10.4 The Hereditary Integral

In the previous section, it was shown that the constitutive relation for a linear viscoelastic material can be expressed in the form of a linear differential equation, Eqn. 10.3.19. Here it is shown that the stress-strain relation can also be expressed in the form of an integral, called the **hereditary integral**. Quite a few different forms of this integral are commonly used; to begin this section, the different forms are first derived for the Maxwell model, before looking at the more general case(s).

10.4.1 An Example: the Maxwell Model

Consider the differential equation for the Maxwell model, Eqn. 10.3.6,

$$\frac{d\sigma}{dt} + \frac{E}{\eta}\sigma = E\frac{d\varepsilon}{dt} \tag{10.4.1}$$

The first order differential equation can be solved using the standard **integrating factor** method. This converts 10.4.1 into an integral equation. Three similar integral equations will be derived in what follows[1].

Hereditary Integral over $[-\infty, \mathbf{t}]$

It is sometimes convenient to regard 10.4.1 as a differential equation over the time interval $[-\infty, t]$, even though the time interval of interest is really $[0, t]$. This can make it easier to deal with sudden "jumps" in stress or strain at time $t = 0$. The initial condition on 10.4.1 is then

$$\sigma(-\infty) = 0\,. \tag{10.4.2}$$

Using the integrating factor $e^{Et/\eta}$, re-write 10.4.1 in the form

$$\frac{d}{dt}\left(e^{Et/\eta}\sigma(t)\right) = Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt} \tag{10.4.3}$$

Integrating both sides over $[-\infty, \hat{t}]$ gives

$$\left(e^{Et/\eta}\sigma\right)_{\hat{t}} - \left(e^{Et/\eta}\sigma\right)_{-\infty} = \int_{-\infty}^{\hat{t}} Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt}dt \tag{10.4.4}$$

or

[1] note that Eqn. 10.4.1 predicts that sudden changes in the strain-rate, $\dot{\varepsilon}$, will lead to sudden changes in the stress-rate, $\dot{\sigma}$, but the stress σ will remain continuous. The strain ε does not appear explicitly in 10.4.1; sudden changes in strain can be dealt with by (i) integrating across the point where the jump occurs, or (ii) using step functions and the integral formulation (see later)

$$\sigma(\hat{t}) = \int_{-\infty}^{\hat{t}} E e^{-E(\hat{t}-t)/\eta} \frac{d\varepsilon(t)}{dt} dt \tag{10.4.5}$$

Changing the notation,

$$\boxed{\sigma(t) = \int_{-\infty}^{t} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau} \tag{10.4.6}$$

where $E(t)$, the relaxation modulus for the Maxwell model, is

$$E(t) = E e^{-Et/\eta} \tag{10.4.7}$$

This is known as a hereditary integral; given the **strain history** over $[-\infty, t]$, one can evaluate the stress at the current time. It is the *same* constitutive equation as Eqn. 10.4.1, only in a different form.

Hereditary Integral over $[\mathbf{0}, \mathbf{t}]$

The hereditary integral can also be expressed in terms of an integral over $[0, t]$. Let there be a sudden non-zero strain $\varepsilon(0)$ at $t = 0$, with the strain possibly varying, but continuously, thereafter. The strain, which in Eqn. 10.4.6 is to be regarded as a single function over $[-\infty, t]$ with a jump at $t = 0$, is sketched in Fig. 10.4.1.

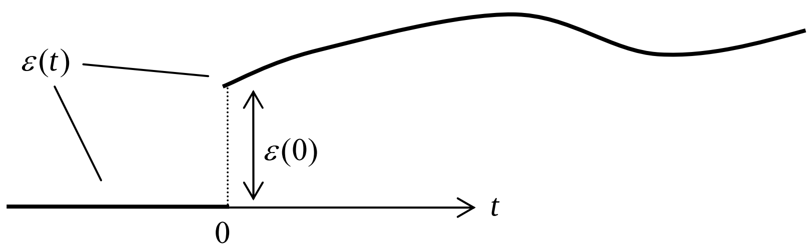

Figure 10.4.1: Strain with a sudden jump to a non-zero strain at $t = 0$

There are two ways to proceed. First, write the integral over three separate intervals:

$$\sigma(t) = \lim_{\vartheta \to 0} \left\{ \int_{-\infty}^{-\vartheta} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{-\vartheta}^{+\vartheta} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{+\vartheta}^{t} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \right\} \tag{10.4.8}$$

With $\varepsilon(t) = 0$ over $[-\infty, -\vartheta]$, the first integral is zero. With a jump in strain only at $t = 0$, the integrand in the third integral remains finite. The second integral can be evaluated by considering the function $f(t)$ illustrated in Fig. 10.4.2, a straight line with slope $\varepsilon(0)/2\vartheta$. As $\vartheta \to 0$, it approaches the actual strain function $\varepsilon(t)$, which jumps to $\varepsilon(0)$ at $t = 0$. Then

$$\lim_{\vartheta \to 0} \int_{-\vartheta}^{+\vartheta} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau = \lim_{\vartheta \to 0} \frac{\varepsilon(0)}{2\vartheta}\eta e^{-Et/\eta}\left(e^{+E\vartheta/\eta} - e^{-E\vartheta/\eta}\right) \qquad (10.4.9)$$

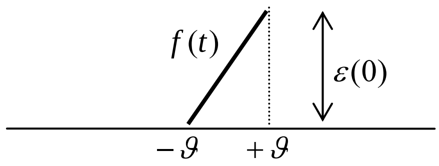

Figure 10.4.2: A function used to approximate the strain for a sudden jump

Using the approximation $e^x \approx 1 + x$ for small x, the value of this integral is $\varepsilon(0)Ee^{-Et/\eta}$. Thus Eqn. 10.4.6 can be expressed as

$$\boxed{\sigma(t) = E(t)\varepsilon(0) + \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau} \qquad (10.4.10)$$

By "0" here in the lower limit of the integral, one means 0^+, just after any possible non-zero initial strain. In that sense, the strain $\varepsilon(t)$ in Eqn. 10.4.10 is to be regarded as a continuous function, i.e. with no jumps over $[0,t]$. Jumps in strain after $t = 0$ can be dealt with in a similar manner.

A second and more elegant way to arrive at Eqn. 10.4.10 is to re-express the above analysis in terms of the Heaviside step function $H(t)$ and the Dirac delta function $\delta(t)$ (see the Appendix to this section for a discussion of these functions).

The function sketched in Fig. 10.4.1 can be expressed as $H(t)\varepsilon(t)$ where now $\varepsilon(t)$ is to be regarded as a continuous function over $[-\infty,t]$ – the jump is now contained within the step function $H(t)$. Eqn. 10.4.6 now becomes

$$\sigma(t) = \int_{-\infty}^t E(t-\tau)H(\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau + \int_{-\infty}^t E(t-\tau)\varepsilon(\tau)\frac{dH(\tau)}{d\tau}d\tau \qquad (10.4.11)$$

The first integral becomes the integral in 10.4.10. From the brief discussion in the Appendix to this section, the second integral becomes

$$\int_{-\infty}^t E(t-\tau)\varepsilon(\tau)\frac{dH(\tau)}{d\tau}d\tau = \int_{-\infty}^t E(t-\tau)\varepsilon(\tau)\delta(\tau)d\tau = E(t)\varepsilon(0) \qquad (10.4.12)$$

A Third Hereditary Integral

Finally, the integral can also be expressed as a function of $\varepsilon(t)$, rather than its derivative. To achieve this, one can integrate 10.4.10 by parts:

$$\sigma(t) = E(0)\varepsilon(t) + \int_0^t \frac{dE(t-\tau)}{d(t-\tau)}\varepsilon(\tau)d\tau \qquad (10.4.13)$$

This can be expressed as

$$\boxed{\sigma(t) = E(0)\varepsilon(t) - \int_0^t R(t-\tau)\varepsilon(\tau)d\tau} \qquad (10.4.14)$$

where $R(t) = -dE(t)/dt$.

Note that integration by parts is only possible when there are no "jumps" in the functions under the integral sign and this is assumed for the integrand in 10.4.10. If there are jumps, the integral can either be split into separate integrals as in 10.4.8, or the functions can be represented in terms of step functions, which automatically account for jumps.

The formulae 10.4.6, 10.4.10 and 10.4.14 give the stress as functions of the strain. Similar formulae can be derived for the strain in terms of the stress (see the Appendix to this Section).

Relaxation Test

To illustrate the use of the hereditary integral formulae, consider a relaxation test, where the strain history is given by

$$\varepsilon(t) = \begin{cases} 0, & t < 0 \\ \varepsilon_0, & \text{otherwise} \end{cases} \qquad (10.4.15)$$

Expressing the strain history as $\varepsilon(t) = \varepsilon_0 H(t)$, Eqn. 10.4.6 gives

$$\sigma(t) = \varepsilon_0 \int_{-\infty}^t E(t-\tau)\delta(\tau)d\tau = \varepsilon_0 E(t) \qquad (10.4.16)$$

From 10.4.10, with the derivative in the integrand zero, one has $\sigma(t) = E(t)\varepsilon(0) = \varepsilon_0 E(t)$. Finally, from 10.4.14, with $R(t) = +(E^2/\eta)e^{-Et/\eta}$, one again has

$$\sigma(t) = E\varepsilon_0 - \varepsilon_0 \int_0^t \frac{E^2}{\eta} e^{-E(t-\tau)/\eta} d\tau = \varepsilon_0 E e^{-Et/\eta} = \varepsilon_0 E(t) \qquad (10.4.17)$$

10.4.2 Hereditary Integrals: General Formulation

Although derived for the Maxwell mode, these formulae Eqns. 10.4.6, 10.4.10, 10.4.14, are in fact quite general, for example they can be derived from the differential equation for the Kelvin model (see Appendix to this section).

The hereditary integrals were derived directly from the Maxwell model differential equation so as to emphasis that they are one and the same constitutive equation. Here they are derived more generally from first principles.

The strain due to a constant step load $\sigma(0)$ applied at time $t=0$ is by definition $\varepsilon(t)=\sigma(0)J(t)$, where $J(t)$ is the creep compliance function. The strain due to a second load, $\Delta\sigma$ say, applied at some later time τ, is $\varepsilon(t)=\Delta\sigma J(t-\tau)$. The total strain due to both loads is[2], Fig. 10.4.3,

$$\varepsilon(t)=\sigma(0)J(t)+\Delta\sigma J(t-\tau) \tag{10.4.18}$$

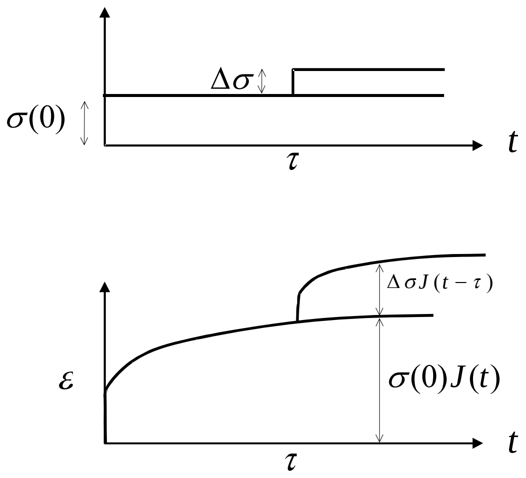

Figure 10.4.3: Superposition of loads

Generalising to an indefinite number of applied loads of infinitesimal magnitude, $d\sigma_i$, one has

$$\varepsilon(t)=\sigma(0)J(t)+\sum_{i=1}^{\infty}d\sigma_i J(t-\tau_i) \tag{10.4.19}$$

In the limit, the summation becomes the integral $\int J(t-\tau)d\sigma$, or[3] (see Fig. 10.4.4)

$$\boxed{\varepsilon(t)=\sigma(0)J(t)+\int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau} \quad \textbf{Hereditary Integral (for Strain)} \tag{10.4.20}$$

---

[2] this is again an application of the linear superposition principle, mentioned in §6.1.2; because the material is linear (and only because it is linear), the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause"

[3] this integral equation allows for a sudden non-zero stress at $t=0$. Other jumps in stress at later times can be allowed for in a similar manner – one would split the integral into separate integrals at the point where the jump occurs

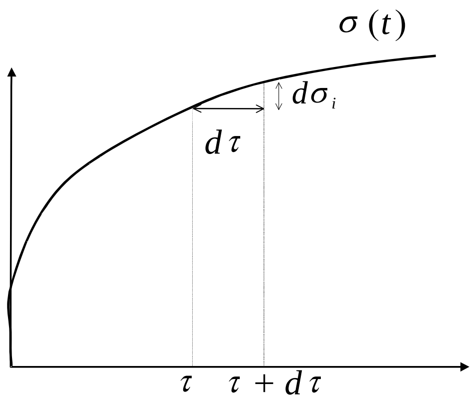

Figure 10.4.4: Formation of the hereditary integral

One can also derive a corresponding hereditary integral in terms of the relaxation modulus {▲Problem 1}:

$$\sigma(t) = \varepsilon(0)E(t) + \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau \qquad \textbf{Hereditary Integral (for Stress)} \qquad (10.4.21)$$

This is Eqn. 10.4.10, which was derived specifically from the Maxwell model.

The hereditary integrals only require a knowledge of the creep function (or relaxation function). One does not need to construct a rheological model (with springs/dashpots) to determine a creep function. For example, the creep function for a material may be determined from test-data from a creep test. The hereditary integral formulation is thus not restricted to particular combinations of springs and dash-pots.

Example

Consider the Maxwell model and the two load histories shown in Fig. 10.4.5. The maximum stress is the same in both, $\hat{\sigma}$, but load (1) is applied more gradually.

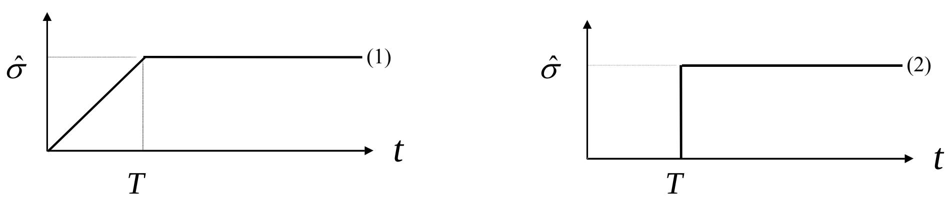

Figure 10.4.5: two stress histories

Examine load (1) first. The stress history is

$$\sigma(t) = \begin{cases} \dfrac{\hat{\sigma}}{T}t, & t < T \\ \\ \hat{\sigma}, & t > T \end{cases}$$

In the hereditary integral 10.4.20, the creep compliance function $J(t)$ is given by 10.3.8, $J(t) = t/\eta + 1/E$, and the stress is zero at time zero, so $\sigma(0) = 0$. The strain is then

$0 < t < T: \; d\sigma/dt = \hat{\sigma}/T$

$$\varepsilon(t) = \sigma(0)J(t) + \int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau = \frac{\hat{\sigma}}{T}\int_0^t \left[\frac{t-\tau}{\eta} + \frac{1}{E}\right]d\tau = \frac{\hat{\sigma}}{T}\left[\frac{t^2}{2\eta} + \frac{t}{E}\right]$$

$T < t: \; d\sigma/dt = 0$

$$\begin{aligned} \varepsilon(t) &= \sigma(0)J(t) + \int_0^T J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau + \int_T^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \\ &= \int_0^T J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau = \frac{\hat{\sigma}}{T}\int_0^{\bar{\tau}} \left[\frac{t-\tau}{\eta} + \frac{1}{E}\right]d\tau = \hat{\sigma}\left[\frac{t-T/2}{\eta} + \frac{1}{E}\right] \end{aligned}$$

For load history (2),

$$\sigma(t) = \begin{cases} 0, & t < T \\ \\ \hat{\sigma}, & t > T \end{cases}$$

The strain is then $\varepsilon(t) = 0$ for $t < T$. The hereditary integral 10.4.20 allows for a jump at $t = 0$. For a jump from zero stress to a non-zero stress at $t = T$ it can be modified to

$$\varepsilon(t) = \sigma(T)J(t-T) = \hat{\sigma}\left[\frac{t-T}{\eta} + \frac{1}{E}\right]$$

which is less than the strain due to load (1). (Alternatively, one could use the Heaviside step function and let $\sigma(t) = \hat{\sigma}H(t-T)$ in 10.4.20, leading to the same result,

$$\varepsilon(t) = \sigma(0)J(t) + \hat{\sigma}\int_0^t J(t-\tau)\delta(\tau - T)d\tau = \hat{\sigma}J(t-T)\,.)$$

This example illustrates two points:

(1) the material has a "memory". It remembers the previous loading history, responding differently to different loading histories

(2) the rate of loading is important in viscoelastic materials. This result agrees with an observed phenomenon: the strain in viscoelastic materials is larger for stresses which grow gradually to their final value, rather than when applied more quickly[4].

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[4] for the Maxwell model, if one applied the second load at time $t = T/2$, so that the total stress applied in (1) and (2) was the same, one would have obtained the same response after time *T*, but this is not the case in general

10.4.3 Non-linear Hereditary Integrals

The linear viscoelastic models can be extended into the non-linear range in a number of ways. For example, generalising expressions of the form 10.4.14,

$$\sigma(t) = f_1(\varepsilon(t)) + \int_0^t R(t-\tau) f_2(\varepsilon(t)) d\tau \tag{10.4.22}$$

where f_1, f_2 are non-linear functions of the strain history. The relaxation function can also be assumed to be a function of strain as well as time:

$$\sigma(t) = f_1(\varepsilon(t)) + \int_0^t K(t-\tau,\varepsilon) f_2(\varepsilon(t)) d\tau \tag{10.4.23}$$

10.4.4 Problems

1. Derive the hereditary integral 10.4.21,

$$\sigma(t) = \varepsilon(0)E(t) + \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau} d\tau$$

2. Use the hereditary integral form of the constitutive equation for a linear viscoelastic material, Eqn. 10.4.20, to evaluate the response of a material with creep compliance function

$$J(t) = \ln(t+1)$$

to a load $\sigma(t) = \sigma_o(1+Bt)$. Sketch $J(t)$, which of course gives the strain response due to a unit load $\sigma(t) = 1$. Sketch also the load $\sigma(t)$ and the calculated strain $\varepsilon(t)$.
[note: $\int_0^t \ln[(b-x)+1]dx = -(b-t+1)\ln(b-t+1) + (b+1)\ln(b+1) - t$]

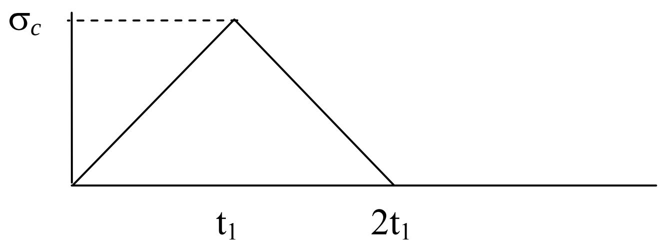
3. A creep test was carried out on a certain linear viscoelastic material and the data was fitted approximately by the function

$$\varepsilon(t) = \hat{\sigma}\left(1 - e^{-2t}\right),$$

where $\hat{\sigma}$ was the constant applied load.
(i) Sketch this strain response over $0 \le t \le 3$ (very roughly, with $\hat{\sigma} = 1$).
(ii) Which of the following three rheological models could be used to model the material:
(a) the full generalized Kelvin chain of Fig. 10.3.9
(b) the Kelvin chain minus the free spring
(c) the generalized Maxwell model minus the free spring and free dash-pot
Give reasons for your choice (and reasons for discounting the other two).
(iii) For the rheological model you chose in part (ii), roughly sketch the response to a standard creep-recovery test (the response during the loading phase has already been done in part (i)).
(iv) Find the material's response to a load $\sigma(t) = t^2 + 1$.

4. Determine the strain response of the Kelvin model to a stress history which is triangular in time:

$$\sigma(t) = \begin{cases} \sigma(t) = 0, & t < 0 \\ \sigma(t) = (\sigma_c / t_1)t, & 0 < t < t_1 \\ \sigma(t) = 2\sigma_c - (\sigma_c / t_1)t, & t_1 < t < 2t_1 \\ \sigma(t) = 0, & 2t_1 < t < \infty \end{cases}$$



10.4.5 Appendix to §10.4

1. The Heaviside Step Function and the Dirac Delta Functions

The **Heaviside step function** $H(t)$ is defined through

$$H(t-a) = \begin{cases} 0, & t < a \\ 1/2, & t = a \\ 1, & t > a \end{cases} \tag{10.4.24}$$

and is illustrated in Fig. 10.4.6a. The derivative of the Heaviside step function, dH/dt, can be evaluated by considering $H(t-a)$ to be the limit of the function $f(t)$ shown in Fig. 10.4.6b as $\vartheta \to 0$. This derivative df/dt is shown in Fig. 10.4.6c and in the limit is

$$\frac{dH(t-a)}{dt} = \lim_{\vartheta \to 0} \frac{df}{dt} = \delta(t-a) \tag{10.4.25}$$

where δ is the **Dirac delta function** defined through (the integral here states that the "area" is unity, as illustrated in Fig. 10.4.6c)

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0 & \text{otherwise} \end{cases}, \qquad \int_{-\infty}^{+\infty} \delta(t-a)\,dt = 1 \tag{10.4.26}$$

Integrals involving delta functions are evaluated as follows: consider the integral

$$\int_{-\infty}^{+\infty} g(t)\delta(t-b)dt \tag{10.4.27}$$

The delta function here is zero, and hence the integrand is zero, everywhere except at $t = b$. Thus the integral is

$$\int_{-\infty}^{+\infty} g(t)\delta(t-b)dt = \int_{-\infty}^{+\infty} g(b)\delta(t-b)dt = g(b)\int_{-\infty}^{+\infty} \delta(t-b)dt = g(b) \tag{10.4.28}$$

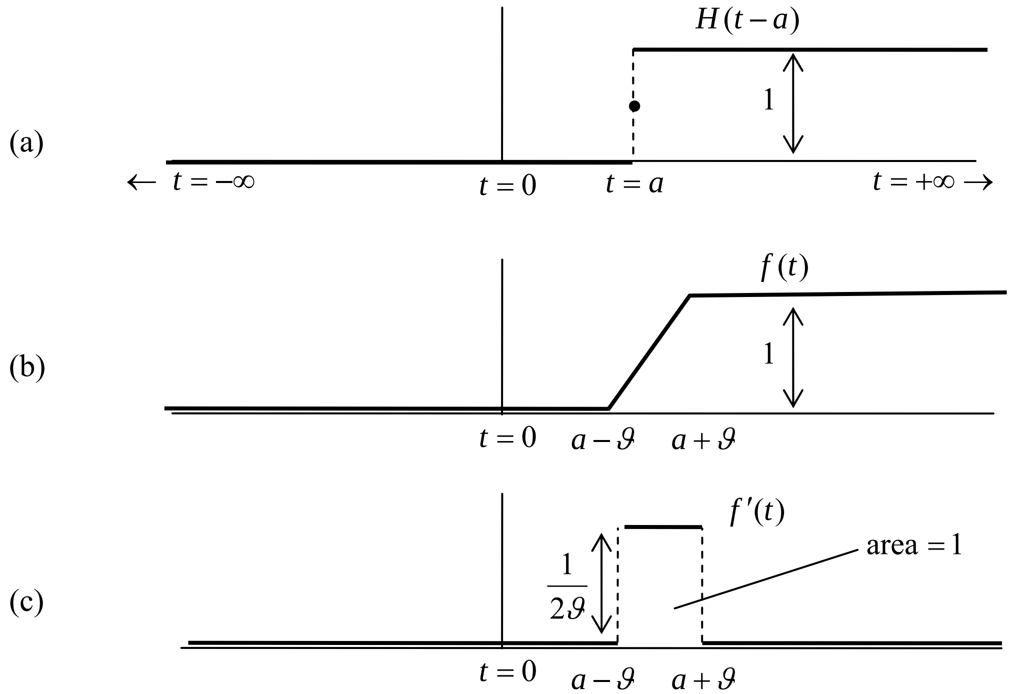


Figure 10.4.6: The Heaviside Step Function and evaluation of its derivative

2. The Maxwell Model: Functions of the Stress

In §10.4.1, the hereditary integrals for the Maxwell model were derived for the stress in terms of integrals of the strain. Here, they are derived for the strain in terms of integrals of the stress.

Consider again the differential equation for the Maxwell model, Eqn. 7.3.6,

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{1}{\eta}\sigma \tag{10.4.29}$$

Direct integration gives

$$\varepsilon(t) = \frac{1}{E}\sigma(t) + \int_{-\infty}^{t} \frac{1}{\eta}\sigma(\tau)d\tau \tag{10.4.30}$$

Integrating by parts leads to

$$\varepsilon(t) = \left(\frac{1}{E} + \frac{t}{\eta}\right)\sigma(t) - \int_{-\infty}^{t} \frac{\tau}{\eta} \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.31}$$

Bringing the first term inside the integral,

$$\varepsilon(t) = \int_{-\infty}^{t} \left(\frac{1}{E} + \frac{t-\tau}{\eta}\right) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.32}$$

or

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.33}$$

where the creep function is $J(t) = 1/E + t/\eta$.

If there is a jump in stress at $t = 0$, 10.4.33 can be expressed as an integral over $[0,t]$ by evaluating the contribution of the jump to the integral in 10.4.33:

$$\begin{aligned}\lim_{\vartheta \to 0} \int_{-\vartheta}^{+\vartheta} \left[\frac{1}{E} + \frac{t-\tau}{\eta}\right] \frac{\sigma(0)}{2\vartheta} d\tau &= \lim_{\vartheta \to 0} \frac{\sigma(0)}{2\vartheta} \left[\frac{1}{E}\tau + \frac{t\tau - \tau^2/2}{\eta}\right]_{-\vartheta}^{+\vartheta} = \lim_{\vartheta \to 0} \frac{\sigma(0)}{2\vartheta} 2\vartheta \left(\frac{1}{E} + \frac{t}{\eta}\right) \\ &= \left(\frac{1}{E} + \frac{t}{\eta}\right)\sigma(0)\end{aligned} \tag{10.4.34}$$

leading to

$$\varepsilon(t) = J(t)\sigma(0) + \int_{0}^{t} J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.35}$$

Alternatively, one could also have simply let $\sigma(t) \to H(t)\sigma(t)$ in 10.4.33, again leading to the term $\int_{-\infty}^{t} J(t-\tau)\sigma(\tau)\delta(\tau) d\tau = J(t)\sigma(0)$.

Finally, integrating by parts, one also has

$$\varepsilon(t) = J(0)\sigma(t) - \int_{0}^{t} S(t-\tau)\sigma(\tau) d\tau \tag{10.4.36}$$

where $S(t) = -dJ(t)/dt$.

3. The Kelvin Model: Functions of the Stress

Consider the differential equation for the Kelvin model, Eqn. 10.3.11,

$$\frac{d\varepsilon}{dt} + \frac{E}{\eta}\varepsilon = \frac{1}{\eta}\sigma \tag{10.4.37}$$

Using the integrating factor $e^{Et/\eta}$, one has

$$\frac{d}{dt}\left(e^{Et/\eta}\varepsilon(t)\right) = \frac{1}{\eta}e^{Et/\eta}\sigma(t) \tag{10.4.38}$$

Integrating both sides over $\left[-\infty, \hat{t}\right]$ gives

$$\left(e^{Et/\eta}\varepsilon\right)_{\hat{t}} - \left(e^{Et/\eta}\varepsilon\right)_{-\infty} = \int_{-\infty}^{\hat{t}} \frac{1}{\eta} e^{Et/\eta}\sigma(t)dt \tag{10.4.39}$$

or

$$\varepsilon(\hat{t}) = \int_{-\infty}^{\hat{t}} \frac{1}{\eta} e^{-E(\hat{t}-t)/\eta}\sigma(t)dt \tag{10.4.40}$$

Changing the notation,

$$\varepsilon(t) = \int_{-\infty}^{t} \frac{1}{\eta} e^{-E(t-\tau)/\eta}\sigma(\tau)d\tau \tag{10.4.41}$$

An integration by parts leads to

$$\varepsilon(t) = \frac{1}{E}\sigma(t) - \int_{-\infty}^{t} \frac{1}{E} e^{-E(t-\tau)/\eta}\frac{d\sigma(\tau)}{d\tau}d\tau \tag{10.4.42}$$

Finally, taking the free term inside the integral:

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \tag{10.4.43}$$

where $J(t) = \left(1 - e^{-Et/\eta}\right)/E$ is the creep compliance function for the Kelvin model.

The other versions of the hereditary integral,. Eqn. 10.4.10, 10.4.14 can be derived from this as before.