7.5 Elastic Buckling

The initial theory of the buckling of columns was worked out by Euler in 1757, a nice example of a theory preceding the application, the application mainly being for the later "invented" metal and concrete columns in modern structures.

7.5.1 Columns and Buckling

A **column** is a long slender bar under axial compression, Fig. 7.5.1. A column can be horizontal, vertical or inclined; in the latter cases it is termed a **strut**.

The column under axial compression responds elastically in exactly the same way as the axial bar of §7.1. For example, it decreases in length under a compressive force P by an amount given by Eqn. 7.1.5, $\Delta = PL/EA$. However, when the compressive force is large enough, the column will **buckle** with lateral deflection. This possibility is the subject of this section.

Euler's Theory of Buckling

Consider an elastic column of length L, pin-ended so free to rotate at its ends, subjected to an axial load P, Fig. 7.5.1. Assume that it undergoes a lateral deflection denoted by v. Moment equilibrium of a section of the deflected column cut at a typical point x, and using the moment-curvature Eqn. 7.4.37, results in

$$-Pv(x) = M(x) = EI\frac{d^2v}{dx^2} \tag{7.5.1}$$

Hence the deflection v satisfies the differential equation

$$\frac{d^2v}{dx^2} + k^2v(x) = 0 \tag{7.5.2}$$

where

$$k^2 = \frac{P}{EI} \tag{7.5.3}$$

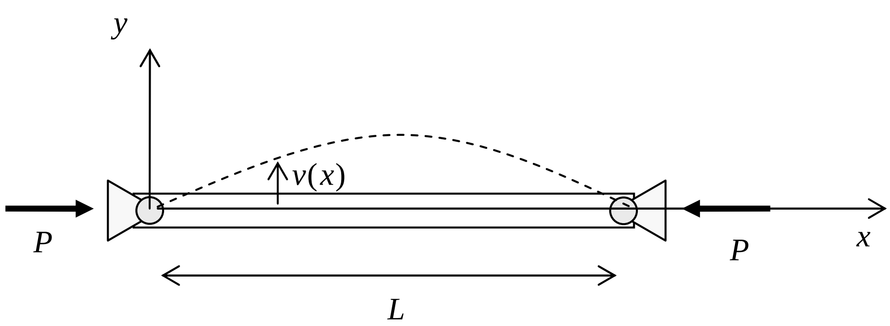

Fig. 7.5.1: a column with deflection *v*

The ordinary differential equation 7.5.2 is linear, homogeneous and with constant coefficients. Its solution can be found in any standard text on differential equations and is given by (for $k^2 > 0$)

$$v(x) = A\cos(kx) + B\sin(kx) \tag{7.5.4}$$

where A and B are as yet unknown constants. The boundary conditions for pinned-ends are

$$v(0) = 0, \qquad v(L) = 0 \tag{7.5.5}$$

The first condition requires A to be zero and the second leads to

$$B\sin(kL) = 0 \tag{7.5.6}$$

It follows that either:
(a) $B = 0$, in which case $v(x) = 0$ for all x and the column is not deflected

or

(b) $\sin(kL) = 0$, which holds when kL is an integer number of π's, i.e.

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots, \tag{7.5.7}$$

As mentioned, the solution (a) is governed by the axial deformation theory discussed in §7.1. Concentrating on (b), the corresponding solution for the deflection is

$$v_n(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{7.5.8}$$

The parameter k is defined by Eqn. 7.5.3, so that, using 7.5.7,

$$P_n = EI\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{7.5.9}$$

It has hence been shown that buckling, i.e. $v \neq 0$, can only occur at a discreet set of applied loads - the **buckling loads** - given by 7.5.9. In practice the most important buckling load is the first, corresponding to $n = 1$, since this will be the first of the loads reached as the applied load P is increased from zero; this is called the **critical buckling load**:

$$\boxed{P_c = EI\left(\frac{\pi}{L}\right)^2} \tag{7.5.10}$$

with associated deflection

$$v_1(x) = B\sin\left(\frac{\pi x}{L}\right) \qquad (7.5.11)$$

The column hence deforms into a single sine wave, which is termed the **mode** or **mode shape** of the deflected column. Note that B, the amplitude of the deflection, can not be determined by this model. This is a consequence of assuming the deflection is small; of **linearising** the problem (which is inherent in the derivation of the moment-deflection curve, Eqn. 7.5.1). A more exact finite deformation theory has been worked out and is called the **theory of the elastica**, but this is not pursued here.

This mathematical structure, where one finds one can only get non-zero solutions of an equation for certain values of a parameter is very common in engineering and theoretical physics. The critical values of the parameter, in this case *k*, are termed the **eigenvalues** of the problem, and the corresponding non-zero solutions, $v(x)$, are the **eigenfunctions**.

The second moment of area *I* has dimensions of $(\text{length})^4$, and for columns is often written in the form $I = Ar^2$ where *A* is the cross-sectional area of the column and the length *r* is called the **radius of gyration**. For example in the case of a circular shaft of radius *a*, $I = \pi a^4/4$ (see Eqn. 7.4.23) so $r = a/2$.

Failure of the Column

The expression 7.5.10 for the critical buckling load can be written in terms of the radius of gyration:

$$P_{cr} = EAr^2\left(\frac{\pi}{L}\right)^2 \qquad \text{or} \qquad \frac{\sigma_{cr}}{E} = \frac{\pi^2}{(L/r)^2} \qquad (7.5.12)$$

where σ_{cr} is the mean compressive stress on the loaded end of the column.

The second equation in 7.5.12 is the most convenient non-dimensional form of presenting theoretical and experimental results for buckling problems. The ratio L/r is called the **slenderness ratio**.

Failure of the column will occur in purely axial compression if the stress in the column reaches the yield stress of the material (see §5.2). On the other hand, if the critical buckling stress σ_{cr} is less than the yield stress, then the column will fail by buckling before the yield stress is reached.

Eqn. 7.5.12 is plotted in Fig. 7.5.2. The yield stress of the material is denoted by *Y*. A critical slenderness ratio is denoted by $(L/r)_{cr}$. For slenderness ratios less than the critical value, that is, for relatively squat columns, the stress in the column will reach the yield stress before buckling occurs.

For example, consider a steel column for which $E = 210\ \text{GPa}$ and $Y = 210\ \text{MPa}$. The critical value of the slenderness ratio is then $L/r = 99.35$, which is a length to diameter ratio of about 25 for a circular column. Buckling will then occur in such columns which have $L/r > 99.35$, for sufficiently high applied axial compressive force.

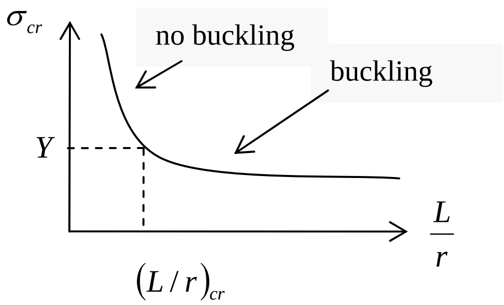

Fig. 7.5.2: critical values of the slenderness ratio

7.5.2 A General Approach to Buckling

The model developed above only applies to columns simply supported at each end. To discuss the more general case one can return to the formulation of the bending of a beam discussed in §7.4.3, but include also axial forces. Fig. 7.4.18 is reproduced as Fig. 7.5.3 but now with compressive axial forces, the forces offset by a small increment in deflection Δv.

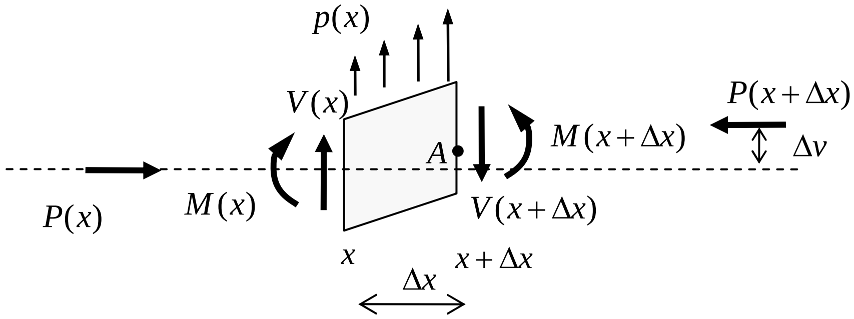

Figure 7.5.3: forces and moments acting on a column

Resolving vertically, one again arrives at Eqn. 7.4.10:

$$\frac{dV}{dx} = p(x) \qquad (7.5.13)$$

Resolving horizontally, one simply gets $P(x) = P(x + \Delta x)$, so that P is constant. Taking moments, one has, instead of 7.4.13,

$$\frac{dM}{dx} + P\frac{dv}{dx} = V \qquad (7.5.14)$$

Note the extra term involving *P*, which is not present in pure bending theory. Eliminating *M* between 7.5.14 and the moment-curvature equation 7.4.37 then leads to an expression for the shear force:

$$\frac{V}{EI} = \frac{d^3 v}{dx^3} + \frac{P}{EI}\frac{dv}{dx} \tag{7.5.15}$$

Note that, in the beam theory, where $P = 0$, the third derivative of the deflection is zero whenever the shear force is zero, in particular at a free, i.e. unsupported, end. Here, however, it is no longer true that the third derivative is zero.

The final differential equation is now obtained by differentiating 7.5.15 and using 7.5.13:

$$\frac{d^4 v}{dx^4} + \frac{P}{EI}\frac{d^2 v}{dx^2} = \frac{p}{EI} \tag{7.5.16}$$

Concentrating on the buckling behaviour and so neglecting the transverse load $p(x)$ [1], one arrives at the differential equation

$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0 \tag{7.5.17}$$

where again $k^2 = P/EI$ (Eqn. 7.5.3). Eqn. 7.5.17 is a homogeneous fourth-order differential equation and its solution is

$$v(x) = A\cos(kx) + B\sin(kx) + Cx + D \tag{7.5.18}$$

The four constants are determined by the end conditions on $v(x)$, two conditions at each end. There are three cases:

(1) Pinned end:
boundary conditions are $v = 0$ and $M = 0$; from the moment-curvature equation, $M = 0$ can be replaced with $d^2 v / dx^2 = 0$

(2) Fixed end:
boundary conditions are $v = 0$, $dv/dx = 0$

(3) Free end:
Boundary conditions are $M = 0$ and $V = 0$; again, this implies that $d^2 v / dx^2 = 0$ and, from Eqn. 7.5.15, $V = 0$ can be replaced with $d^3 v / dx^3 + k^2 (dv/dx) = 0$

The case of pinned-pinned results again in the Euler solution given above. Consider now the case where one end is clamped and the other, loaded, end, is unrestrained ("fixed-free"), Fig. 7.5.4.

[1] bars subjected to both axial compressive loads and transverse loads are called **beam-columns**

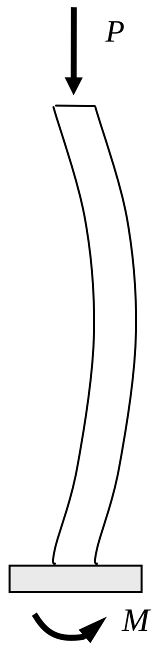

Fig. 7.5.4: a fixed-free column

At the clamped end, $v(0) = v'(0) = 0$, giving

$$A + D = 0, \qquad C + kB = 0 \tag{7.5.19}$$

At the free end, $v''(L) = 0$ and $v'''(L) + k^2 v'(L) = 0$, leading to

$$A\cos(kx) + B\sin(kx) = 0, \qquad C = 0 \tag{7.5.20}$$

Thus, from 7.5.19, *B* too is zero and *A* satisfies

$$A\cos(kL) = 0 \tag{7.5.21}$$

Buckling hence can only occur when $\cos(kL) = 0$, i.e. when

$$kL = \pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots \tag{7.5.22}$$

Using the definition of the parameter *k* the buckling loads are given by

$$P = EI\left[\frac{\pi\left(n + \frac{1}{2}\right)}{L}\right]^2, \quad n = 0, 1, 2, \dots \tag{7.5.23}$$

with the critical buckling load now

$$P_{cr} = EI\left(\frac{\pi}{2L}\right)^2 \tag{7.5.24}$$

which is one quarter of the value for a pinned strut, Eqn. 7.5.10. The buckling modes are given by 7.5.18:

$$v(x) = D\left\{1 - \cos\left[\left(n + \tfrac{1}{2}\right)\pi \frac{x}{L}\right]\right\}, \quad n = 0, 1, 2, \ldots \tag{7.5.25}$$

The first three modes are sketched in Fig. 7.5.5; again, the amplitude is unknown, only the shape.

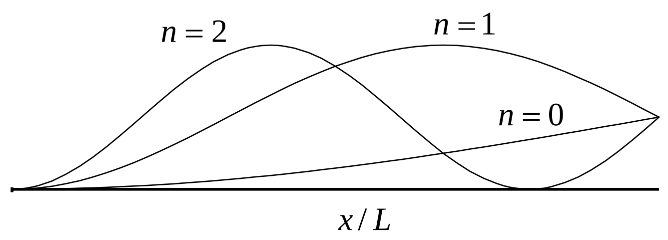

Figure 7.5.5: mode shapes for the fixed-free column

Other cases of end-support can be treated in the same way. Results for the critical buckling stress for various cases are sketched in Fig. 7.5.6.

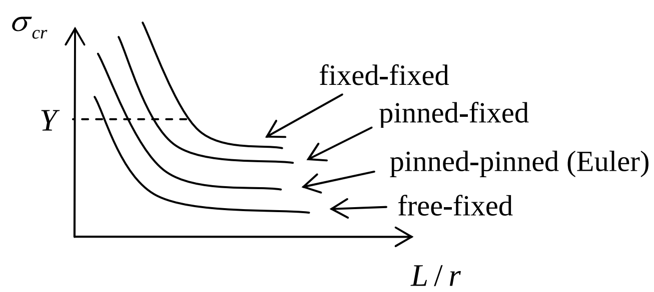

Fig. 7.5.6: critical values of the slenderness ratio for different end-cases