10.3b Retardation and Relaxation Spectra

Generalised models can contain many parameters and will exhibit a whole array of relaxation and retardation times. For example, consider two Kelvin units in series, as in the generalised Kelvin chain; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. Using the methods discussed in §10.4-§10.5, it can be shown that he constitutive equation is

$$\sigma + \frac{\eta_1 + \eta_2}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1\eta_2 + E_2\eta_1}{E_1 + E_2}\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_1 + E_2}\ddot{\varepsilon} \qquad (10.3.24)$$

Consider the case of specified stress, so that this is a second order differential equation in $\varepsilon(t)$. The homogeneous solution is {▲Problem 3}

$$\varepsilon_h(t) = Ae^{-t/t_R^1} + Be^{-t/t_R^2} \qquad (10.3.25)$$

where $t_R^1 = \eta_1 / E_1,\, t_R^2 = \eta_2 / E_2$ are the eigenvalues of 10.3.24. For a constant load σ_0, the full solution is {▲Problem 3}

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1}\left(1 - e^{-t/t_R^1}\right) + \frac{1}{E_2}\left(1 - e^{-t/t_R^2}\right)\right] \qquad (10.3.26)$$

Thus, whereas the single Kelvin unit has a single retardation time, Eqn. 10.3.13, this model has two retardation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the creep compliance of the model.

Note that, for constant strain, the model predicts a static response with no stress relaxation (as in the single Kelvin model).

In a similar way, for *N* units, it can be shown that the response of the generalised Kelvin chain to a constant load σ_0 is, neglecting the effect of the free spring/dashpot, of the form

$$\varepsilon(t) = \sigma_0 \sum_{i=1}^{N} \frac{1}{E_i}\left(1 - e^{-t/t_R^i}\right), \quad t_R^i = \frac{\eta_i}{E_i} \qquad (10.3.27)$$

where E_i, η_i are the spring stiffness and dashpot viscosity of Kelvin element i, $i = 1 \ldots N$, Fig. 10.3.9. The response of real materials can be modelled by allowing for a number of different retardation times of different orders of magnitude, e.g. $t_R^i = \{\ldots, 10^{-1}, 1, 10^1, 10^2, \ldots\}$.

If one considers many elements, Eqn. 10.3.27 can be expressed as

$$\varepsilon(t) = \sigma_0 \sum_{i=1}^{N} \Delta\varphi\left(t_R^i\right)\left(1 - e^{-t/t_R^i}\right), \quad \Delta\varphi\left(t_R^i\right) = \frac{1}{\eta_i} t_R^i \qquad (10.3.28)$$

If one is to obtain the same order of magnitude of strain for applied stress, these $\Delta\phi$'s will have to get smaller and smaller for increasing number of Kelvin units. In the limit as $N \to \infty$, letting $d\varphi = (d\varphi / dt_R)dt_R$, one has, changing the dummy variable of integration from dt_R to λ, and letting $\phi(t_R) = d\varphi / dt_R$,

$$\varepsilon(t) = \sigma_0 \int_0^\infty \phi(\lambda)\left(1 - e^{-t/\lambda}\right) d\lambda \tag{10.3.29}$$

The representation 10.3.29 allows for a continuous retardation time, in contrast to the discrete times of the model 10.3.27. The function $\phi(\lambda)$ is called the **retardation spectrum** of the model. Different responses can be modelled by simply choosing different forms for the retardation spectrum.

An alternative form of Eqn. 10.3.29 is often used, using the fact that $d\lambda / d(\ln\lambda) = \lambda$:

$$\varepsilon(t) = \sigma_0 \int_0^\infty \overline{\phi}(\lambda)\left(1 - e^{-t/\lambda}\right) d(\ln\lambda) \tag{10.3.30}$$

where $\overline{\phi} = \lambda\phi$.

A similar analysis can be carried out for the Generalised Maxwell model. For two Maxwell elements in parallel, the constitutive equation can be shown to be

$$\sigma + \frac{E_1\eta_2 + E_2\eta_1}{E_1E_2}\dot{\sigma} + \frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma} = (\eta_1 + \eta_2)\dot{\varepsilon} + \frac{E_1 + E_2}{E_1E_2}\eta_1\eta_2\ddot{\varepsilon} \tag{10.3.31}$$

Consider the case of specified strain, so that this is a second order differential equation in $\sigma(t)$. The homogeneous solution is, analogous to 10.3.25, {▲Problem 4}

$$\sigma_h(t) = Ae^{-t/t_R^1} + Be^{-t/t_R^2} \tag{10.3.32}$$

where again $t_R^1 = \eta_1 / E_1,\ t_R^2 = \eta_2 / E_2$, and are the eigenvalues of 10.3.31. For a constant strain ε_0, the full solution is {▲Problem 4}

$$\sigma(t) = \varepsilon_0\left[E_1e^{-t/t_R^1} + E_2e^{-t/t_R^2}\right] \tag{10.3.33}$$

Thus, whereas the single Maxwell unit has a single relaxation time, Eqn. 10.3.9, this model has two relaxation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the relaxation modulus of the model.

By considering a model with an indefinite number of Maxwell units in parallel, each with vanishingly small elastic moduli ΔE_i, one has the expression analogous to 10.3.29,

$$\sigma(t) = \varepsilon_0 \int_0^{\infty} \vartheta(t_R) e^{-t/t_R} dt_R \tag{10.3.34}$$

and $\vartheta(t_R)$ is called the **relaxation spectrum** of the model.

To complete this section, note that, for the two Maxwell units in parallel, a constant stress σ_0 leads to the creep strain {▲Problem 5}

$$\varepsilon(t) = \sigma_0 \left[\frac{1}{E_1 + E_2} e^{-t/t_R} + \left(\frac{\eta_1 / E_1 + \eta_2 / E_2}{\eta_1 + \eta_2} - \frac{t_R}{\eta_1 + \eta_2} \right)\left(1 - e^{-t/t_R}\right) + \frac{t}{\eta_1 + \eta_2} \right],$$
$$t_R = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2} \frac{E_1 + E_2}{E_1 E_2} \tag{10.3.35}$$

Problems

3. Consider two Kelvin units in series, as in the generalised Kelvin chain; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. The constitutive equation is given by Eqn. 10.3.24.
   (a) The homogeneous equation is of the form $A\ddot{\varepsilon} + B\dot{\varepsilon} + C\varepsilon = 0$. By considering the characteristic equation $A\lambda^2 + B\lambda + C = 0$, show that the eigenvalues are $\lambda = -E_1 / \eta_1, -E_2 / \eta_2$ and hence that the homogeneous solution is 10.3.25.
   (b) Consider now a constant load σ_0. Show that the particular solution is $\varepsilon(t) = \sigma_0 (E_1 + E_2)/ E_1 E_2$.
   (c) One initial condition of the problem is that $\varepsilon(0) = 0$. The second condition results from the fact that only the dashpots react at time $t = 0$ (equivalently, one can integrate the constitutive equation across $t = 0$ as in the footnote in §10.3.2). Show that this condition leads to $\dot{\varepsilon}(0) = \sigma_0 (\eta_1 + \eta_2)/ \eta_1 \eta_2$.
   (d) Use the initial conditions to show that the constants in 10.3.24 are given by $A = -\sigma_0 / E_1, B = -\sigma_0 / E_2$ and hence that the complete is given by 10.3.26.
   (e) Consider again the constitutive equation 10.3.24. What values do the constants E_2, η_2 take so that it reduces to the single Kelvin model, Eqn. 10.3.11.

4. Consider two Maxwell units in parallel, as in the generalised Maxwell model; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. The constitutive equation is given by Eqn. 10.3.31.
   (a) Suppose we have a prescribed strain history and we want to determine the stress. The homogeneous equation is of the form $A\ddot{\sigma} + B\dot{\sigma} + C\sigma = 0$. By considering the characteristic equation $A\lambda^2 + B\lambda + C = 0$, show that the eigenvalues are $t_R^1 = \eta_1 / E_1, t_R^2 = \eta_2 / E_2$ and hence that the homogeneous solution is 10.3.32.
   (b) Consider now a constant load ε_0. Show that the particular solution is zero.
   (c) One initial condition results from the fact that only the springs react at time $t = 0$, which leads to the condition $\sigma(0) = \varepsilon_0 (E_1 + E_2)$. A second condition can be

obtained by integrating the constitutive equation across $t = 0$ as in the footnote in §10.3.2. Show that this leads to the condition $\dot{\sigma}(0^+) = -\left(E_1^2/\eta_1 + E_2^2/\eta_2\right)\varepsilon_0$.

(d) Use the initial conditions to show that the constants in 10.3.32 are given by $A = E_1\varepsilon_0$, $B = E_2\varepsilon_0$ and hence that the complete solution is given by 10.3.33.

(e) Consider again the constitutive equation 10.3.31. What values do the constants E_2, η_2 take so that it reduces to the single Maxwell model, Eqn. 10.3.6.

5. Consider again the two Maxwell units in parallel, as in Problem 4. This time consider a stress-driven problem.

(a) From the constitutive equation 10.3.31, the differential equation to be solved is of the form $A\ddot{\varepsilon} + B\dot{\varepsilon} = \ldots$. By considering the characteristic equation $A\lambda^2 + B\lambda = 0$, show that the eigenvalues are

$$\lambda_1 = 0, \; \lambda_2 = -\frac{(\eta_1 + \eta_2)E_1E_2}{\eta_1\eta_2(E_1 + E_2)}$$

and hence that the homogeneous solution is $\varepsilon(t) = C_1 + C_2 e^{-t/t_R}$ where $t_R = -1/\lambda_2$.

(b) Consider now a constant stress σ_0. By using the condition that only the springs react at time $t = 0$, show that the particular solution is $\sigma_0 t/(\eta_1 + \eta_2)$.

(c) One initial condition results from the fact that only the springs react at time $t = 0$, which leads to the condition $\varepsilon(0) = \sigma_0/(E_1 + E_2)$. A second condition can be obtained by integrating the constitutive equation across $t = 0$ as in the footnote in §10.3.2. Be careful to consider all terms in 10.3.31. Show that this leads to the condition

$$\dot{\varepsilon}(0^+) = (\sigma_0/t_R)\left[(\eta_1/E_1 + \eta_2/E_2)/(\eta_1 + \eta_2) - 1/(E_1 + E_2)\right].$$

(d) Use the initial conditions to show that the complete solution is given by 10.3.35.