

## 10.3 Rheological Models

In this section, a number of one-dimensional linear viscoelastic models are discussed.

### 10.3.1 Mechanical (rheological) models

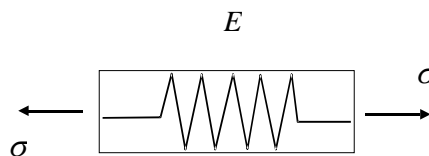
The word viscoelastic is derived from the words "viscous" + "elastic"; a viscoelastic material exhibits both viscous and elastic behaviour – a bit like a fluid and a bit like a solid. One can build up a model of linear viscoelasticity by considering combinations of the linear elastic spring and the linear viscous dash-pot. These are known as **rheological models** or **mechanical models**.

#### The Linear Elastic Spring

The constitutive equation for a material which responds as a linear elastic spring of stiffness  $E$  is (see Fig. 10.3.1)

$$\varepsilon = \frac{1}{E} \sigma \quad (10.3.1)$$

The response of this material to a creep-recovery test is to undergo an instantaneous elastic strain upon loading, to maintain that strain so long as the load is applied, and then to undergo an instantaneous de-straining upon removal of the load.



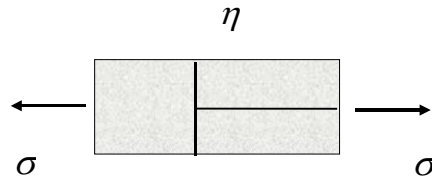
**Figure 10.3.1: the linear elastic spring**

#### The Linear Viscous Dash-pot

Imagine next a material which responds like a viscous dash-pot; the dash-pot is a piston-cylinder arrangement, filled with a viscous fluid, Fig. 10.3.2 – a strain is achieved by dragging the piston through the fluid. By definition, the dash-pot responds with a strain-rate proportional to stress:

$$\dot{\varepsilon} = \frac{1}{\eta} \sigma \quad (10.3.2)$$

where  $\eta$  is the **viscosity** of the material. This is the typical response of many **fluids**; the larger the stress, the faster the straining (as can be seen by pushing your hand through water at different speeds).

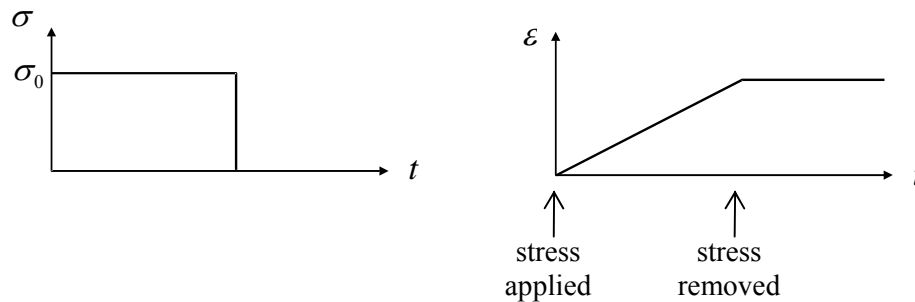


**Figure 10.3.2: the linear dash-pot**

The strain due to a suddenly applied load  $\sigma_o$  may be obtained by integrating the constitutive equation 10.3.2. Assuming zero initial strain, one has

$$\varepsilon = \frac{\sigma_o}{\eta} t \quad (10.3.3)$$

The strain is seen to increase linearly and without bound so long as the stress is applied, Fig. 10.3.3. Note that there is no movement of the dash-pot at the onset of load; it takes time for the strain to build up. When the load is removed, there is no stress to move the piston back through the fluid, so that any strain built up is permanent. The slope of the creep-line is  $\sigma_o / \eta$ .



**Figure 10.3.3: Creep-Recovery Response of the Dash-pot**

The relationship between the stress and strain during the creep-test may be expressed in the form

$$\varepsilon(t) = \sigma_o J(t), \quad J(t) = \frac{t}{\eta} \quad (10.3.4)$$

$J$  here is called the **creep (compliance) function** ( $J = 1/E$  for the elastic spring).

### 10.3.2 The Maxwell Model

Consider next a spring and dash-pot in series, Fig. 10.3.4. This is the **Maxwell model**. One can divide the total strain into one for the spring ( $\varepsilon_1$ ) and one for the dash-pot ( $\varepsilon_2$ ).

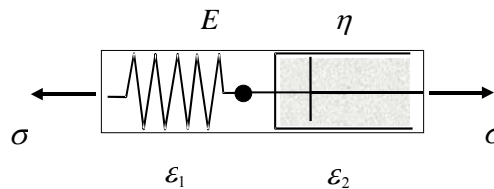
Equilibrium requires that the stress be the same in both elements. One thus has the following three equations in four unknowns<sup>1</sup>:

$$\varepsilon_1 = \frac{1}{E}\sigma, \quad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \quad \varepsilon = \varepsilon_1 + \varepsilon_2 \quad (10.3.5)$$

To eliminate  $\varepsilon_1$  and  $\varepsilon_2$ , differentiate the first and third equations, and put the first and second into the third:

$$\boxed{\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}} \quad \text{Maxwell Model} \quad (10.3.6)$$

This constitutive equation has been put in what is known as **standard form** – stress on left, strain on right, increasing order of derivatives from left to right, and coefficient of  $\sigma$  is 1.



**Figure 10.3.4: the Maxwell Model**

### Creep-Recovery Response

Consider now a creep test. Physically, when the Maxwell model is subjected to a stress  $\sigma_0$ , the spring will stretch immediately and the dash-pot will take time to react. Thus the initial strain is  $\varepsilon(0) = \sigma_0 / E$ . Using this as the initial condition, an integration of 10.3.6 (with a zero stress-rate<sup>2</sup>) leads to

<sup>1</sup> If one considers an actual spring of length  $L_s$  and a dashpot of length  $L_d$  as in Fig. 10.3.4, and corresponding elongations  $d_1$  and  $d_2$  due to strains  $\varepsilon_1$  and  $\varepsilon_2$ , the total elongation would be  $\varepsilon(L_s + L_d) = \varepsilon_s L_s + \varepsilon_d L_d$ , which is not quite the same as Eqn. 10.3.5c. It is best to think of the Maxwell model as: the total strain at a material particle can be decomposed additively according to Eqn. 10.3.5c, with the separate strains being linear elastic and viscous; Fig. 10.3.4 is simply an attempt to visualise this concept.

<sup>2</sup> there is a jump in stress from zero to  $\sigma_0$  when the load is applied, implying an infinite stress-rate  $\dot{\sigma}$ .

One is not really interested in this jump here because the corresponding jump in strain can be predicted from the physical response of the spring. One is more interested in what happens just "after" the load is applied. In that sense, when one speaks of initial strains and stress-rates, one means their values at  $0^+$ , just after  $t = 0$ ; the stress-rate is zero from  $0^+$  on. To be more precise, one can deal with the sudden jump in stress by integrating the constitutive equation across the point  $t = 0$  as follows:

$$\begin{aligned} (E/\eta) \int_{-\Delta\tau}^{+\Delta\tau} \sigma(t) dt + \int_{-\Delta\tau}^{+\Delta\tau} \dot{\sigma}(t) dt &= E \int_{-\Delta\tau}^{+\Delta\tau} \dot{\varepsilon}(t) dt \\ \rightarrow (E/\eta) \int_{-\Delta\tau}^{+\Delta\tau} \sigma(t) dt + [\sigma(+\Delta\tau) - \sigma(-\Delta\tau)] &= E [\varepsilon(+\Delta\tau) - \varepsilon(-\Delta\tau)] \end{aligned}$$

In the limit as  $\Delta\tau \rightarrow 0$ , the integral tends to zero ( $\sigma$  is finite), the values of stress and strain at  $0^-$ , i.e. in the limit as  $\Delta\tau \rightarrow 0$  from the left, are zero. All that remains are the values to the right, giving

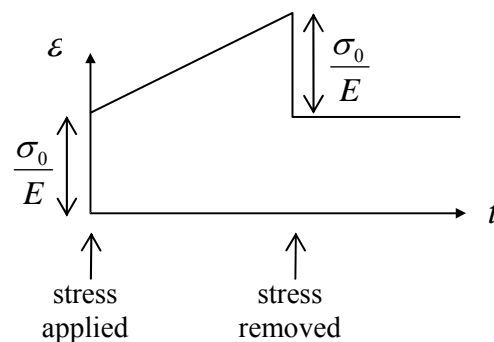
$$\begin{aligned}\dot{\varepsilon} &= \frac{\sigma_o}{\eta} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{\eta}t + C \\ &\quad \rightarrow \quad \varepsilon(t) = \sigma_o \left( \frac{1}{\eta}t + \frac{1}{E} \right)\end{aligned}\tag{10.3.7}$$

The creep-response can again be expressed in terms of a creep compliance function:

$$\varepsilon(t) = \sigma_o J(t) \quad \text{where} \quad J(t) = \frac{t}{\eta} + \frac{1}{E}\tag{10.3.8}$$

When the load is removed, the spring again reacts immediately, but the dash-pot has no tendency to recover. Hence there is an immediate elastic recovery  $\sigma_o / E$ , with the creep strain due to the dash-pot remaining. The full creep and recovery response is shown in Fig. 10.3.5.

The Maxwell model predicts creep, but not of the ever-decreasing strain-rate type. There is no anelastic recovery, but there is the elastic response and a permanent strain.



**Figure 10.3.5: Creep-Recovery Response of the Maxwell Model**

### Stress Relaxation

In the stress relaxation test, the material is subjected to a constant strain  $\varepsilon_0$  at  $t = 0$ . The Maxwell model then leads to {▲Problem 1}

$$\sigma(t) = \varepsilon_0 E(t) \quad \text{where} \quad E(t) = E e^{-t/t_R}, \quad t_R = \frac{\eta}{E}\tag{10.3.9}$$

Analogous to the creep function  $J$  for the creep test,  $E(t)$  is called the **relaxation modulus** function.

$\sigma(0^+) = E\varepsilon(0^+)$ , as expected. One can deal with this sudden behaviour more easily using integral formulations or with the Laplace Transform (see §10.4, §10.5)

The parameter  $t_R$  is called the **relaxation time** of the material and is a measure of the time taken for the stress to relax; the shorter the relaxation time, the more rapid the stress relaxation.

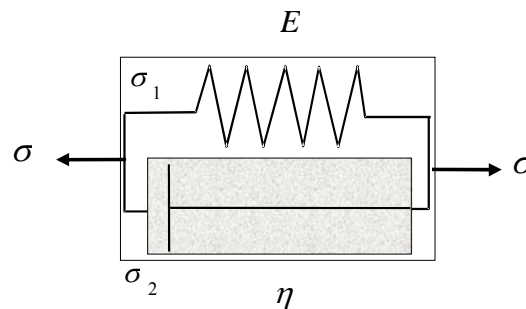
### 10.3.3 The Kelvin (Voigt) Model

Consider next the other two-element model, the **Kelvin** (or **Voigt**) **model**, which consists of a spring and dash-pot in parallel, Fig. 10.3.6. It is assumed there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash-pot. This time,

$$\varepsilon = \frac{1}{E} \sigma_1, \quad \dot{\varepsilon} = \frac{1}{\eta} \sigma_2, \quad \sigma = \sigma_1 + \sigma_2 \quad (10.3.10)$$

where  $\sigma_1$  is the stress in the spring and  $\sigma_2$  is the dash-pot stress. Eliminating  $\sigma_1, \sigma_2$  leaves the constitutive law

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}} \quad \text{Kelvin (Voigt) Model} \quad (10.3.11)$$



**Figure 10.3.6: the Kelvin (Voigt) Model**

#### Creep-Recovery Response

If a load  $\sigma_0$  is applied suddenly to the Kelvin model, the spring will want to stretch, but is held back by the dash-pot, which cannot react immediately. Since the spring does not change length, the stress is initially taken up by the dash-pot. The creep curve thus starts with an initial slope  $\sigma_0 / \eta$ .

Some strain then occurs and so some of the stress is transferred from the dash-pot to the spring. The slope of the creep curve is now  $\sigma_2 / \eta$ , where  $\sigma_2$  is the stress in the dash-pot, with  $\sigma_2$  ever-decreasing. In the limit when  $\sigma_2 = 0$ , the spring takes all the stress and thus the maximum strain is  $\sigma_0 / E$ .

Solving the first order non-homogeneous differential equation 10.3.11 with the initial condition  $\varepsilon(0) = 0$  gives

$$\varepsilon(t) = \frac{\sigma_o}{E} \left(1 - e^{-(E/\eta)t}\right) \quad (10.3.12)$$

which agrees with the above physical reasoning; the creep compliance function is now

$$J(t) = \frac{1}{E} \left(1 - e^{-t/t_R}\right), \quad t_R = \frac{\eta}{E} \quad (10.3.13)$$

The parameter  $t_R$ , in contrast to the relaxation time of the Maxwell model, is here called the **retardation time** of the material and is a measure of the time taken for the creep strain to accumulate; the shorter the retardation time, the more rapid the creep straining.

When the Kelvin model is unloaded, the spring will want to contract but again the dash pot will hold it back. The spring will however eventually pull the dash-pot back to its original zero position given time and full recovery occurs.

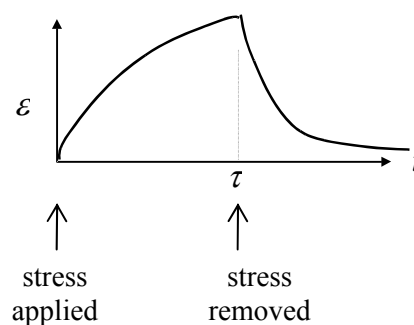
Suppose the material is unloaded at time  $t = \tau$ . The constitutive law, with zero stress, reduces to  $0 = E\varepsilon + \eta\dot{\varepsilon}$ . Solving leads to

$$\varepsilon(t) = Ce^{-(E/\eta)t} \quad (10.3.14)$$

where  $C$  is a constant of integration. The  $t$  here is measured from the point where "zero load" begins. If one wants to measure time from the onset of load,  $t$  must be replaced with  $t - \tau$ . From Eqn. 10.3.12, the strain at  $t = \tau$  is  $\varepsilon(\tau) = (\sigma_o / E)(1 - e^{-(E/\eta)\tau})$ . Using this as the initial condition, one finds that

$$\varepsilon(t) = \frac{\sigma_o}{E} e^{-(E/\eta)t} \left(e^{(E/\eta)\tau} - 1\right), \quad t > \tau \quad (10.3.15)$$

The creep and recovery response is shown in Fig. 10.3.7. There is a transient-type creep and anelastic recovery, but no instantaneous or permanent strain.



**Figure 10.3.7: Creep-Recovery Response of the Kelvin (Voigt) Model**

## Stress Relaxation

Consider next a stress-relaxation test. Setting the strain to be a constant  $\varepsilon_0$ , the constitutive law 10.3.11 reduces to  $\sigma = E\varepsilon_0$ . Thus the stress is taken up by the spring and is constant, so there is in fact no stress relaxation over time. Actually, in order that the Kelvin model undergoes an instantaneous strain of  $\varepsilon_0$ , an infinite stress needs to be applied, since the dash-pot will not respond instantaneously to a finite stress<sup>3</sup>.

### 10.3.4 Three – Element Models

The Maxwell and Kelvin models are the simplest viscoelastic models. More realistic material responses can be modelled using more elements. The four possible three-element models are shown in Fig. 10.3.8 below. The models of Fig. 10.3.8a-b are referred to as “solids” since they react instantaneously as elastic materials and recover completely upon unloading. The models of Figs. 10.3.8c-d are referred to as “fluids” since they involve dashpots at the initial loading phase and do not recover upon unloading.

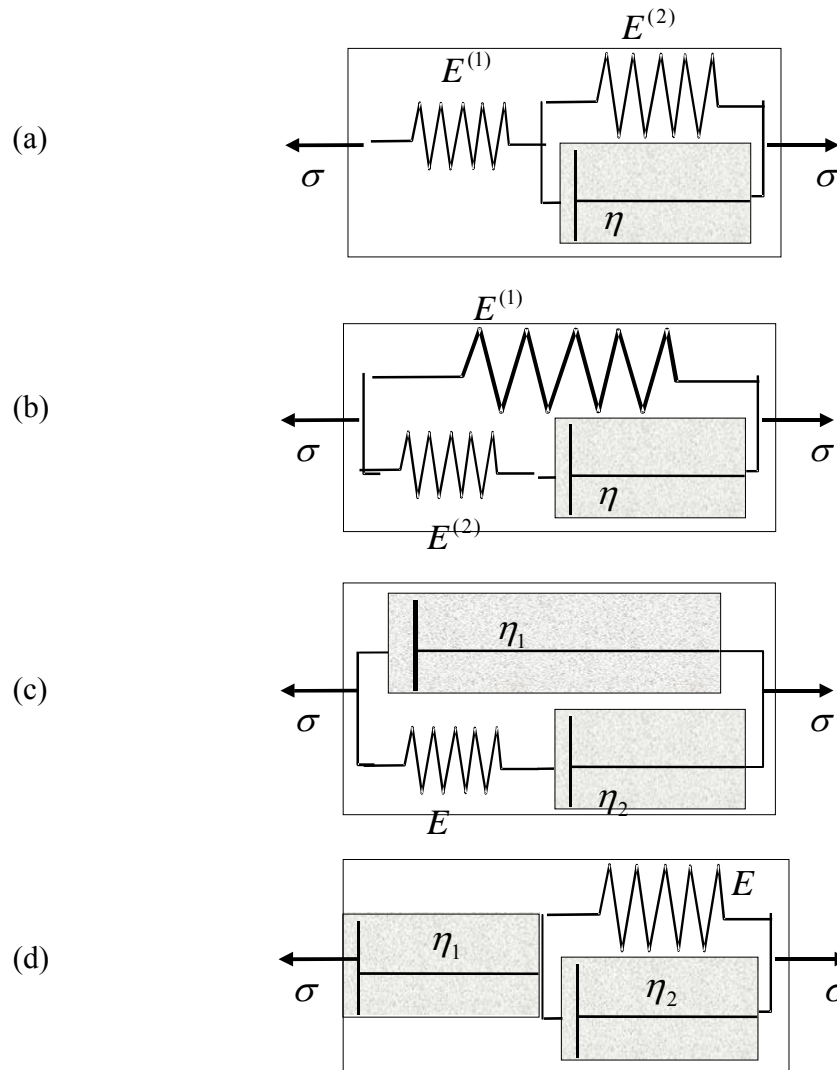
The differential constitutive relations for the Maxwell and Kelvin models were not to difficult to derive. However, even with three elements, deriving them can be a difficult task. This is because one needs to eliminate variables from a set of equations, one or more of which is a differential equation (for example, see 10.3.5). The task is more easily accomplished using integral formulations and the Laplace transform, which are discussed in §10.4-§10.5.

Only results are given here: the constitutive relations for the four models shown in Fig. 10.3.8 are

$$\begin{array}{l}
 \text{(a)} \quad \sigma + \frac{\eta}{E_1 + E_2} \dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2} \varepsilon + \frac{\eta E_1}{E_1 + E_2} \dot{\varepsilon} \\
 \text{(b)} \quad \sigma + \frac{\eta}{E_2} \dot{\sigma} = E_1 \varepsilon + \frac{\eta(E_1 + E_2)}{E_2} \dot{\varepsilon} \\
 \text{(c)} \quad \sigma + \frac{\eta_2}{E} \dot{\sigma} = (\eta_1 + \eta_2) \dot{\varepsilon} + \frac{\eta_1 \eta_2}{E} \ddot{\varepsilon} \\
 \text{(d)} \quad \sigma + \frac{\eta_1 + \eta_2}{E} \dot{\sigma} = \eta_1 \dot{\varepsilon} + \frac{\eta_1 \eta_2}{E} \ddot{\varepsilon}
 \end{array} \tag{10.3.16}$$

The response of these models can be determined by specifying stress (strain) and solving the differential equations 10.3.16 for strain (stress).

<sup>3</sup> the stress required is  $\sigma(0) = \eta \varepsilon_0 \delta(t)$ , where  $\delta(t)$  is the Dirac delta function (this can be determined using the integral representations of §10.4)



**Figure 10.3.8: Three-element Models: (a) Standard Solid I, (b) Standard Solid II, (c) Standard Fluid I, (d) Standard Fluid II**

### 10.3.5 The Creep Compliance and the Relaxation Modulus

The creep compliance function and the relaxation modulus have been mentioned in the context of the two-element models discussed above. More generally, they are defined as follows: the creep compliance is the strain due to unit stress:

$$\boxed{\varepsilon(t) = \sigma_o J(t), \quad \varepsilon(t) = J(t) \text{ when } \sigma_o = 1} \quad \text{Creep Compliance} \quad (10.3.17)$$

The relaxation modulus is the stress due to unit strain:

$$\boxed{\sigma(t) = \varepsilon_o E(t), \quad \sigma(t) = E(t) \text{ when } \varepsilon_o = 1} \quad \text{Relaxation Modulus} \quad (10.3.18)$$

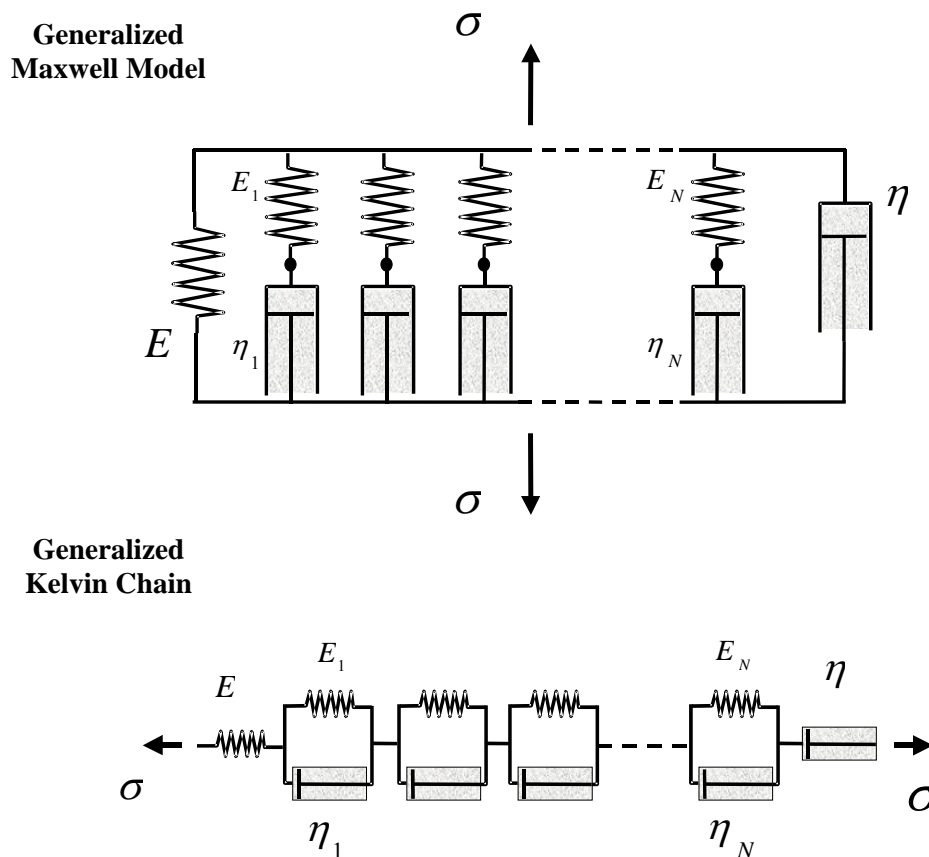
Whereas the creep function describes the response of a material to a creep test, the relaxation modulus describes the response to a stress-relaxation test.



### 10.3.6 Generalized Models

More complex models can be constructed by using more and more elements. A complex viscoelastic rheological model will usually be of the form of the **generalized Maxwell model** or the **generalized Kelvin chain**, shown in Fig. 10.3.9. The generalized Maxwell model consists of  $N$  different Maxwell units in parallel, each unit with different parameter values. The absence of the isolated spring would ensure fluid-type behaviour, whereas the absence of the isolated dash-pot would ensure an instantaneous response. The generalised Kelvin chain consists of a chain of Kelvin units and again the isolated spring may be omitted if a fluid-type response is required.

In general, the more elements one has, the more accurate a model will be in describing the response of real materials. That said, the more complex the model, the more material parameters there are which need to be evaluated by experiment – the determination of a large number of material parameters might be a difficult, if not an impossible, task.



**Figure 10.3.9: Generalised Viscoelastic Models**

In general, a linear viscoelastic constitutive equation will be of the general form

$$p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + p_3\dddot{\sigma} + p_4\sigma^{(IV)} + \dots = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + q_3\dddot{\varepsilon} + q_4\varepsilon^{(IV)} + \dots \quad (10.3.19)$$

The more elements (springs/dashpots) one uses, the higher the order of the differential equation.

Eqn. 10.3.19 is sometimes written in the short-hand notation

$$\mathbf{P}\sigma = \mathbf{Q}\varepsilon \quad (10.3.20)$$

where  $\mathbf{P}$  and  $\mathbf{Q}$  are the linear differential operators

$$\mathbf{P} = \sum_{i=0}^n p_i \frac{\partial^i}{\partial t^i}, \quad \mathbf{Q} = \sum_{i=0}^n q_i \frac{\partial^i}{\partial t^i} \quad (10.3.21)$$

A viscoelastic model can be created by simply entering values for the coefficients  $p_i$ ,  $q_i$ , in 10.3.19, without referring to any particular rheological spring – dashpot arrangement. In that sense, springs and dashpots are not necessary for a model, all one needs is a differential equation of the form 10.3.19. However, the use of springs and dashpots is helpful as it gives one a physical feel for the way a material might respond, rather than simply using an abstract mathematical expression such as 10.3.19.

### 10.3.7 Non-Linear Models

More realistic material responses can be achieved by using non-linear models. For example, the springs of the previous section can be replaced with more general non-linear stress-strain relations of the form:

$$\sigma = E\varepsilon^n \quad (10.3.22)$$

Various non-linear expressions for dash-pots can also be used, for example,

$$\sigma = A\varepsilon^n \dot{\varepsilon}, \quad \sigma = Ae^{-\varepsilon/\varepsilon_0} \dot{\varepsilon} \quad (10.3.23)$$

Material data will certainly be better matched by such non-linear expressions; however, of course, they will result in non-linear differential equations which will be more difficult to solve than their linear counterparts.

### 10.3.8 Problems

1. Derive the Relaxation Modulus  $E(t)$  for the Maxwell material.
2. What are the values of the coefficients  $p_i$ ,  $q_i$  in the general differential equation 10.3.19 for
  - (a) the Maxwell model and the Kelvin model?
  - (b) The three-element models