9.1 Failure of Elastic Materials

In terms of material behavior, **failure** means *a change in the normal constitutive behavior* of a material, usually in response to excessive loads or deformations that cause irreparable changes to the microstructure. For example, compressed rock will respond elastically up to a certain point but, if the load is high enough, the rock will crush with permanent deformations (see Fig. 5.2.10b). A model of crushing rock will involve a non-elastic constitutive law and is hence beyond the scope of elasticity theory. However, at issue here is the attempt to predict when the material first ceases to respond elastically, not what happens after it does so. The failure of a specimen of rock under uniaxial tension[1] can be predicted if a tension test has been carried out on a similar rock – it will fail when the applied tension reaches the yield strength (see §5.2.1). However, the question to be addressed here is how to predict the failure of a component which is loaded in a complex way, with a consequent complex three-dimensional state of stress state at any material particle.

The theory of **stress modulated failure** assumes that failure occurs once some function of the stresses reaches some critical value. This function of the stress, or **stress metric**, might be the maximum principal stress, the maximum shear stress or some more complicated function of the stress components. Once the stress metric exceeds the critical value, the material no longer behaves elastically.

This section examines the case of failure being a transition from elastic to inelastic behaviour. Another type of failure, which occurs before this transition point is reached, is the case of buckling of a column, considered in section 7.5.

9.1.1 Failure Theories

Three theories of material failure will be discussed in what follows. They are used principally in predicting the failure of the (hard) engineering materials, but can be used or modified for many other types of material.

1. Maximum Principal Stress Theory

Consider a very brittle material, such as a ceramic or glass, or cold metal. Such a material will fracture with a "clean break". If there is no permanent deformation, the stress-strain curve in a tension test up to the point of failure will look something like that in Fig. 9.1.1. The clean break can be hypothesised to be brought about by the stresses acting normal to the fracture surface, as sketched in Fig. 9.1.1a; when these stresses reach the failure stress σ_f, the material "breaks".

Using this argument for a complex three-dimensional component, Fig. 9.1.1b, one can hypothesise that the material will fail at any location where the normal stress of largest magnitude, i.e. the maximum principal stress σ_1, reaches the appropriate failure stress σ_f of the material:

[1] i.e. acting along one axis, so one-dimensional

$$\sigma_1 = \sigma_f \tag{9.1.1}$$

Assuming the material fails similarly in tension as in compression, one can write $|\sigma_1| = \sigma_f$.

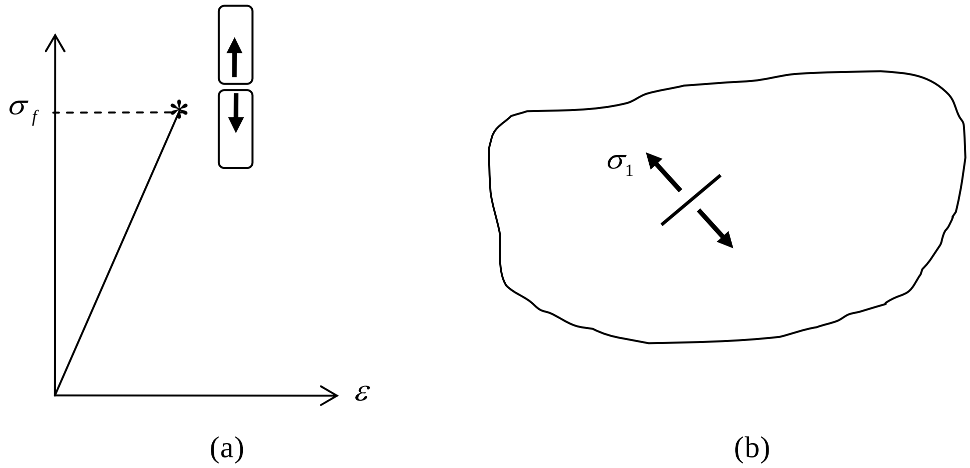


Figure 9.1.1: Brittle failure; (a) stress-strain curve in a tension test, (b) fracture occurring in a three-dimensional component

2. The Maximum Shear Stress (Tresca) Theory

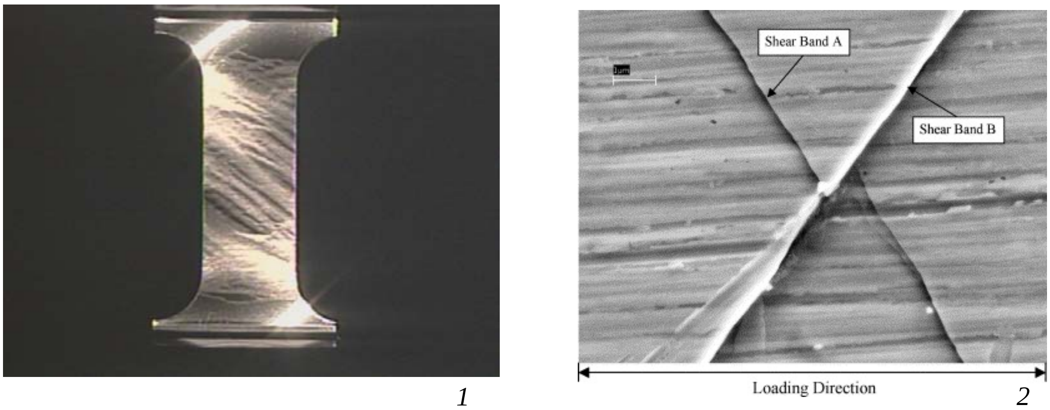
Consider now a different type of failure. Instead of the clean fracture described above, the material deforms in a more complex way, and develops visible deformation bands. The bands are called **Lüders** (**slip**) **bands** (in the case of ductile steels and other metals) and **shear bands** (in the case of more brittle materials). These bands appear at roughly 45 degrees to the direction of loading. Lüders bands in a steel and shear bands in an amorphous Zirconium alloy[2] after tensile testing are shown in Fig. 9.1.2.

This evidence points to a shearing-type failure, with the metal sheared along these bands/planes to failure. The hypothesis here, then, is that the material fails when the shear stress acting on these planes is large enough to shear the material along these planes. In the tension test, let the applied tension be Y at the point when the elastic limit (or the yield stress) is reached, Fig. 9.1.3a. The shear stress in the specimen is given by Eqn. 3.3.1 and the maximum shear stress $\tau_{\max} = Y/2$ occurs at 45 degrees to the direction of loading.

Using this argument for a complex three-dimensional component, Fig. 9.1.3b, one can hypothesise that the material will fail at any location where the maximum shear stress reaches one half the tensile yield stress Y of the material (see Eqns. 3.5.9. *et seq.*):

$$\max\left(|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\right) = Y \tag{9.1.2}$$

[2] these metals have a non-crystalline disordered molecular structure, like a glass, and behave in a more brittle fashion than the standard crystalline metals



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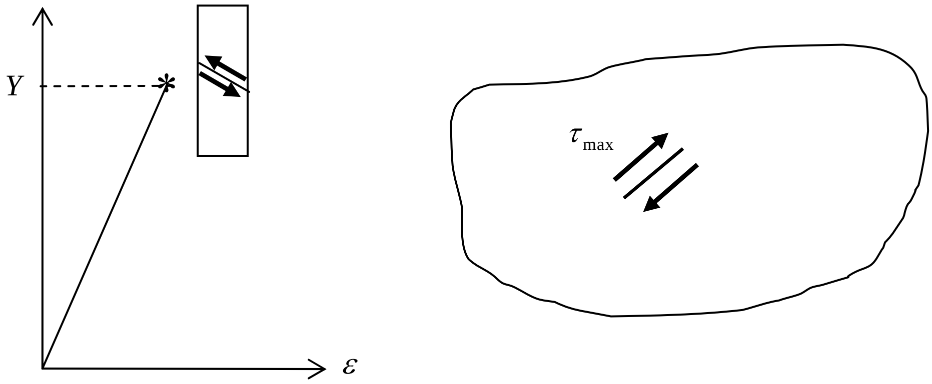
(a) (b)

Figure 9.1.2: Tension test; (a) Lüders bands in a steel, (b) shear bands in an amorphous metal



(a) (b)

Figure 9.1.3: Failure through shear; (a) stress-strain curve in a tension test, (b) shear failure in a three-dimensional component

3. The Von Mises Theory

The **Von Mises theory** predicts that failure of a material subjected to any state of stress occurs when the following expression, involving the sum of the squares of the differences between the principal stresses, is satisfied

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}=Y \qquad (9.1.3)$$

where Y is the yield stress in a tension test. Although it appears quite different, this criterion is very similar to the Tresca criterion – they both give similar predictions, the Tresca criterion being slightly more conservative, i.e. the Tresca criterion will predict that a material will fail at a lower stress than the Von Mises.

The Von Mises criterion is usually used in preference to the Tresca criterion; one reason is that one does not have to deal with the cumbersome absolute signs in Eqn. 9.1.2, and also the Von Mises criterion is usually more accurate, particularly for ductile metals.

The Von Mises criterion was really developed from theoretical grounds and "works". Well after the criterion was proposed, the following physical interpretations were made: the first is that the quantity on the left hand side of Eqn. 9.1.3 is proportional to the **octahedral shear stress** $\tau_{\rm oct}$ (Eqn. 9.1.3 can be expressed as $\frac{3}{\sqrt{2}}\tau_{\rm oct} = Y$). This is the shear stress acting on planes which make equal angles to the principal stress axes, as shown in Fig. 9.1.4.

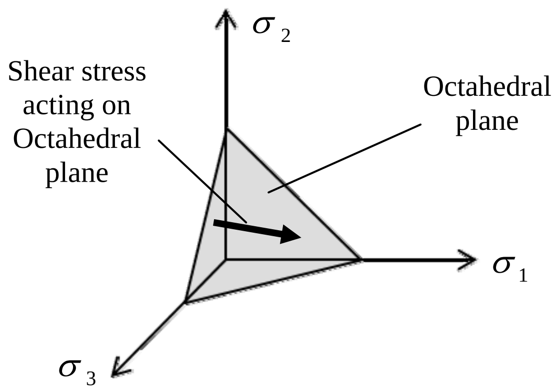


Figure 9.1.4: Interpretation of the Von Mises criterion as the shear stress acting on the Octahedral Plane

A second interpretation is that, when a linear elastic material deforms, its deformation can be decomposed into the addition of a pure volume change (as in a uniform pressure) and a distortion (change in shape). The strain energy in a deforming material subject to an arbitrary stress state can be found from Eqn. 8.2.19. Expressing these equations in terms of principal stresses, one can show that the left hand side of Eqn. 9.1.3 is proportional to the distortional component of strain energy u_d (Eqn. 9.1.3 can be expressed as $\sqrt{3Eu_d / (1+\nu)} = Y$).

Graphical Interpretation of the Failure Theories

The three failure theories can conveniently be displayed on a single graph. Assuming plane stress conditions, with $\sigma_3 = 0$, the two principal stresses σ_1 and σ_2 can be plotted against each other in what is known as **stress space**, Fig. 9.1.5. Each closed curve is called the **failure locus** of the associated failure theory. When the stress state is such that the point (σ_1, σ_2) lies inside the locus, then the material remains elastic. If the stress state is such that (σ_1, σ_2) reaches the locus, then the failure criterion is satisfied and failure occurs.

As mentioned, what happens once failure occurs is beyond elasticity theory. "Beyond failure" of ductile metals and other materials which undergo plasticity is examined in Chapter 11.

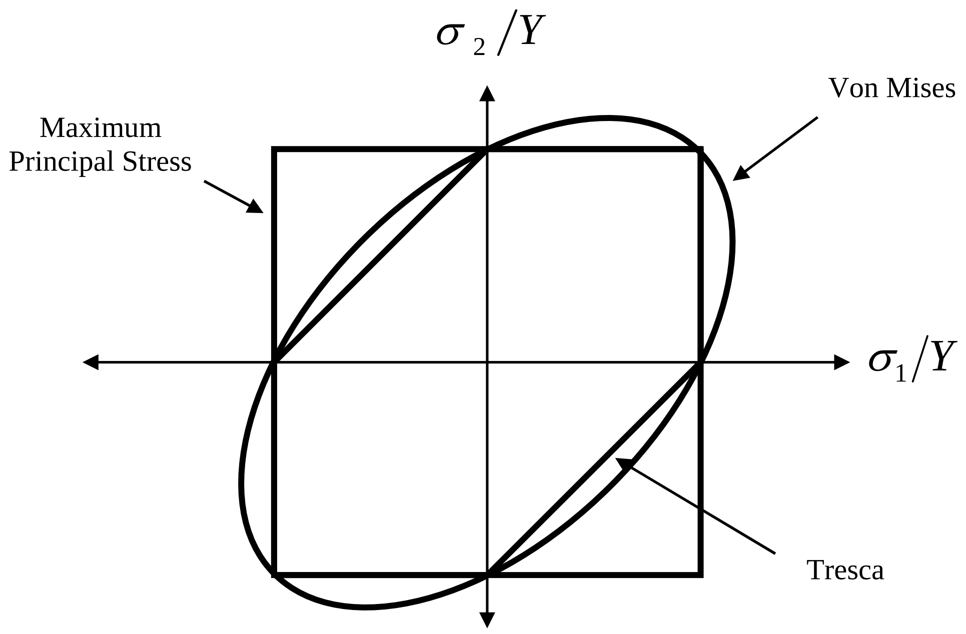


Figure 9.1.5: Failure theories in stress space

Images used:
1. http://vimeo.com/4586024
2. Yang B *et al*, Temperature evolution during fatigue damage, Intermetallics, 13(3-4), 419-428, 2005.