}}{\partial \mathbf{a}} = \mathbf{I}, \quad \frac{\partial a_i}{\partial b_m} \frac{\partial b_m}{\partial a_j} = \delta_{ij} \quad (1.15.2) \]

To show this, with \( a_i = a_i(b_j), b_i = b_j(a_i) \), note that the differential can be written as

\[ da_i = \frac{\partial a_i}{\partial b_j} db_j = \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} da_i = da_i \left( \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} \right) + da_j \left( \frac{\partial a_j}{\partial b_i} \frac{\partial b_i}{\partial a_j} \right) + da_3 \left( \frac{\partial a_3}{\partial b_i} \frac{\partial b_i}{\partial a_3} \right) \]

Since \( da_1, da_2, da_3 \) are independent, one may set \( da_2 = da_3 = 0 \), so that

\[ \frac{\partial a_i}{\partial b_j} \frac{\partial b_j}{\partial a_i} = 1 \]

Similarly, the terms inside the other brackets are zero and, in this way, one finds Eqn. 1.15.2.

1.15.2 Scalar-valued functions of a tensor

Consider a scalar valued function of a (second-order) tensor

\[ \phi = \phi(T), \quad T = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]

This is a function of nine independent variables, \( \phi = \phi(T_{ij}) \), so there are nine different partial derivatives:
The partial derivative of $\phi$ with respect to $T$ is defined to be a second-order tensor with these partial derivatives as its components:

$$\frac{\partial \phi}{\partial T} = e_i \otimes e_j$$  \hspace{1cm} \text{Partial Derivative with respect to a Tensor} \hspace{1cm} (1.15.3)$$

The quantity $\frac{\partial \phi(T)}{\partial T}$ is also called the gradient of $\phi$ with respect to $T$.

Thus differentiation with respect to a second-order tensor raises the order by 2. This agrees with the idea of the gradient of a scalar field where differentiation with respect to a vector raises the order by 1.

**Derivatives of the Trace and Invariants**

Consider now the trace: the derivative of $\text{tr}A$, with respect to $A$ can be evaluated as follows:

$$\frac{\partial}{\partial A} \text{tr}A = \frac{\partial A_{11}}{\partial A} + \frac{\partial A_{22}}{\partial A} + \frac{\partial A_{33}}{\partial A}$$

$$= \frac{\partial A_{11}}{\partial A} e_i \otimes e_j + \frac{\partial A_{22}}{\partial A} e_i \otimes e_j + \frac{\partial A_{33}}{\partial A} e_i \otimes e_j$$  \hspace{1cm} (1.15.4)$$

$$= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

$$= I$$

Similarly, one finds that \{▲Problem 1\}

$$\frac{\partial (\text{tr}A^2)}{\partial A} = 2(\text{tr}A)I$$  \hspace{1cm} (1.15.5)$$

$$\frac{\partial ((\text{tr}A)^3)}{\partial A} = 3((\text{tr}A)^2)I$$

**Derivatives of Trace Functions**

From these and 1.11.17, one can evaluate the derivatives of the invariants \{▲Problem 2\}:

$$\frac{\partial I_A}{\partial A} = I$$

$$\frac{\partial I_A^T}{\partial A} = I_A I - A^T$$

$$\frac{\partial I_A}{\partial A} = (A^T)^2 - I_A A^T + I_A I = I_A A^{-T}$$  \hspace{1cm} \text{Derivatives of the Invariants} \hspace{1cm} (1.15.6)$$
Derivative of the Determinant

An important relation is

\[
\frac{\partial}{\partial \mathbf{A}} (\det \mathbf{A}) = (\det \mathbf{A}) \mathbf{A}^{-T}
\]  

(1.15.7)

which follows directly from 1.15.6c.

Other Relations

The total differential can be written as

\[
d\phi = \frac{\partial \phi}{\partial T_{11}} dT_{11} + \frac{\partial \phi}{\partial T_{12}} dT_{12} + \frac{\partial \phi}{\partial T_{13}} dT_{13} + \cdots
\]

\[
\equiv \frac{\partial \phi}{\partial \mathbf{T}} : d\mathbf{T}
\]

(1.15.8)

This total differential gives an approximation to the total increment in \( \phi \) when the increments of the independent variables \( T_{ij}, \cdots \) are small.

The second partial derivative is defined similarly:

\[
\frac{\partial^2 \phi}{\partial \mathbf{T} \partial \mathbf{T}} \equiv \frac{\partial \phi}{\partial T_{ij}} \frac{\partial}{\partial T_{pq}} e_i \otimes e_j \otimes e_p \otimes e_q,
\]

(1.15.9)

the result being in this case a fourth-order tensor.

Consider a scalar-valued function of a tensor, \( \phi(\mathbf{A}) \), but now suppose that the components of \( \mathbf{A} \) depend upon some scalar parameter \( t \): \( \phi = \phi(\mathbf{A}(t)) \). By means of the chain rule of differentiation,

\[
\dot{\phi} = \frac{\partial \phi}{\partial A_{ij}} \frac{dA_{ij}}{dt}
\]

(1.15.10)

which in symbolic notation reads (see Eqn. 1.10.10e)

\[
\frac{d\phi}{dt} = \frac{\partial \phi}{\partial \mathbf{A}} : \frac{d\mathbf{A}}{dt} = \text{tr} \left[ \left( \frac{\partial \phi}{\partial \mathbf{A}} \right)^T \frac{d\mathbf{A}}{dt} \right]
\]

(1.15.11)

Identities for Scalar-valued functions of Symmetric Tensor Functions

Let \( \mathbf{C} \) be a symmetric tensor, \( \mathbf{C} = \mathbf{C}^T \). Then the partial derivative of \( \phi = \phi(\mathbf{C}(\mathbf{T})) \) with respect to \( \mathbf{T} \) can be written as {\( \blacktriangle \) Problem 3}
(1) \[ \frac{\partial \phi}{\partial T} = 2T \frac{\partial \phi}{\partial C} \quad \text{for } C = T^T T \]

(2) \[ \frac{\partial \phi}{\partial T} = 2 \frac{\partial \phi}{\partial T} C \quad \text{for } C = TT^T \quad (1.15.12) \]

(3) \[ \frac{\partial \phi}{\partial T} + \frac{\partial \phi}{\partial C} = \frac{\partial \phi}{\partial C} T + \frac{\partial \phi}{\partial C} T \quad \text{for } C = TT^T \text{ and symmetric } T \]

**Scalar-valued functions of a Symmetric Tensor**

Consider the expression

\[ B = \frac{\partial \phi(A)}{\partial A}, \quad B_y = \frac{\partial \phi(A_y)}{\partial A_y} \quad (1.15.13) \]

If \( A \) is a symmetric tensor, there are a number of ways to consider this expression: two possibilities are that \( \phi \) can be considered to be

(i) a symmetric function of the 9 variables \( A_y \)

(ii) a function of 6 independent variables: \( \phi = \phi(A_{11}, \bar{A}_{12}, \bar{A}_{13}, A_{22}, A_{23}, A_{33}) \)

where

\[ \bar{A}_{12} = \frac{1}{2} (A_{12} + A_{21}) = A_{12} = A_{21} \]
\[ \bar{A}_{13} = \frac{1}{2} (A_{13} + A_{31}) = A_{13} = A_{31} \]
\[ \bar{A}_{23} = \frac{1}{2} (A_{23} + A_{32}) = A_{23} = A_{32} \]

Looking at (i) and writing \( \phi = \phi(A_{11}, A_{12} (\bar{A}_{12}), \ldots, A_{21} (\bar{A}_{12}), \ldots) \), one has, for example,

\[ \frac{\partial \phi}{\partial A_{12}} = \frac{\partial \phi}{\partial A_{12}} \frac{\partial A_{12}}{\partial \bar{A}_{12}} + \frac{\partial \phi}{\partial A_{21}} \frac{\partial A_{21}}{\partial \bar{A}_{12}} = \frac{\partial \phi}{\partial A_{12}} + \frac{\partial \phi}{\partial A_{21}} = 2 \frac{\partial \phi}{\partial A_{12}}, \]

the last equality following from the fact that \( \phi \) is a symmetrical function of the \( A_y \).

Thus, depending on how the scalar function is presented, one could write

(i) \[ B_{11} = \frac{\partial \phi}{\partial A_{11}}, \quad B_{12} = \frac{\partial \phi}{\partial A_{12}}, \quad B_{13} = \frac{\partial \phi}{\partial A_{13}}, \quad \text{etc.} \]

(ii) \[ B_{11} = \frac{\partial \phi}{\partial A_{11}}, \quad B_{12} = \frac{1}{2} \frac{\partial \phi}{\partial A_{12}}, \quad B_{13} = \frac{1}{2} \frac{\partial \phi}{\partial A_{13}}, \quad \text{etc.} \]
1.15.3 Tensor-valued functions of a tensor

The derivative of a (second-order) tensor $A$ with respect to another tensor $B$ is defined as

$$\frac{\partial A}{\partial B} = \frac{\partial A_{pq}}{\partial B_{ij}} e_i \otimes e_j \otimes e_p \otimes e_q$$  \hspace{1cm} (1.15.14)$$

and forms therefore a fourth-order tensor. The total differential $dA$ can in this case be written as

$$dA = \frac{\partial A}{\partial B} : dB$$  \hspace{1cm} (1.15.15)$$

Consider now

$$\frac{\partial A}{\partial A} = \frac{\partial A_{ij}}{\partial A_{kl}} e_i \otimes e_j \otimes e_k \otimes e_l$$

The components of the tensor are independent, so

$$\frac{\partial A_{11}}{\partial A_{11}} = 1, \quad \frac{\partial A_{12}}{\partial A_{11}} = 0, \quad \cdots \quad \text{etc.} \quad \frac{\partial A_{mn}}{\partial A_{pq}} = \delta_{mp} \delta_{nq}$$  \hspace{1cm} (1.15.16)$$

and so

$$\frac{\partial A}{\partial A} = e_i \otimes e_j \otimes e_i \otimes e_j = I$$,  \hspace{1cm} (1.15.17)$$

the fourth-order identity tensor of Eqn. 1.12.4.

**Example**

Consider the scalar-valued function $\phi$ of the tensor $A$ and vector $v$ (the “dot” can be omitted from the following and similar expressions),

$$\phi(A, v) = v \cdot Av$$

The gradient of $\phi$ with respect to $v$ is

$$\frac{\partial \phi}{\partial v} = \frac{\partial v}{\partial v} \cdot Av + v \cdot \frac{\partial A}{\partial v} \cdot v = Av + vA = (A + A^T)v$$

On the other hand, the gradient of $\phi$ with respect to $A$ is

$$\frac{\partial \phi}{\partial A} = v \cdot \frac{\partial A}{\partial A} v = v \cdot I v = v \otimes v$$
Consider now the derivative of the inverse, $\frac{\partial A^{-1}}{\partial A}$. One can differentiate $A^{-1}A = 0$ using the product rule to arrive at

$$\frac{\partial A^{-1}}{\partial A} A = -A^{-1} \frac{\partial A}{\partial A}$$

One needs to be careful with derivatives because of the position of the indices in 1.15.14); it looks like a post-operation of both sides with the inverse leads to $\frac{\partial A^{-1}}{\partial A} = -A^{-1}(\partial A / \partial A)A^{-1} = -A^{-1}A^{-1}e_i \otimes e_j \otimes e_k \otimes e_l$. However, this is not correct (unless $A$ is symmetric). Using the index notation (there is no clear symbolic notation), one has

$$\frac{\partial A^{-1}}{\partial A} A_{ij} = -A^{-1} \frac{\partial A_{ij}}{\partial A} \left( e_i \otimes e_j \otimes e_k \otimes e_l \right)$$

1.15.4 The Directional Derivative

The directional derivative was introduced in §1.6.11. The ideas introduced there can be extended to tensors. For example, the directional derivative of the trace of a tensor $A$, in the direction of a tensor $T$, is

$$\frac{\partial}{\partial A} \left( \text{tr}A[T] = \frac{d}{d\varepsilon} \right)_{\varepsilon=0} \text{tr}(A + \varepsilon T) = \frac{d}{d\varepsilon} \right)_{\varepsilon=0} (\text{tr}A + \varepsilon \text{tr}T) = \text{tr}T$$

As a further example, consider the scalar function $\phi(A) = u \cdot Av$, where $u$ and $v$ are constant vectors. Then

$$\frac{\partial}{\partial A} \phi(A, u, v) = \frac{d}{d\varepsilon} \right)_{\varepsilon=0} [u \cdot (A + \varepsilon T)v] = u \cdot Tv$$

Also, the gradient of $\phi$ with respect to $A$ is

$$\frac{\partial}{\partial A} (u \cdot Av) = u \otimes v$$
and it can be seen that this is an example of the more general relation

$$\partial_A \phi[T] = \frac{\partial \phi}{\partial A} : T$$

(1.15.22)

which is analogous to 1.6.41. Indeed,

$$\partial_A \phi[w] = \frac{\partial \phi}{\partial x} \cdot w$$

$$\partial_A \phi[T] = \frac{\partial \phi}{\partial A} : T$$

(1.15.23)

$$\partial_A \phi[w] = \frac{\partial v}{\partial u} w$$

**Example (the Directional Derivative of the Determinant)**

It was shown in §1.6.11 that the directional derivative of the determinant of the $2 \times 2$ matrix $A$, in the direction of a second matrix $T$, is

$$\partial_A (\det A)[T] = A_{11}T_{22} + A_{22}T_{11} - A_{12}T_{21} - A_{21}T_{12}$$

This can be seen to be equal to $\det A (A^{-T} : T)$, which will now be proved more generally for tensors $A$ and $T$:

$$\partial_A (\det A)[T] = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \det(A + \varepsilon T)$$

$$= \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \det\left[A(I + \varepsilon A^{-1}T)\right]$$

$$= \det A \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \det(I + \varepsilon A^{-1}T)$$

The last line here follows from (1.10.16a). Now the characteristic equation for a tensor $B$ is given by (1.11.4, 1.11.5),

$$(\lambda_i - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0 = \det(B - \lambda I)$$

where $\lambda_i$ are the three eigenvalues of $B$. Thus, setting $\lambda = -1$ and $B = \varepsilon A^{-1}T$, ...
\[ \partial_A \left( \det A \right) \left[ T \right] = \det A \left( \frac{d}{d\epsilon} \left. \left( 1 + \lambda_1 \right) \left( 1 + \lambda_2 \right) \left( 1 + \lambda_3 \right) \right|_{A^{-T}} \right) \]

\[ = \det A \left( \frac{d}{d\epsilon} \left. \left( 1 + \epsilon \lambda_1 \right) \left( 1 + \epsilon \lambda_2 \right) \left( 1 + \epsilon \lambda_3 \right) \right|_{A^{-T}} \right) \]

\[ = \det A \left( \lambda_1 \left|_{A^{-T}} + \lambda_2 \left|_{A^{-T}} + \lambda_3 \left|_{A^{-T}} \right. \right. \right) \]

\[ = \det A \operatorname{tr}\left( A^{-1} T \right) \]

and, from (1.10.10e),

\[ \partial_A \left( \det A \right) \left[ T \right] = \det A \left( A^{-T} : T \right) \quad (1.15.24) \]

Example (the Directional Derivative of a vector function)

Consider the \( n \) homogeneous algebraic equations \( f(x) = 0 \):

\[ f_1(x_1, x_2, \cdots, x_n) = 0 \]
\[ f_2(x_1, x_2, \cdots, x_n) = 0 \]
\[ \vdots \]
\[ f_n(x_1, x_2, \cdots, x_n) = 0 \]

The directional derivative of \( f \) in the direction of some vector \( u \) is

\[ \partial_x f(x)[u] = \frac{d}{d\epsilon} \left. f(z(\epsilon)) \right|_{\epsilon=0} \quad (z = x + \epsilon u) \]

\[ = \left( \frac{\partial f(z)}{\partial z} \frac{dz}{d\epsilon} \right)_{\epsilon=0} \]

\[ = Ku \quad (1.15.25) \]

where \( K \), called the tangent matrix of the system, is

\[ K = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad \partial_x f[u] = \left( \operatorname{grad} f \right) u \]

which can be compared to (1.15.23c).

Properties of the Directional Derivative

The directional derivative is a linear operator and so one can apply the usual product rule. For example, consider the directional derivative of \( A^{-1} \) in the direction of \( T \):
\[ \partial_A (A^{-1}T) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} (A + \varepsilon T)^{-1} \]

To evaluate this, note that \( \partial_A (A^{-1}A[T]) = \partial_A (I[T]) = 0 \), since \( I \) is independent of \( A \). The product rule then gives \( \partial_A (A^{-1}A[T])A = -A^{-1} \partial_A (A[T]) \), so that

\[ \partial_A (A^{-1}T) = -A^{-1} \partial_A A[T]A^{-1} = -A^{-1}TA^{-1} \quad (1.15.26) \]

Another important property of the directional derivative is the **chain rule**, which can be applied when the function is of the form \( f(x) = \hat{f}(B(x)) \). To derive this rule, consider (see §1.6.11)

\[ f(x + u) \approx f(x) + \partial_x f[u], \quad (1.15.27) \]

where terms of order \( o(u) \) have been neglected, i.e.

\[ \lim_{|u| \to 0} \frac{o(u)}{|u|} = 0. \]

The left-hand side of the previous expression can also be written as

\[ \hat{f}(B(x + u)) \approx \hat{f}(B(x) + \partial_x B[u]) \]
\[ \approx \hat{f}(B(x)) + \partial_B \hat{f}(B) \partial_x B[u] \]

Comparing these expressions, one arrives at the chain rule,

\[ \partial_x f[u] = \partial_B \hat{f}(B) \partial_x B[u] \quad \text{Chain Rule} \quad (1.15.28) \]

As an application of this rule, consider the directional derivative of \( \det A^{-1} \) in the direction \( T \); here, \( f \) is \( \det A^{-1} \) and \( \hat{f} = \hat{f}(B(A)) \). Let \( B = A^{-1} \) and \( \hat{f} = \det B \). Then, from Eqns. 1.15.24, 1.15.25, 1.10.3h, \( f \),

\[ \partial_A (\det A^{-1}T) = \partial_B (\det B) \partial_A A^{-1}[T] \]
\[ = (\det B)(B^T : (-A^{-1}TA^{-1})) \]
\[ = -\det A^{-1}(A^T : (A^{-1}TA^{-1})) \]
\[ = -\det A^{-1}(A^{-T} : T) \quad (1.15.29) \]

### 1.15.5 Formal Treatment of Tensor Calculus

Following on from §1.6.12 and §1.14.6, a scalar function \( f : V^2 \to R \) is **differentiable** at \( A \in V^2 \) if there exists a second order tensor \( Df(A) \in V^2 \) such that
\[ f(A + H) = f(A) + Df(A): H + o(\|H\|) \quad \text{for all} \quad H \in V^2 \]  \hspace{1cm} (1.15.30)

In that case, the tensor \( Df(A) \) is called the \textbf{derivative} of \( f \) at \( A \). It follows from this that \( Df(A) \) is that tensor for which

\[ \partial_A f[B] = Df(A): B = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} f(A + \varepsilon B) \quad \text{for all} \quad B \in V^2 \]  \hspace{1cm} (1.15.31)

For example, from 1.15.24,

\[ \partial_A(\det A[T]) = \det A(A^{-T} : T) = (\det A A^{-T}) : T \]  \hspace{1cm} (1.15.32)

from which it follows, from 1.15.31, that

\[ \frac{\partial}{\partial A} \det A = \det A A^{-T} \]  \hspace{1cm} (1.15.33)

which is 1.15.7.

Similarly, a tensor-valued function \( T : V^2 \rightarrow V^2 \) is \textbf{differentiable} at \( A \in V^2 \) if there exists a fourth order tensor \( DT(A) \in V^4 \) such that

\[ T(A + H) = T(A) + DT(A)H + o(\|H\|) \quad \text{for all} \quad H \in V^2 \]  \hspace{1cm} (1.15.34)

In that case, the tensor \( DT(A) \) is called the \textbf{derivative} of \( T \) at \( A \). It follows from this that \( DT(A) \) is that tensor for which

\[ \partial_A T[B] = DT(A): B = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} T(A + \varepsilon B) \quad \text{for all} \quad B \in V^2 \]  \hspace{1cm} (1.15.35)

1.15.6 \textbf{Problems}

1. Evaluate the derivatives (use the chain rule for the last two of these)

\[ \frac{\partial(\text{tr}A^2)}{\partial A} \quad \frac{\partial(\text{tr}A^3)}{\partial A} \quad \frac{\partial((\text{tr}A)^2)}{\partial A} \quad \frac{\partial((\text{tr}A)^2)}{\partial A} \]

2. Derive the derivatives of the invariants, Eqn. 1.15.5. [Hint: use the Cayley-Hamilton theorem, Eqn. 1.11.15, to express the derivative of the third invariant in terms of the third invariant.]

3. (a) Consider the scalar valued function \( \phi = \phi(C(F)) \), where \( C = F^T F \). Use the chain rule

\[ \frac{\partial \phi}{\partial F} = \frac{\partial \phi}{\partial C_{mn}} \frac{\partial C_{mn}}{\partial F_{ij}} e_i \otimes e_j \]

to show that
\[ \frac{\partial \phi}{\partial F} = 2F \frac{\partial \phi}{\partial C}, \quad \frac{\partial \phi}{\partial F_{ij}} = 2F_{ik} \frac{\partial \phi}{\partial C_{kj}} \]

(b) Show also that
\[ \frac{\partial \phi}{\partial U} = 2U \frac{\partial \phi}{\partial C} = 2 \frac{\partial \phi}{\partial C} U \]

for \( C = UU \) with \( U \) symmetric.

[Hint: for (a), use the index notation: first evaluate \( \partial C_{mn} / \partial F_{ij} \) using the product rule, then evaluate \( \partial \phi / \partial F_{ij} \) using the fact that \( C \) is symmetric.]

4. Show that
   (a) \( \frac{\partial A^{-1}}{\partial A} : B = -A^{-1}BA^{-1} \),   
   (b) \( \frac{\partial A^{-1}}{\partial A} : A \otimes A^{-1} = -A^{-1} \otimes A^{-1} \)

5. Show that
\[ \frac{\partial A}{\partial A} : B = B^T \]

6. By writing the norm of a tensor \( |A| \), 1.10.14, where \( A \) is symmetric, in terms of the trace (see 1.10.10), show that
\[ \frac{\partial |A|}{\partial A} = A |A| \]

7. Evaluate
   (i) \( \partial \lambda \left( A^2 \right)^T \]
   (ii) \( \partial \lambda \left( \text{tr} A^2 \right) \) (see 1.10.10e)

8. Derive 1.15.29 by using the definition of the directional derivative and the relation 1.15.7, \( \partial (\text{det} A) / \partial A = (\text{det} A) A^{-T} \).