1.1 Vector Algebra

1.1.1 Scalars

A physical quantity which is completely described by a single real number is called a **scalar**. Physically, it is something which has a magnitude, and is completely described by this magnitude. Examples are **temperature, density** and **mass**. In the following, lowercase (usually Greek) letters, e.g. α, β, γ , will be used to represent scalars.

1.1.2 Vectors

The concept of the **vector** is used to describe physical quantities which have both a magnitude and a direction associated with them. Examples are **force**, **velocity**, **displacement** and **acceleration**.

Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow, Fig. 1.1.1a.

Analytically, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **r**, **q**.

The **magnitude** (or **length**) of a vector is denoted by $|\mathbf{a}|$ or a. It is a scalar and must be non-negative. Any vector whose length is 1 is called a **unit vector**; unit vectors will usually be denoted by **e**.

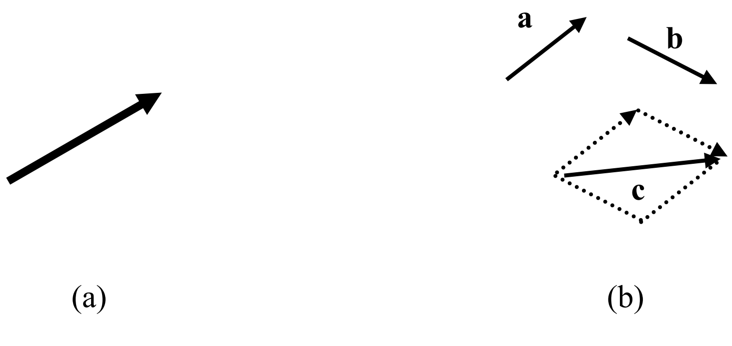


Figure 1.1.1: (a) a vector; (b) addition of vectors

1.1.3 Vector Algebra

The operations of addition, subtraction and multiplication familiar in the algebra of numbers (or scalars) can be extended to an algebra of vectors.

The following definitions and properties fundamentally *define* the vector:

1. Sum of Vectors:

   The addition of vectors **a** and **b** is a vector **c** formed by placing the initial point of **b** on the terminal point of **a** and then joining the initial point of **a** to the terminal point of **b**. The sum is written $\mathbf{c} = \mathbf{a} + \mathbf{b}$. This definition is called the parallelogram law for vector addition because, in a geometrical interpretation of vector addition, **c** is the diagonal of a parallelogram formed by the two vectors **a** and **b**, Fig. 1.1.1b. The following properties hold for vector addition:

   $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ … commutative law

   $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ … associative law

2. The Negative Vector:

   For each vector **a** there exists a **negative vector**. This vector has direction opposite to that of vector **a** but has the same magnitude; it is denoted by $-\mathbf{a}$. A geometrical interpretation of the negative vector is shown in Fig. 1.1.2a.

3. Subtraction of Vectors and the Zero Vector:

   The **subtraction** of two vectors **a** and **b** is defined by $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$, Fig. 1.1.2b. If $\mathbf{a} = \mathbf{b}$ then $\mathbf{a} - \mathbf{b}$ is defined as the **zero vector** (or **null vector**) and is represented by the symbol **o**. It has zero magnitude and unspecified direction. A **proper vector** is any vector other than the null vector. Thus the following properties hold:

   $$\mathbf{a} + \mathbf{o} = \mathbf{a}$$
   $$\mathbf{a} + (-\mathbf{a}) = \mathbf{o}$$

4. Scalar Multiplication:

   The product of a vector **a** by a scalar α is a vector $\alpha\mathbf{a}$ with magnitude $|\alpha|$ times the magnitude of **a** and with direction the same as or opposite to that of **a**, according as α is positive or negative. If $\alpha = 0$, $\alpha\mathbf{a}$ is the null vector. The following properties hold for scalar multiplication:

   $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$ … distributive law, over addition of scalars

   $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ … distributive law, over addition of vectors

   $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$ … associative law for scalar multiplication

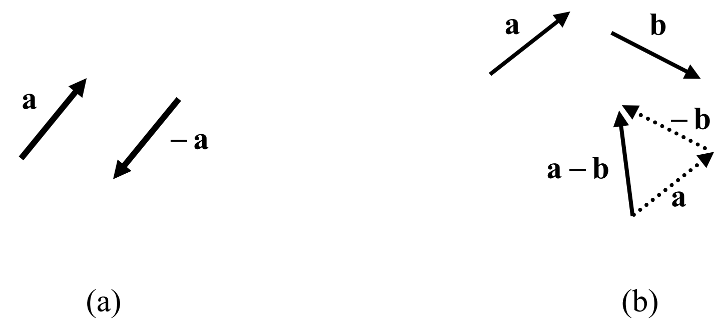


Figure 1.1.2: (a) negative of a vector; (b) subtraction of vectors

Note that when two vectors **a** and **b** are equal, they have the same direction and magnitude, regardless of the position of their initial points. Thus $\mathbf{a} = \mathbf{b}$ in Fig. 1.1.3. A particular position in space is not assigned here to a vector – it just has a magnitude and a direction. Such vectors are called **free**, to distinguish them from certain special vectors to which a particular position in space is actually assigned.

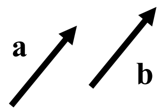


Figure 1.1.3: equal vectors

The vector as something with "magnitude and direction" and defined by the above rules is an element of one case of the mathematical structure, the **vector space**. The vector space is discussed in the next section, §1.2.

1.1.4 The Dot Product

The **dot product** of two vectors **a** and **b** (also called the **scalar product**) is denoted by $\mathbf{a} \cdot \mathbf{b}$. It is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \tag{1.1.1}$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$, Fig. 1.1.4.

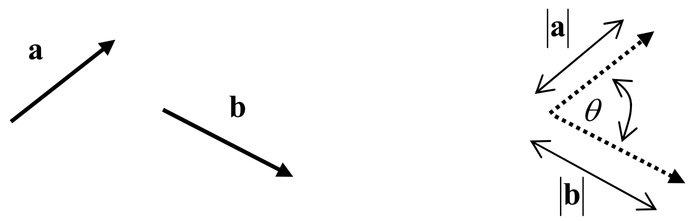


Figure 1.1.4: the dot product

An important property of the dot product is that if for two (proper) vectors **a** and **b**, the relation $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** and **b** are perpendicular. The two vectors are said to be **orthogonal**. Also, $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos(0)$, so that the length of a vector is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Another important property is that the **projection** of a vector **u** along the direction of a unit vector **e** is given by $\mathbf{u} \cdot \mathbf{e}$. This can be interpreted geometrically as in Fig. 1.1.5.

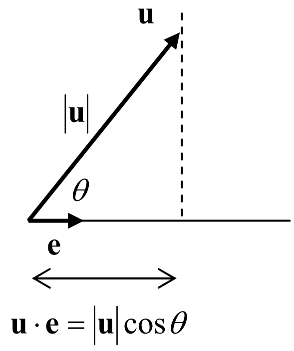


Figure 1.1.5: the projection of a vector along the direction of a unit vector

It follows that any vector **u** can be decomposed into a component parallel to a (unit) vector **e** and another component perpendicular to **e**, according to

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{e})\mathbf{e} + [\mathbf{u} - (\mathbf{u}\cdot\mathbf{e})\mathbf{e}] \tag{1.1.2}$$

The dot product possesses the following properties (which can be proved using the above definition) {▲Problem 6}:

(1) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (commutative)
(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(\alpha\mathbf{b})$
(4) $\mathbf{a}\cdot\mathbf{a} \geq 0$; and $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{o}$

1.1.5 The Cross Product

The **cross product** of two vectors **a** and **b** (also called the **vector product**) is denoted by $\mathbf{a}\times\mathbf{b}$. It is a vector with magnitude

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \tag{1.1.3}$$

with θ defined as for the dot product. It can be seen from the figure that the magnitude of $\mathbf{a}\times\mathbf{b}$ is equivalent to the area of the parallelogram determined by the two vectors **a** and **b**.

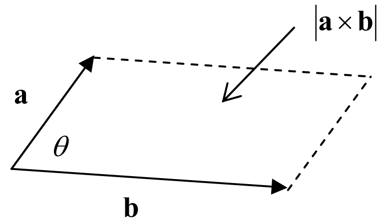


Figure 1.1.6: the magnitude of the cross product

The direction of this new vector is perpendicular to both **a** and **b**. Whether $\mathbf{a}\times\mathbf{b}$ points "up" or "down" is determined from the fact that the three vectors **a**, **b** and $\mathbf{a}\times\mathbf{b}$ form a **right handed system**. This means that if the thumb of the right hand is pointed in the direction of $\mathbf{a}\times\mathbf{b}$, and the open hand is directed in the direction of **a**, then the curling of the fingers of the right hand so that it closes should move the fingers through the angle θ, $0 \le \theta \le \pi$, bringing them to **b**. Some examples are shown in Fig. 1.1.7.

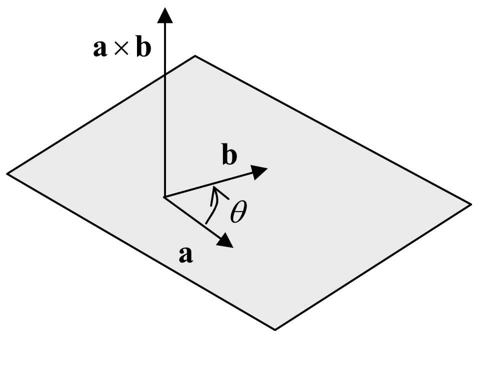


Figure 1.1.7: examples of the cross product

The cross product possesses the following properties (which can be proved using the above definition):

(1) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ (*not* commutative)
(2) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(\alpha\mathbf{b})$
(4) $\mathbf{a}\times\mathbf{b} = \mathbf{o}$ if and only if **a** and **b** $(\neq \mathbf{o})$ are parallel ("linearly dependent")

The Triple Scalar Product

The **triple scalar product**, or **box product**, of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined by

$$\boxed{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}} \quad \textbf{Triple Scalar Product} \tag{1.1.4}$$

Its importance lies in the fact that, if the three vectors form a right-handed triad, then the volume *V* of a parallelepiped spanned by the three vectors is equal to the box product.

To see this, let **e** be a unit vector in the direction of $\mathbf{u}\times\mathbf{v}$, Fig. 1.1.8. Then the projection of **w** on $\mathbf{u}\times\mathbf{v}$ is $h = \mathbf{w}\cdot\mathbf{e}$, and

$$\begin{aligned} \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{w}\cdot(|\mathbf{u}\times\mathbf{v}|\mathbf{e}) \\ &= |\mathbf{u}\times\mathbf{v}|h \\ &= V \end{aligned} \tag{1.1.5}$$

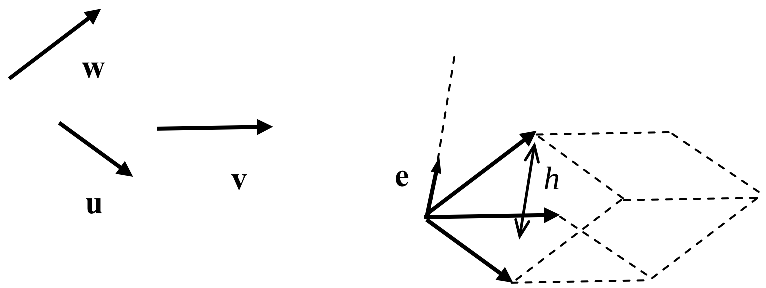


Figure 1.1.8: the triple scalar product

Note: if the three vectors do not form a right handed triad, then the triple scalar product yields the negative of the volume. For example, using the vectors above, $(\mathbf{w}\times\mathbf{v})\cdot\mathbf{u}=-V$.

1.1.6 Vectors and Points

Vectors are objects which have magnitude and direction, but they do not have any specific location in space. On the other hand, a **point** has a certain position in space, and the only characteristic that distinguishes one point from another is its position. Points cannot be "added" together like vectors. On the other hand, a vector **v** can be added to a point **p** to give a new point **q**, $\mathbf{q}=\mathbf{v}+\mathbf{p}$, Fig. 1.1.9. Similarly, the "difference" between two points gives a vector, $\mathbf{q}-\mathbf{p}=\mathbf{v}$. Note that the notion of point as defined here is slightly different to the familiar point in space with axes and origin – the concept of origin is not necessary for these points and their simple operations with vectors.

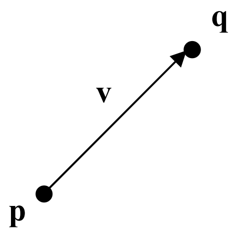
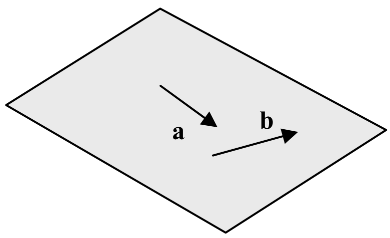


Figure 1.1.9: adding vectors to points

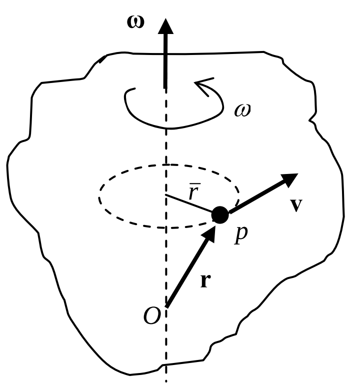
1.1.7 Problems

1. Which of the following are scalars and which are vectors?
   (i) weight
   (ii) specific heat
   (iii) momentum
   (iv) energy
   (v) volume
2. Find the magnitude of the sum of three unit vectors drawn from a common vertex of a cube along three of its sides.

3. Consider two **non-collinear** (not parallel) vectors **a** and **b**. Show that a vector **r** lying in the same plane as these vectors can be written in the form $\mathbf{r} = p\mathbf{a} + q\mathbf{b}$, where p and q are scalars. [Note: one says that all the vectors **r** in the plane are specified by the **base** vectors **a** and **b**.]
4. Show that the dot product of two vectors **u** and **v** can be interpreted as the magnitude of **u** times the component of **v** in the direction of **u**.
5. The work done by a force, represented by a vector **F**, in moving an object a given distance is the product of the component of force in the given direction times the distance moved. If the vector **s** represents the direction and magnitude (distance) the object is moved, show that the work done is equivalent to $\mathbf{F} \cdot \mathbf{s}$.
6. Prove that the dot product is commutative, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. [Note: this is equivalent to saying, for example, that the work done in problem 5 is also equal to the component of **s** in the direction of the force, times the magnitude of the force.]
7. Sketch $\mathbf{b} \times \mathbf{a}$ if **a** and **b** are as shown below.



8. Show that $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$.
9. Suppose that a rigid body rotates about an axis O with angular speed ω, as shown below. Consider a point p in the body with position vector **r**. Show that the velocity **v** of p is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is the vector with magnitude ω and whose direction is that in which a right-handed screw would advance under the rotation. [Note: let s be the arc-length traced out by the particle as it rotates through an angle θ on a circle of radius $\bar{r}$, then $v = |\mathbf{v}| = \bar{r}\omega$ (since $s = \bar{r}\theta,\ ds/dt = \bar{r}(d\theta/dt)$).]



10. Show, geometrically, that the dot and cross in the triple scalar product can be interchanged: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
11. Show that the **triple vector product** $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ lies in the plane spanned by the vectors **a** and **b**.