1.5 Coordinate Transformation of Vector Components

Very often in practical problems, the components of a vector are known in one coordinate system but it is necessary to find them in some other coordinate system.

For example, one might know that the force \( f \) acting “in the \( x_1 \) direction” has a certain value, Fig. 1.5.1 – this is equivalent to knowing the \( x_1 \) component of the force, in an \( x_1 - x_2 \) coordinate system. One might then want to know what force is “acting” in some other direction – for example in the \( x'_1 \) direction shown – this is equivalent to asking what the \( x'_1 \) component of the force is in a new \( x'_1 - x'_2 \) coordinate system.

![Figure 1.5.1: a vector represented using two different coordinate systems](image)

The relationship between the components in one coordinate system and the components in a second coordinate system are called the transformation equations. These transformation equations are derived and discussed in what follows.

1.5.1 Rotations and Translations

Any change of Cartesian coordinate system will be due to a translation of the base vectors and a rotation of the base vectors. A translation of the base vectors does not change the components of a vector. Mathematically, this can be expressed by saying that the components of a vector \( \mathbf{a} \) are \( \mathbf{e}_i \cdot \mathbf{a} \), and these three quantities do not change under a translation of base vectors. Rotation of the base vectors is thus what one is concerned with in what follows.

1.5.2 Components of a Vector in Different Systems

Vectors are mathematical objects which exist independently of any coordinate system. Introducing a coordinate system for the purpose of analysis, one could choose, for example, a certain Cartesian coordinate system with base vectors \( \mathbf{e}_i \) and origin \( o \), Fig.
1.5.2. In that case the vector can be written as \( \mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \), and \( u_1, u_2, u_3 \) are its components.

Now a second coordinate system can be introduced (with the same origin), this time with base vectors \( \mathbf{e}'_1 \). In that case, the vector can be written as \( \mathbf{u} = u'_1 \mathbf{e}'_1 + u'_2 \mathbf{e}'_2 + u'_3 \mathbf{e}'_3 \), where \( u'_1, u'_2, u'_3 \) are its components in this second coordinate system, as shown in the figure. Thus the same vector can be written in more than one way:

\[
\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 = u'_1 \mathbf{e}'_1 + u'_2 \mathbf{e}'_2 + u'_3 \mathbf{e}'_3
\]

The first coordinate system is often referred to as “the \( ox_1x_2x_3 \) system” and the second as “the \( ox'_1x'_2x'_3 \) system”.

Note that the new coordinate system is obtained from the first one by a rotation of the base vectors. The figure shows a rotation \( \theta \) about the \( x_3 \) axis (the sign convention for rotations is positive counterclockwise).

**Two Dimensions**

Concentrating for the moment on the two dimensions \( x_1 - x_2 \), from trigonometry (refer to Fig. 1.5.3),

\[
\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 = [OB] - [AB] \mathbf{e}_1 + [BD] + [CP] \mathbf{e}_2
\]

\[
= [\cos \theta u'_1 - \sin \theta u'_2] \mathbf{e}_1 + [\sin \theta u'_1 + \cos \theta u'_2] \mathbf{e}_2
\]

and so
In matrix form, these transformation equations can be written as

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  u'_1 \\
  u'_2
\end{bmatrix}
\]

Figure 1.5.3: geometry of the 2D coordinate transformation

The \(2 \times 2\) matrix is called the \textbf{transformation} or \textbf{rotation matrix} \([Q]\). By pre-multiplying both sides of these equations by the inverse of \([Q]\), \([Q^{-1}]\), one obtains the transformation equations transforming from \([u_1 \ u_2]^T\) to \([u'_1 \ u'_2]^T\):

\[
\begin{bmatrix}
  u'_1 \\
  u'_2
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

An important property of the transformation matrix is that it is \textbf{orthogonal}, by which is meant that

\[
[Q^{-1}] = [Q]^T \quad \text{Orthogonality of Transformation/Rotation Matrix} \quad (1.5.1)
\]

\textbf{Three Dimensions}

It is straightforward to show that, in the full three dimensions, Fig. 1.5.4, the components in the two coordinate systems are related through

\[
\begin{bmatrix}
  x'_1 \\
  x'_2 \\
  x'_3
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix} =
\begin{bmatrix}
  \phi & 0 & 0 \\
  0 & \cos \theta & -\sin \theta \\
  0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 1 \\
  \cos \phi & -\sin \phi & 0 \\
  \sin \phi & \cos \phi & 0
\end{bmatrix}
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & \sin \theta & 0 \\
  -\sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \theta \\
  \phi \\
  \psi
\end{bmatrix}
\]
where $\cos(x_i, x'_j)$ is the cosine of the angle between the $x_i$ and $x'_j$ axes. These nine quantities are called the direction cosines of the coordinate transformation. Again denoting these by the letter $Q$, $Q_{11} = \cos(x_1, x'_1)$, $Q_{12} = \cos(x_1, x'_2)$, etc., so that

$$Q_{ij} = \cos(x_i, x'_j), \quad (1.5.2)$$

one has the matrix equations

$$\begin{bmatrix}
u_1 \\
u_2 \\
u_3
ddot\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33} \end{bmatrix}\begin{bmatrix}u'_1 \\
u'_2 \\
u'_3\end{bmatrix}$$

or, in element form and short-hand matrix notation,

$$u_i = Q_{ij}u'_j \quad \ldots \quad [u] = [Q][u'] \quad (1.5.3)$$

![Figure 1.5.4: two different coordinate systems in a 3D space](image)

Note: some authors define the matrix of direction cosines to consist of the components $Q_{ij} = \cos(x'_i, x_j)$, so that the subscript $i$ refers to the new coordinate system and the $j$ to the old coordinate system, rather than the other way around as used here.

**Transformation of Cartesian Base Vectors**

The direction cosines introduced above also relate the base vectors in any two Cartesian coordinate systems. It can be seen that

$$\mathbf{e}_i \cdot \mathbf{e}'_j = Q_{ij} \quad (1.5.4)$$
This relationship is illustrated in Fig. 1.5.5 for \( i = 1 \).

![Figure 1.5.5: direction cosines](image)

**Formal Derivation of the Transformation Equations**

In the above, the transformation equations \( u_i = Q_{ij} u'_j \) were derived geometrically. They can also be derived algebraically using the index notation as follows: start with the relations \( u = u_k e_k = u'_j e'_j \) and post-multiply both sides by \( e_i \) to get (the corresponding matrix representation is to the right also, see Problem 3 in §1.4.3):

\[
\begin{align*}
  u_k e_k \cdot e_j &= u'_j e'_j \cdot e_j, \\
  \rightarrow u_k \delta_{ki} &= u'_j Q_{ij} \\
  \rightarrow u_i &= u'_j Q_{ij} \\
  \rightarrow u_i &= Q_{ij} u'_j
\end{align*}
\]

The inverse equations are {\footnotesize Problem 3}

\[
\begin{align*}
  u'_i &= Q_{ji} u_j \\
  \rightarrow [u'] &= [Q^T] [u]
\end{align*}
\]  \hspace{1cm} (1.5.5)

**Orthogonality of the Transformation Matrix \([Q]\)**

As in the two dimensional case, the transformation matrix is orthogonal, \([Q^T] = [Q^{-1}]\). This follows from 1.5.3, 1.5.5.

**Example**

Consider a Cartesian coordinate system with base vectors \( e_i \). A coordinate transformation is carried out with the new basis given by
\[ \mathbf{e}'_1 = n^{(1)}_1 \mathbf{e}_1 + n^{(1)}_2 \mathbf{e}_2 + n^{(1)}_3 \mathbf{e}_3, \]
\[ \mathbf{e}'_2 = n^{(2)}_1 \mathbf{e}_1 + n^{(2)}_2 \mathbf{e}_2 + n^{(2)}_3 \mathbf{e}_3, \]
\[ \mathbf{e}'_3 = n^{(3)}_1 \mathbf{e}_1 + n^{(3)}_2 \mathbf{e}_2 + n^{(3)}_3 \mathbf{e}_3. \]

What is the transformation matrix?

Solution

The transformation matrix consists of the direction cosines \( Q_{ij} = \cos(x_i, x'_j) = \mathbf{e}_i \cdot \mathbf{e}'_j \), so

\[
\begin{bmatrix}
  u_1' \\
  u_2' \\
  u_3'
\end{bmatrix} =
\begin{bmatrix}
  n^{(1)}_1 & n^{(2)}_1 & n^{(3)}_1 \\
  n^{(1)}_2 & n^{(2)}_2 & n^{(3)}_2 \\
  n^{(1)}_3 & n^{(2)}_3 & n^{(3)}_3
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3
\end{bmatrix}
\]

\[ \mathbf{u}' = \mathbf{Q} \mathbf{u}. \]

1.5.3 Problems

1. The angles between the axes in two coordinate systems are given in the table below.

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x'_1 )</td>
<td>135°</td>
<td>60°</td>
<td>120°</td>
</tr>
<tr>
<td>( x'_2 )</td>
<td>90°</td>
<td>45°</td>
<td>45°</td>
</tr>
<tr>
<td>( x'_3 )</td>
<td>45°</td>
<td>60°</td>
<td>120°</td>
</tr>
</tbody>
</table>

Construct the corresponding transformation matrix \( [\mathbf{Q}] \) and verify that it is orthogonal.

2. The \( \alpha x'_1 x'_2 x'_3 \) coordinate system is obtained from the \( \alpha x_1 x_2 x_3 \) coordinate system by a positive (counterclockwise) rotation of \( \Theta \) about the \( x_3 \) axis. Find the (full three dimensional) transformation matrix \( [\mathbf{Q}] \). A further positive rotation \( \beta \) about the \( x_2 \) axis is then made to give the \( \alpha x'_1 x'_2 x'_3 \) coordinate system. Find the corresponding transformation matrix \( [\mathbf{P}] \). Then construct the transformation matrix \( [\mathbf{R}] \) for the complete transformation from the \( \alpha x_1 x_2 x_3 \) to the \( \alpha x'_1 x'_2 x'_3 \) coordinate system.

3. Beginning with the expression \( u_j \mathbf{e}_j \cdot \mathbf{e}'_i = u'_k \mathbf{e}'_k \cdot \mathbf{e}'_i \), formally derive the relation \( u'_i = Q_{ij} u_j \) (\( [\mathbf{u}'] = [\mathbf{Q}^T] \mathbf{u} \)).