Answers to Selected Problems: Chapter 1

1.1

2. $\sqrt{3}$

1.3

1. -10
3. $19/9$
4. 90°
6. $2\mathbf{e}_1 + 5\mathbf{e}_2 + 6\mathbf{e}_3$

1.5

1. $$\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix}$$

1.6

2. $3t^2\mathbf{e}_1 - 4t^3\mathbf{e}_2 + 6t\mathbf{e}_3$
3. (iii) $2\mathbf{x}$, (iv) $\mathbf{x}/|\mathbf{x}|$
5. $x_2 x_3 + x_1$

   $-(1 - x_1 x_2)\mathbf{e}_2 + (x_2 - x_1 x_3)\mathbf{e}_3$

1.7

1. 303
9. 2π

1.8

1. No
3. when $\mathbf{a} = \mathbf{b}$

1.9

2. No
4. $A_{ijk} B_{jk} \mathbf{e}_i$ or simply $A_{ijk} B_{jk}$.
6. $\mathbf{D} \cdot \mathbf{F} = 12\mathbf{e}_1 \otimes \mathbf{e}_3 + 15\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3 - 15\mathbf{e}_3 \otimes \mathbf{e}_2 + 5\mathbf{e}_3 \otimes \mathbf{e}_3$

   $\mathbf{F} : \mathbf{D} = 17$
7. $4\mathbf{e}_1 + 10\mathbf{e}_2 + 5\mathbf{e}_3$

8. (a) a scalar, equals the trace of a second-order tensor
   (b) 3 functions of the 27 components of a third-order tensor
   (c) 9 components of a second-order tensor
   (d) scalar

9. $a_i b_j c_i d_j$

1.10

1.11

1. The principal invariants are

$$\mathrm{I}_{\mathbf{T}} = \mathrm{tr}\mathbf{T} = 3$$
$$\mathrm{II}_{\mathbf{T}} = \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{T})^2 - \mathrm{tr}(\mathbf{T}^2)\right] = 2$$
$$\mathrm{III}_{\mathbf{T}} = \det\mathbf{T} = 0$$

and the eigenvalues are $0,1,2$

7. (c) Spectral decomposition is $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvectors are $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

U is the square root of this.

1.12

1.13

1. (b) $[\mathbf{Q}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (c) $\mathbf{u} = (-6\cos\theta - 3\sin\theta)\mathbf{e}_1' + (6\sin\theta - 3\cos\theta)\mathbf{e}_2' + \mathbf{e}_3'$

1.14

1. (a) $\mathrm{grad}\mathbf{v} = \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 0 & 2x_3 \\ 0 & 2x_2 & 0 \end{bmatrix}$

   (b) $(\nabla \otimes \mathbf{v})\mathbf{v} = \begin{bmatrix} 2x_1^3 \\ 2x_3 x_2^2 \\ 2x_2 x_3^2 \end{bmatrix}$

2. $\nabla^2\mathbf{u} = \mathbf{o}$

3. $\text{grad}\mathbf{u} = \mathbf{e}_3 \otimes \mathbf{e}_1$

1.15

7. (i) $\mathbf{AT} + \mathbf{TA}$

(ii) $\mathbf{A} : \mathbf{T}^{\mathrm{T}} + \mathbf{T} : \mathbf{A}^{\mathrm{T}}$

1.16

12. Parabolic Cylindrical Coordinates

(i)

$$h_1 = \sqrt{\Theta_1^2 + \Theta_2^2}, \quad h_2 = \sqrt{\Theta_1^2 + \Theta_2^2}, \quad h_3 = 1$$

(ii) The Jacobian is $J = \Theta_1^2 + \Theta_2^2$

(iii) $(\Delta s)^2 = \left(\Theta_1^2 + \Theta_2^2\right) d\Theta_1^2 + \left(\Theta_1^2 + \Theta_2^2\right) d\Theta_2^2 + d\Theta_3^2$

$$\Delta S_1 = \sqrt{\Theta_1^2 + \Theta_2^2}\,\Delta\Theta_2 \Delta\Theta_3$$

$$\Delta S_2 = \sqrt{\Theta_1^2 + \Theta_2^2}\,\Delta\Theta_1 \Delta\Theta_3$$

$$\Delta S_3 = \left(\Theta_1^2 + \Theta_2^2\right) \Delta\Theta_1 \Delta\Theta_2$$

$$\Delta V = \left(\Theta_1^2 + \Theta_2^2\right) \Delta\Theta_1 \Delta\Theta_2 \Delta\Theta_3$$

13. Elliptical Cylindrical Coordinates:

(i) $h_1 = \sqrt{\sinh^2 \Theta_1 + \sin^2 \Theta_2}, \quad h_2 = \sqrt{\sinh^2 \Theta_1 + \sin^2 \Theta_2}, \quad h_3 = 1$

(ii) The Jacobian is $J = \sinh^2 \Theta_1 + \sin^2 \Theta_2$

(iii) $(\Delta s)^2 = \left(\sinh^2 \Theta_1 + \sin^2 \Theta_2\right) d\Theta_1^2 + \left(\sinh^2 \Theta_1 + \sin^2 \Theta_2\right) d\Theta_2^2 + d\Theta_3^2$

$$\Delta S_1 = \sqrt{\sinh^2 \Theta_1 + \sin^2 \Theta_2}\,\Delta\Theta_2 \Delta\Theta_3$$

$$\Delta S_2 = \sqrt{\sinh^2 \Theta_1 + \sin^2 \Theta_2}\,\Delta\Theta_1 \Delta\Theta_3$$

$$\Delta S_3 = \left(\sinh^2 \Theta_1 + \sin^2 \Theta_2\right) \Delta\Theta_1 \Delta\Theta_2$$

$$\Delta V = \left(\sinh^2 \Theta_1 + \sin^2 \Theta_2\right) \Delta\Theta_1 \Delta\Theta_2 \Delta\Theta_3$$

1.17

2. $\overline{g} = Jg$

1.18

13. (a) $\left[\dfrac{\partial x^i}{\partial \Theta^j}\right] = \begin{bmatrix} \Theta^2 & \Theta^1 \\ -2 & 0 \end{bmatrix}$

(b) $$\left[\frac{\partial \Theta^i}{\partial x^j}\right] = \begin{bmatrix} 0 & -\dfrac{1}{2} \\ \dfrac{1}{\Theta^1} & \dfrac{\Theta^2}{2\Theta^1} \end{bmatrix}$$

(c) $$g_{ij} = \begin{bmatrix} \left(\Theta^2\right)^2 + 4 & \Theta^1\Theta^2 \\ \Theta^1\Theta^2 & \left(\Theta^1\right)^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{\Theta^2}{4\Theta^1} \\ -\dfrac{\Theta^2}{4\Theta^1} & \dfrac{1}{\left(\Theta^1\right)^2} + \dfrac{\left(\Theta^2\right)^2}{4\left(\Theta^1\right)^2} \end{bmatrix}$$

(d) $$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^1_{22} = \Gamma^2_{11} = \Gamma^2_{22} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\Theta^1}$$

(e) $$\mathrm{grad}\Phi = \mathbf{g}^1 + \mathbf{g}^2$$