10.5 Linear Viscoelasticity and the Laplace Transform

The Laplace transform is very useful in constructing and analysing linear viscoelastic models.

10.5.1 The Laplace Transform

The formula for the Laplace transform of the derivative of a function is[1]:

$$
\begin{aligned}
L(\dot{f}) &= s\bar{f} - f(0) \\
L(\ddot{f}) &= s^2\bar{f} - sf(0) - \dot{f}(0), \qquad \text{etc.}
\end{aligned}
\tag{10.5.1}
$$

where s is the transform variable, the overbar denotes the Laplace transform of the function, and $f(0)$ is the value of the function at time $t = 0$. The Laplace transform is defined in such a way that $f(0)$ refers to $t = 0^-$, that is, just before time zero. Some other important Laplace transforms are summarised in Table 10.5.1, in which α is a constant.

$f(t)$	$\bar{f}(s)$
α	α / s
$H(t)$	$1/s$
$\delta(t-\tau)$	$e^{-\tau s}$
$\dot{\delta}(t)$	s
$e^{-\alpha t}$	$1/(\alpha + s)$
$\left(1-e^{-\alpha t}\right)/\alpha$	$1/s(\alpha+s)$
$t/\alpha - \left(1-e^{-\alpha t}\right)/\alpha^2$	$1/s^2(\alpha+s)$
t^n	$n!/s^{1+n}, \quad n = 0, 1, \cdots$

Table 10.5.1: Laplace Transforms

Another useful formula is the time-shifting formula:

$$
L\left[f(t-\tau)H(t-\tau)\right] = e^{-\tau s}\bar{f}(s) \tag{10.5.2}
$$

10.5.2 Mechanical models revisited

The Maxwell Model

The Maxwell model is governed by the set of three equations 10.3.5:

[1] this rule actually only works for functions whose derivatives are continuous, although the derivative of the function being transformed may be piecewise continuous. Discontinuities in the function or its derivatives introduce additional terms

$$\varepsilon_1 = \frac{1}{E}\sigma, \quad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \quad \varepsilon = \varepsilon_1 + \varepsilon_2 \qquad (10.5.3)$$

Taking Laplace transforms gives

$$\bar{\varepsilon}_1 = \frac{1}{E}\bar{\sigma}, \quad s\bar{\varepsilon}_2 = \frac{1}{\eta}\bar{\sigma}, \quad \bar{\varepsilon} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \qquad (10.5.4)$$

and it has been assumed that the strain ε_2 is zero at $t = 0^-$. The three differential equations have been reduced to a set of three algebraic equations, which may now be solved to get

$$\bar{\sigma} + \frac{\eta}{E}s\bar{\sigma} = \eta s\bar{\varepsilon} \qquad (10.5.5)$$

Transforming back then gives Eqn. 10.3.6:

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon} \qquad (10.5.6)$$

Now examine the response to a sudden load. When using the Laplace transform, the load is written as $\sigma(t) = \sigma_o H(t)$, where $H(t)$ is the Heaviside step function (see the Appendix to the previous section). Then 10.5.6 reads

$$\sigma_o H(t) + \frac{\eta}{E}\sigma_o \delta(t) = \eta\dot{\varepsilon} \qquad (10.5.7)$$

Using the Laplace transform gives

$$\frac{\sigma_o}{s} + \frac{\eta}{E}\sigma_o = \eta s\bar{\varepsilon} \quad \to \quad \bar{\varepsilon} = \frac{\sigma_o}{E}\frac{1}{s} + \frac{\sigma_o}{\eta}\frac{1}{s^2} \quad \to \quad \varepsilon(t) = \frac{\sigma_o}{E}H(t) + \frac{\sigma_o}{\eta}t \qquad (10.5.8)$$

which is the same result as before, Eqn. 10.3.7-8. Subsequent unloading, at time $t = \tau$ say, can be dealt with most conveniently by superimposing another load $\sigma(t) = -\sigma_o H(t-\tau)$ onto the first. Putting this into the constitutive equation and using the Laplace transform gives

$$\bar{\varepsilon} = -\frac{\sigma_o}{\eta}\frac{1}{s^2}e^{-\tau s} - \frac{\sigma_o}{E}\frac{1}{s}e^{-\tau s} \qquad (10.5.9)$$

Transforming back, again using the time-shifting rule, gives

$$\varepsilon(t) = -\frac{\sigma_o}{\eta}(t-\tau)H(t-\tau) - \frac{\sigma_o}{E}H(t-\tau) \qquad (10.5.10)$$

Adding this to the strain due to the first load then gives the expected result

$$\varepsilon(t)=\begin{cases}\dfrac{\sigma_o}{E}+\dfrac{\sigma_o}{\eta}t, & 0<t<\tau\\[2ex] \dfrac{\sigma_o}{\eta}\tau, & \tau<t\end{cases} \tag{10.5.11}$$

The Kelvin Model

Taking Laplace transforms of the three equations for the Kelvin model, Eqns. 10.3.10, gives $\bar{\sigma}=E\bar{\varepsilon}+\eta s\bar{\varepsilon}$, which yields 10.3.11, $\sigma=E\varepsilon+\eta\dot{\varepsilon}$. The response to a load $\sigma(t)=\sigma_o H(t)$ is

$$\sigma_o H(t)=E\varepsilon+\eta\dot{\varepsilon} \quad\rightarrow\quad \bar{\varepsilon}=\frac{\sigma_o}{\eta}\frac{1}{s(E/\eta+s)} \quad\rightarrow\quad \varepsilon(t)=\frac{\sigma_o}{E}\left(1-e^{-(E/\eta)t}\right) \tag{10.5.12}$$

The response to another load of magnitude $\sigma(t)=-\sigma_o H(t-\tau)$ is

$$\begin{aligned}-\sigma_o H(t-\tau)=E\varepsilon+\eta\dot{\varepsilon} \quad&\rightarrow\quad \bar{\varepsilon}=-\frac{\sigma_o}{\eta}\frac{e^{-\tau s}}{s(E/\eta+s)}\\ &\rightarrow\quad \varepsilon(t)=-\frac{\sigma_o}{E}\left(1-e^{-(E/\eta)(t-\tau)}\right)H(t-\tau)\end{aligned} \tag{10.5.13}$$

The response to both loads now gives the complete creep and recovery response:

$$\varepsilon(t)=\begin{cases}\dfrac{\sigma_o}{E}\left(1-e^{-(E/\eta)t}\right), & 0<t<\tau\\[2ex] \dfrac{\sigma_o}{E}e^{-(E/\eta)t}\left(e^{(E/\eta)\tau}-1\right), & t>\tau\end{cases} \tag{10.5.14}$$

To analyse the response to a suddenly applied strain, substitute $\varepsilon(t)=\varepsilon_o H(t)$ into the constitutive equation $\sigma=E\varepsilon+\eta\dot{\varepsilon}$ to get $\sigma=E\varepsilon_0 H(t)+\eta\varepsilon_0\delta(t)$, which shows that the relaxation modulus of the Kelvin model is

$$E(t)=E+\eta\delta(t) \tag{10.5.15}$$

The Standard Linear Model

Consider next the standard linear model, which consists of a spring in series with a Kelvin unit, Fig. 10.5.1 (see Fig. 10.3.8a). Upon loading one expects the left-hand spring to stretch immediately. The dash pot then takes up the stress, transferring the load to the second spring as it slowly opens over time. Upon unloading one expects the left-hand spring to contract immediately and for the right-hand spring to slowly contract, being held back by the dash-pot.

The equations for this model are, from the figure,

$$\begin{aligned}
\sigma &= \sigma_1 + \sigma_2 \\
\varepsilon &= \varepsilon_1 + \varepsilon_2 \\
\sigma &= E_1 \varepsilon_1 \\
\sigma_1 &= E_2 \varepsilon_2 \\
\sigma_2 &= \eta \dot{\varepsilon}_2
\end{aligned} \tag{10.5.16}$$

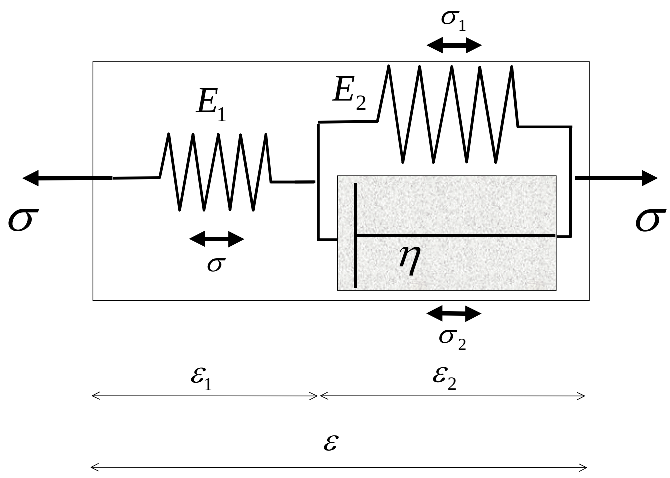

Figure 10.5.1: the standard linear model

One can eliminate the four unknowns from these five equations using the Laplace transform, giving

$$(E_1 + E_2)\bar{\sigma} + \eta s \bar{\sigma} = E_1 E_2 \bar{\varepsilon} + E_1 \eta s \bar{\varepsilon} \tag{10.5.17}$$

which transforms back to (in standard form)

$$\boxed{\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1 \eta}{E_1 + E_2}\dot{\varepsilon}} \tag{10.5.18}$$

which is Eqn. 10.3.16a.

The response to a load $\sigma(t) = \sigma_o H(t)$ is

$$\begin{aligned}
&(E_1 + E_2)\sigma_o H(t) + \eta \sigma_o \delta(t) = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon} \\
&\quad \to \quad \bar{\varepsilon} = \frac{\sigma_o}{E_1} \frac{1}{((E_2/\eta) + s)} + \frac{\sigma_o (E_1 + E_2)}{E_1 \eta} \frac{1}{s((E_2/\eta) + s)} \\
&\quad \to \quad \varepsilon(t) = \sigma_o J(t)
\end{aligned} \tag{10.5.19}$$

and the creep compliance is

$$J(t)=\frac{1}{E_1}e^{-(E_2/\eta)t}+\frac{E_1+E_2}{E_1E_2}\left(1-e^{-(E_2/\eta)t}\right) \tag{10.5.20}$$

Note that $\varepsilon(0)=\sigma_o/E_1$ as expected.

For recovery one can superimpose an opposite load onto the first, at time τ say:

$$\begin{aligned}
&-(E_1+E_2)\sigma_o H(t-\tau)-\eta\sigma_o\delta(t-\tau)=E_1E_2\varepsilon+E_1\eta\dot{\varepsilon}\\
&\quad\to\quad \bar{\varepsilon}=-\frac{\sigma_o}{E_1}\frac{1}{(E_2/\eta+s)}e^{-\tau s}-\sigma_o\left(\frac{E_1+E_2}{E_1\eta}\right)\frac{1}{s(E_2/\eta+s)}e^{-\tau s}\\
&\quad\to\quad \varepsilon(t)=-\sigma_o H(t-\tau)\left\{\frac{1}{E_1}e^{-(E_2/\eta)(t-\tau)}+\left(\frac{E_1+E_2}{E_1E_2}\right)\left(1-e^{-(E_2/\eta)(t-\tau)}\right)\right\}
\end{aligned} \tag{10.5.21}$$

The response after time τ is then

$$\varepsilon(t)=\frac{\sigma_o}{E_2}e^{-(E_2/\eta)t}\left(e^{(E_2/\eta)\tau}-1\right) \tag{10.5.22}$$

This is, as expected, simply the recovery response of the Kelvin unit. The full response is as shown in Fig. 10.5.2. This seems to be fairly close now to the response of a real material as discussed in §10.1, although it does not allow for a permanent strain.

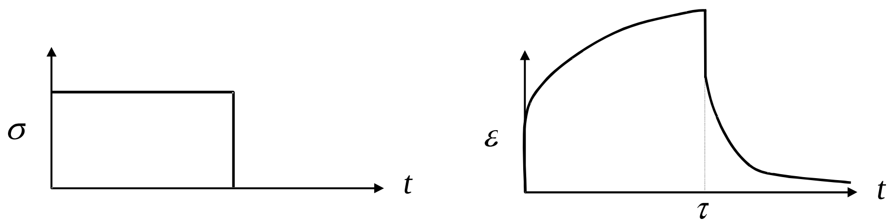

Figure 10.5.2: Creep-recovery response of the standard linear model

Non-constant Loading

The response to a complex loading history can be evaluated by solving the differential constitutive equation (or the corresponding hereditary integral). The differential equation can be most easily solved using Laplace transforms.

Example

Consider the example treated earlier using hereditary integrals, at the end of §10.4.2. Load (1) of Fig. 10.4.5 can be thought of as consisting of the two loads (1a) $\sigma=(\hat{\sigma}/T)t$ and (1b) $\sigma=-(\hat{\sigma}/T)(t-T)H(t-T)$ applied at time $t=T$. Load (2) consists of a constant load applied at time $t=T$.

For load (1a),

$$\frac{\hat{\sigma}}{T}t + \frac{\eta}{E}\frac{\hat{\sigma}}{T} = \eta\dot{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\hat{\sigma}}{\eta T}\frac{1}{s^3} + \frac{1}{E}\frac{\hat{\sigma}}{T}\frac{1}{s^2} \quad \rightarrow \quad \frac{\varepsilon(t)}{\hat{\sigma}} = \frac{1}{ET}t + \frac{1}{2\eta T}t^2$$

which gives the response for $t < T$.

For load (1b) one has [note: $L((t-\tau)\delta(t-\tau)) = 0$]

$$\begin{aligned}
\eta\dot{\varepsilon} &= -\frac{\hat{\sigma}}{T}(t-T)H(t-T) - \frac{\eta}{E}\frac{\hat{\sigma}}{\tau}\left[(t-T)\delta(t-T) + H(t-T)\right] \\
\rightarrow \quad s\bar{\varepsilon} &= -\frac{\hat{\sigma}}{\eta T}e^{-Ts}\frac{1}{s^2} - \frac{1}{E}\frac{\hat{\sigma}}{T}e^{-Ts}\frac{1}{s} \\
\rightarrow \quad \bar{\varepsilon} &= -\frac{\hat{\sigma}}{\eta\tau}e^{-Ts}\frac{1}{s^3} - \frac{1}{E}\frac{\hat{\sigma}}{\tau}e^{-Ts}\frac{1}{s^2} \\
\rightarrow \quad \frac{\varepsilon(t)}{\hat{\sigma}} &= H(t-T)\left\{-\frac{1}{2\eta T}(t-T)^2 - \frac{1}{E\tau}(t-T)\right\}
\end{aligned}$$

The response after time T is then given by adding the two results:

$$\frac{\varepsilon(t)}{\hat{\sigma}} = \frac{1}{E} + \frac{1}{\eta}\left(t - \frac{T}{2}\right)$$

10.5.3 Relationship between Creep and Relaxation

Taking the Laplace transform of the general constitutive equation 10.3.19, $\mathbf{P}\sigma = \mathbf{Q}\varepsilon$, leads to

$$\left(p_o + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \cdots\right)\bar{\sigma} = \left(q_o + q_1 s + q_2 s^2 + q_3 s^3 + q_4 s^4 + \cdots\right)\bar{\varepsilon} \tag{10.5.23}$$

which can also be written in the contracted form

$$P(s)\bar{\sigma} = Q(s)\bar{\varepsilon} \tag{10.5.24}$$

where P and Q are the polynomials

$$P(s) = \sum_{i=0}^{n} p_i s^i, \qquad Q(s) = \sum_{i=0}^{n} q_i s^i \tag{10.5.25}$$

The Laplace transforms of the creep compliance $\left(J(t) \rightarrow \bar{J}(s)\right)$ and relaxation modulus $\left(E(t) \rightarrow \bar{E}(s)\right)$ can be written in terms of these polynomials as follows. First, the strain due to a unit load $\sigma = H(t)$ is $J(t)$. Since $\bar{\sigma} = 1/s$, substitution into the above equation gives

$$\bar{J}(s) = \frac{P(s)}{sQ(s)} \tag{10.5.26}$$

Similarly, the stress due to a unit strain $\varepsilon = H(t)$ is $E(t)$ and so

$$\bar{E}(s) = \frac{Q(s)}{sP(s)} \tag{10.5.27}$$

It follows that

$$\boxed{\bar{J}(s)\bar{E}(s) = \frac{1}{s^2}} \tag{10.5.28}$$

Thus, for a linear viscoelastic material, there is a unique and simple relationship between the creep and relaxation behaviour.

10.5.4 Problems

1. Check that the relation 10.5.28, $\bar{J}(s)\bar{E}(s) = 1/s^2$, holds for the Kelvin model

2. (a) Derive the constitutive relation (in standard form) for the three-element model shown below using the Laplace transform (this is the Standard Fluid II of Fig. 10.3.8d and the constitutive relation is given by Eqn. 10.3.16d)
   (b) Derive the creep compliance $J(t)$ by considering a suddenly applied load.