1.8 Tensors

Here the concept of the **tensor** is introduced. Tensors can be of different **orders** – zeroth-order tensors, first-order tensors, second-order tensors, and so on. Apart from the zeroth and first order tensors (see below), the second-order tensors are the most important tensors from a practical point of view, being important quantities in, amongst other topics, continuum mechanics, relativity, electromagnetism and quantum theory.

1.8.1 Zeroth and First Order Tensors

A **tensor of order zero** is simply another name for a scalar α.

A **first-order tensor** is simply another name for a vector **u**.

1.8.2 Second Order Tensors

Notation

Vectors:	lowercase bold-face Latin letters, e.g. **a**, **r**, **q**
2^{nd} order Tensors:	uppercase bold-face Latin letters, e.g. **F**, **T**, **S**

Tensors as Linear Operators

A *second*-order tensor **T** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or[1]

$$\boxed{\mathbf{T} \cdot \mathbf{u} = \mathbf{v} \quad \text{or} \quad \mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (1.8.1)$$

The second-order tensor **T** is a **linear operator** (or **linear transformation**)[2], which means that

$$\begin{aligned} \mathbf{T}(\mathbf{a} + \mathbf{b}) &= \mathbf{Ta} + \mathbf{Tb} && \ldots \text{ distributive} \\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

This linearity can be viewed geometrically as in Fig. 1.8.1.

Note: the vector may also be defined in this way, as a mapping **u** that acts on a vector **v**, this time generating a scalar α, $\mathbf{u} \cdot \mathbf{v} = \alpha$. This transformation (the dot product) is linear (see properties (2,3) in §1.1.4). Thus a first-order tensor (vector) maps a first-order tensor into a zeroth-order tensor (scalar), whereas a second-order tensor maps a first-order tensor into a first-order tensor. It will be seen that a third-order tensor maps a first-order tensor into a second-order tensor, and so on.

[1] both these notations for the tensor operation are used; here, the convention of omitting the "dot" will be used

[2] An operator or transformation is a special function which maps elements of one type into elements of a similar type; here, vectors into vectors

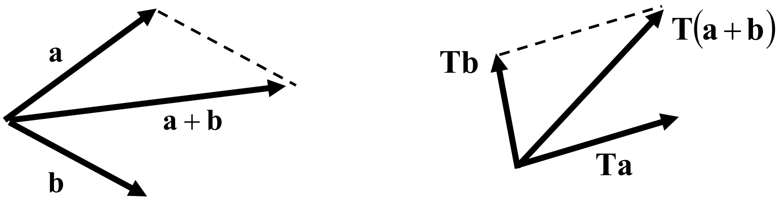

Figure 1.8.1: Linearity of the second order tensor

Further, two tensors **T** and **S** are said to be equal if and only if

$$\mathbf{Sv} = \mathbf{Tv}$$

for all vectors **v**.

Example (of a Tensor)

Suppose that **F** is an operator which transforms every vector into its mirror-image with respect to a given plane, Fig. 1.8.2. **F** transforms a vector into another vector and the transformation is linear, as can be seen geometrically from the figure. Thus **F** is a second-order tensor.

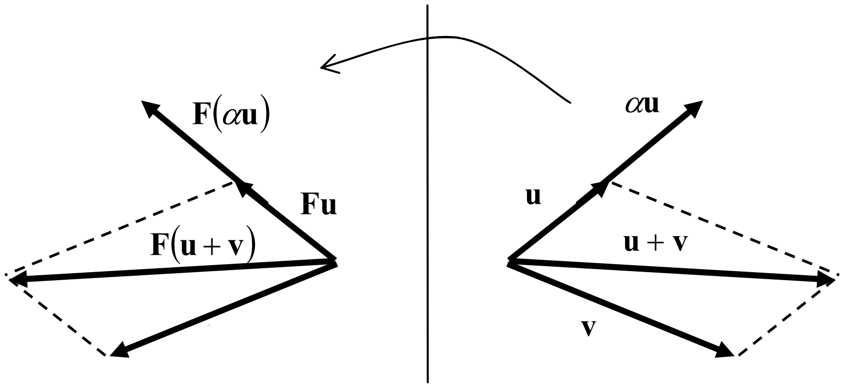

Figure 1.8.2: Mirror-imaging of vectors as a second order tensor mapping

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Example (of a Tensor)

The combination $\mathbf{u}\times$ linearly transforms a vector into another vector and is thus a second-order tensor[3]. For example, consider a force **f** applied to a spanner at a distance **r** from the centre of the nut, Fig. 1.8.3. Then it can be said that the tensor $(\mathbf{r}\times)$ maps the force **f** into the (moment/torque) vector $\mathbf{r}\times\mathbf{f}$.

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[3] Some authors use the notation $\tilde{\mathbf{u}}$ to denote $\mathbf{u}\times$

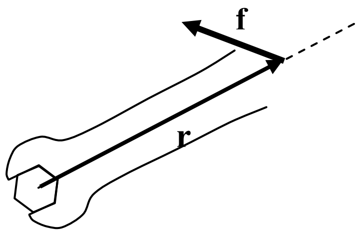

Figure 1.8.3: the force on a spanner

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1.8.3 The Dyad (the tensor product)

The vector *dot product* and vector *cross product* have been considered in previous sections. A third vector product, the **tensor product** (or **dyadic product**), is important in the analysis of tensors of order 2 or more. The tensor product of two vectors **u** and **v** is written as[4]

$$\boxed{\mathbf{u}\otimes\mathbf{v}} \qquad \textbf{Tensor Product} \qquad (1.8.2)$$

This tensor product is itself a tensor of order two, and is called **dyad**:

$\mathbf{u}\cdot\mathbf{v}$	is a scalar	(a zeroth order tensor)
$\mathbf{u}\times\mathbf{v}$	is a vector	(a first order tensor)
$\mathbf{u}\otimes\mathbf{v}$	is a dyad	(a second order tensor)

It is best to *define* this dyad by what it *does*: it transforms a vector **w** into another vector with the direction of **u** according to the rule[5]

$$\boxed{(\mathbf{u}\otimes\mathbf{v})\mathbf{w}=\mathbf{u}(\mathbf{v}\cdot\mathbf{w})} \qquad \textbf{The Dyad Transformation} \qquad (1.8.3)$$

This relation defines the symbol "$\otimes$".

The length of the new vector is $|\mathbf{u}|$ times $\mathbf{v}\cdot\mathbf{w}$, and the new vector has the same direction as $\mathbf{u}$, Fig. 1.8.4. It can be seen that the dyad is a second order tensor, because it operates linearly on a vector to give another vector {▲Problem 2}.

Note that the dyad is not commutative, $\mathbf{u}\otimes\mathbf{v}\neq\mathbf{v}\otimes\mathbf{u}$. Indeed it can be seen clearly from the figure that $(\mathbf{u}\otimes\mathbf{v})\mathbf{w}\neq(\mathbf{v}\otimes\mathbf{u})\mathbf{w}$.

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[4] many authors omit the $\otimes$ and write simply **uv**

[5] note that it is the two vectors that are beside each other (separated by a bracket) that get "dotted" together

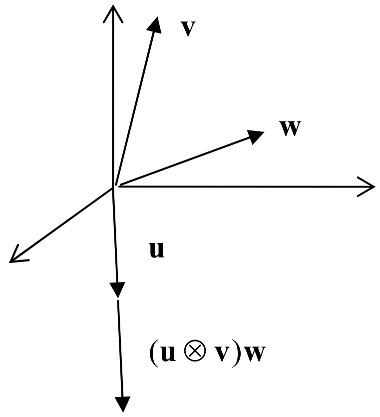

Figure 1.8.4: the dyad transformation

The following important relations follow from the above definition {▲Problem 4},

$$\begin{aligned}(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x}) &= (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x}) \\ \mathbf{u}(\mathbf{v}\otimes\mathbf{w}) &= (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\end{aligned} \tag{1.8.4}$$

It can be seen from these that the operation of the dyad on a vector is not commutative:

$$\mathbf{u}(\mathbf{v}\otimes\mathbf{w}) \neq (\mathbf{v}\otimes\mathbf{w})\mathbf{u} \tag{1.8.5}$$

Example (The Projection Tensor)

Consider the dyad $\mathbf{e}\otimes\mathbf{e}$. From the definition 1.8.3, $(\mathbf{e}\otimes\mathbf{e})\mathbf{u} = (\mathbf{e}\cdot\mathbf{u})\mathbf{e}$. But $\mathbf{e}\cdot\mathbf{u}$ is the projection of **u** onto a line through the unit vector **e**. Thus $(\mathbf{e}\cdot\mathbf{u})\mathbf{e}$ is the vector projection of **u** on **e**. For this reason $\mathbf{e}\otimes\mathbf{e}$ is called the **projection tensor**. It is usually denoted by **P**.

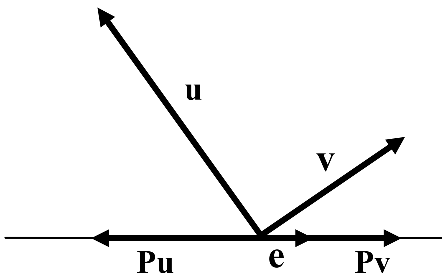

Figure 1.8.5: the projection tensor

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1.8.4 Dyadics

A **dyadic** is a linear combination of these dyads (with scalar coefficients). An example might be

$$5(\mathbf{a}\otimes\mathbf{b})+3(\mathbf{c}\otimes\mathbf{d})-2(\mathbf{e}\otimes\mathbf{f})$$

This is clearly a second-order tensor. It will be seen in §1.9 that *every second-order tensor can be represented by a* dyadic, that is

$$\mathbf{T}=\alpha(\mathbf{a}\otimes\mathbf{b})+\beta(\mathbf{c}\otimes\mathbf{d})+\gamma(\mathbf{e}\otimes\mathbf{f})+\cdots \tag{1.8.6}$$

Note: second-order tensors cannot, in general, be written as a dyad, $\mathbf{T}=\mathbf{a}\otimes\mathbf{b}$ – when they can, they are called **simple tensors**.

Example (Angular Momentum and the Moment of Inertia Tensor)

Suppose a rigid body is rotating so that every particle in the body is instantaneously moving in a circle about some axis fixed in space, Fig. 1.8.6.

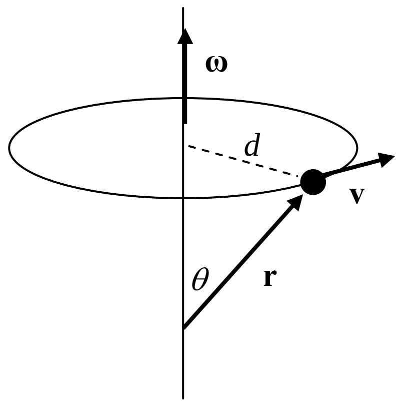

Figure 1.8.6: a particle in motion about an axis

The body's angular velocity $\boldsymbol{\omega}$ is defined as the vector whose magnitude is the angular speed ω and whose direction is along the axis of rotation. Then a particle's linear velocity is

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$$

where $v=wd$ is the linear speed, d is the distance between the axis and the particle, and $\mathbf{r}$ is the position vector of the particle from a fixed point O on the axis. The particle's **angular momentum** (or moment of momentum) **h** about the point O is defined to be

$$\mathbf{h}=m\mathbf{r}\times\mathbf{v}$$

where m is the mass of the particle. The angular momentum can be written as

$$\mathbf{h} = \hat{\mathbf{I}}\boldsymbol{\omega} \tag{1.8.8}$$

where $\hat{\mathbf{I}}$, a second-order tensor, is the **moment of inertia** of the particle about the point O, given by

$$\hat{\mathbf{I}} = m\left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right) \tag{1.8.9}$$

where $\mathbf{I}$ is the identity tensor, i.e. $\mathbf{Ia} = \mathbf{a}$ for all vectors $\mathbf{a}$.

To show this, it must be shown that $\mathbf{r} \times \mathbf{v} = \left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right)\boldsymbol{\omega}$. First examine $\mathbf{r} \times \mathbf{v}$. It is evidently a vector perpendicular to both $\mathbf{r}$ and $\mathbf{v}$ and in the plane of $\mathbf{r}$ and $\boldsymbol{\omega}$; its magnitude is

$$|\mathbf{r} \times \mathbf{v}| = |\mathbf{r}||\mathbf{v}| = |\mathbf{r}|^2 |\boldsymbol{\omega}| \sin\theta$$

Now (see Fig. 1.8.7)

$$\begin{aligned}\left(|\mathbf{r}|^2 \mathbf{I} - \mathbf{r} \otimes \mathbf{r}\right)\boldsymbol{\omega} &= |\mathbf{r}|^2 \boldsymbol{\omega} - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}) \\ &= |\mathbf{r}|^2 |\boldsymbol{\omega}| \left(\mathbf{e}_\omega - \cos\theta \mathbf{e}_r\right)\end{aligned}$$

where $\mathbf{e}_\omega$ and $\mathbf{e}_r$ are unit vectors in the directions of $\boldsymbol{\omega}$ and $\mathbf{r}$ respectively. From the diagram, this is equal to $|\mathbf{r}|^2 |\boldsymbol{\omega}| \sin\theta \mathbf{e}_h$. Thus both expressions are equivalent, and one can indeed write $\mathbf{h} = \hat{\mathbf{I}}\boldsymbol{\omega}$ with $\hat{\mathbf{I}}$ defined by Eqn. 1.8.9: the second-order tensor $\hat{\mathbf{I}}$ maps the angular velocity vector $\boldsymbol{\omega}$ into the angular momentum vector $\mathbf{h}$ of the particle.

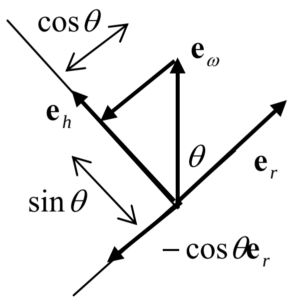

Figure 1.8.7: geometry of unit vectors for angular momentum calculation

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1.8.5 The Vector Space of Second Order Tensors

The vector space of vectors and associated spaces were discussed in §1.2. Here, spaces of second order tensors are discussed.

As mentioned above, the second order tensor is a mapping on the vector space V,

$$\mathbf{T}: V \to V \tag{1.8.10}$$

and follows the rules

$$\begin{aligned} \mathbf{T}(\mathbf{a}+\mathbf{b}) &= \mathbf{Ta}+\mathbf{Tb} \\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta}) \end{aligned} \tag{1.8.11}$$

for all $\mathbf{a}, \mathbf{b} \in V$ and $\alpha \in R$.

Denote the set of all second order tensors by V^2. Define then the sum of two tensors $\mathbf{S}, \mathbf{T} \in V^2$ through the relation

$$(\mathbf{S}+\mathbf{T})\mathbf{v} = \mathbf{Sv}+\mathbf{Tv} \tag{1.8.12}$$

and the product of a scalar $\alpha \in R$ and a tensor $\mathbf{T} \in V^2$ through

$$(\alpha\mathbf{T})\mathbf{v} = \alpha\mathbf{Tv} \tag{1.8.13}$$

Define an identity tensor $\mathbf{I} \in V^2$ through

$$\mathbf{Iv} = \mathbf{v}, \quad \text{for all } \mathbf{v} \in V \tag{1.8.14}$$

and a zero tensor $\mathbf{O} \in V^2$ through

$$\mathbf{Ov} = \mathbf{o}, \quad \text{for all } \mathbf{v} \in V \tag{1.8.15}$$

It follows from the definition 1.8.11 that V^2 has the structure of a real vector space, that is, the sum $\mathbf{S}+\mathbf{T} \in V^2$, the product $\alpha\mathbf{T} \in V^2$, and the following 8 axioms hold:

1. for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V^2$, one has $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$
2. there exists an element $\mathbf{O} \in V^2$ such that $\mathbf{T}+\mathbf{O} = \mathbf{O}+\mathbf{T} = \mathbf{T}$ for every $\mathbf{T} \in V^2$
3. for each $\mathbf{T} \in V^2$ there exists an element $-\mathbf{T} \in V^2$, called the negative of **T**, such that $\mathbf{T}+(-\mathbf{T}) = (-\mathbf{T})+\mathbf{T} = 0$
4. for any $\mathbf{S}, \mathbf{T} \in V^2$, one has $\mathbf{S}+\mathbf{T} = \mathbf{T}+\mathbf{S}$
5. for any $\mathbf{S}, \mathbf{T} \in V^2$ and scalar $\alpha \in R$, $\alpha(\mathbf{S}+\mathbf{T}) = \alpha\mathbf{S}+\alpha\mathbf{T}$
6. for any $\mathbf{T} \in V^2$ and scalars $\alpha, \beta \in R$, $(\alpha+\beta)\mathbf{T} = \alpha\mathbf{T}+\beta\mathbf{T}$
7. for any $\mathbf{T} \in V^2$ and scalars $\alpha, \beta \in R$, $\alpha(\beta\mathbf{T}) = (\alpha\beta)\mathbf{T}$
8. for the unit scalar $1 \in R$, $1\mathbf{T} = \mathbf{T}$ for any $\mathbf{T} \in V^2$.

1.8.6 Problems

1. Consider the function **f** which transforms a vector **v** into $\mathbf{a}\cdot\mathbf{v}+\beta$. Is **f** a tensor (of order one)? [Hint: test to see whether the transformation is linear, by examining $\mathbf{f}(\alpha\mathbf{u}+\mathbf{v})$.]
2. Show that the dyad is a linear operator, in other words, show that $(\mathbf{u}\otimes\mathbf{v})(\alpha\mathbf{w}+\beta\mathbf{x})=\alpha(\mathbf{u}\otimes\mathbf{v})\mathbf{w}+\beta(\mathbf{u}\otimes\mathbf{v})\mathbf{x}$
3. When is $\mathbf{a}\otimes\mathbf{b}=\mathbf{b}\otimes\mathbf{a}$?
4. Prove that
   (i) $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})=(\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x})$ [Hint: post-"multiply" both sides of the definition (1.8.3) by $\otimes\mathbf{x}$; then show that $((\mathbf{u}\otimes\mathbf{v})\mathbf{w})\otimes\mathbf{x}=(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})$.]
   (ii) $\mathbf{u}(\mathbf{v}\otimes\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ [hint: pre "multiply" both sides by $\mathbf{x}\otimes$ and use the result of (i)]
5. Consider the dyadic (tensor) $\mathbf{a}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{b}$. Show that this tensor orthogonally projects every vector **v** onto the plane formed by **a** and **b** (sketch a diagram).
6. Draw a sketch to show the meaning of $\mathbf{u}\cdot(\mathbf{P}\mathbf{v})$, where **P** is the projection tensor. What is the order of the resulting tensor?
7. Prove that $\mathbf{a}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a}=(\mathbf{b}\times\mathbf{a})\times$.