7.2 Torsion

In this section, the geometry to be considered is that of a long slender circular bar and the load is one which twists the bar. Such problems are important in the analysis of twisting components, for example lug wrenches and transmission shafts.

7.2.1 Basic relations for Torsion of Circular Members

The theory of torsion presented here concerns **torques**[1] which twist the members but which *do not induce any warping*, that is, cross sections which are perpendicular to the axis of the member remain so after twisting. Further, radial lines remain straight and radial as the cross-section rotates – they merely rotate with the section.

For example, consider the member shown in Fig. 7.2.1, built-in at one end and subject to a torque T at the other. The x axis is drawn along its axis. The torque shown is positive, following the right-hand rule (see §7.1.4). The member twists under the action of the torque and the radial plane $ABCD$ moves to $ABC'D$.

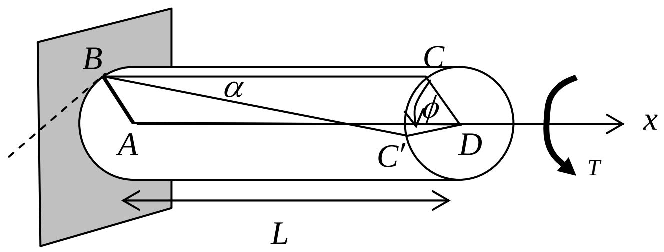

Figure 7.2.1: A cylindrical member under the action of a torque

Whereas in the last section the measure of deformation was elongation of the axial members, here an appropriate measure is the amount by which the member twists, the rotation angle ϕ. The rotation angle will vary along the member – the sign convention is that ϕ is positive in the same direction as positive T as indicated by the arrow in Fig. 7.2.1. Further, whereas the measure of strain used in the previous section was the normal strain ε_{xx}, here it will be the engineering shear strain γ_{xy} (twice the tensorial shear strain ε_{xy}). A relationship between γ (dropping the subscripts) and ϕ will next be established.

As the line BC deforms into BC', Fig. 7.2.1, it undergoes an angle change α. As defined in §4.1.2, the shear strain γ is the change in the original right angle formed by BC and a tangent at B (indicated by the dotted line – this is the y axis to be used in γ_{xy}). If α is small, then

$$\gamma \approx \alpha \approx \tan\alpha \approx \frac{CC'}{BC} \approx \frac{R\varphi(L)}{L} \tag{7.2.1}$$

[1] the term torque is usually used instead of moment in the context of twisting shafts such as those considered in this section

where L is the length, R the radius of the member and $\phi(L)$ means the magnitude of ϕ at L. Note that the strain is constant along the length of the member although ϕ is not. Considering a general cross-section within the member, as in Fig. 7.2.2, one has

$$\gamma \approx \alpha \approx \frac{R\varphi(x)}{x} \tag{7.2.2}$$

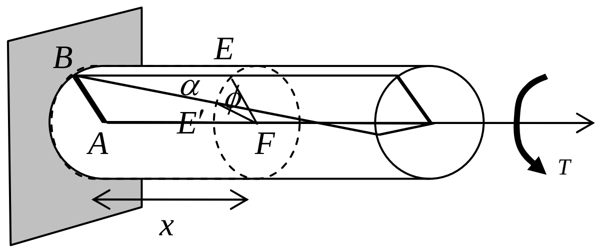

Figure 7.2.2: A section of a twisting cylindrical member

The shear strain at an arbitrary radial location r, $0 < r < R$, is

$$\gamma(r) = \frac{r\phi(x)}{x} \tag{7.2.3}$$

showing that the shear strain varies from zero at the centre of the shaft to a maximum $R\phi(L)/L$ $\left(= R\phi(x)/x\right)$ on the outer surface of the shaft.

The only strain is this shear strain and so the only stress which will arise is a shear stress τ. From Hooke's Law

$$\tau = G\gamma \tag{7.2.4}$$

where G is the shear modulus (the μ of Eqn. 6.1.5). Following the shear strain, the shear stress is zero at the centre of the shaft and a maximum on the outer surface.

Considering a free-body diagram of any portion of the shaft of Fig. 7.2.1, a torque T acts on all cross-sections. This torque must equal the resultant of the shear stresses acting over the section, as schematically illustrated in Fig. 7.2.3a.

The elemental force acting over an element of area dA is τdA and so the resultant moment about $r = 0$ is

$$T = \int_{dA} r\tau(r)dr \tag{7.2.5}$$

But γ / r is a constant and so therefore also is τ / r (provided G is) and Eqn. 7.2.5 can be re-written as

$$T = \frac{\tau(r)}{r}\left[\int_A r^2 dA\right] = \frac{\tau(r)J}{r} \qquad (7.2.6)$$

The quantity in square brackets is called the **polar moment of inertia** of the cross-section (also called the **polar second moment of area**) and is denoted by J:

$$\boxed{J = \int_A r^2 dA} \qquad \textbf{Polar Moment of Area} \qquad (7.2.7)$$

where dA is an element of area and the integration is over the complete cross-section.

For the circular cross-section under consideration, the area element has sides dr and $rd\theta$, Fig. 7.2.3c, so

$$J = \int_0^{2\pi}\int_0^R r^3 dr d\theta = 2\pi\int_0^R r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \qquad (7.2.8)$$

where D is the diameter.

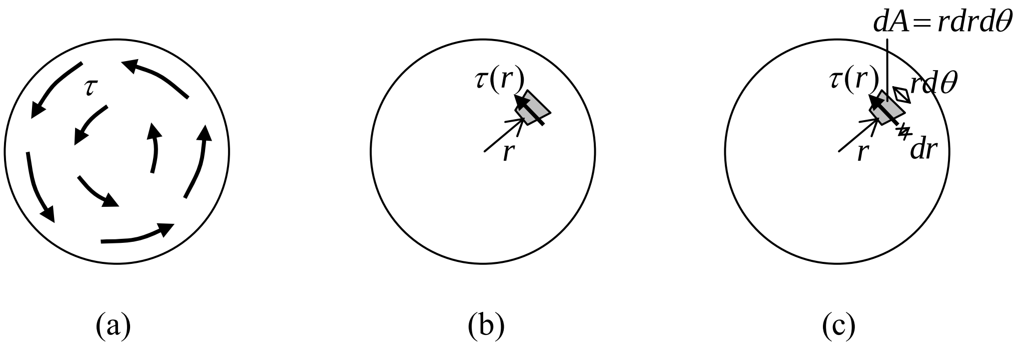

Figure 7.2.3: Shear stresses acting over a cross-section; (a) shear stress, (b,c) moment for an elemental area

From Eqn. 7.2.6, the shear stress at any radial location is given by

$$\boxed{\tau(r) = \frac{rT}{J}} \qquad (7.2.9)$$

From Eqn. 7.2.1, 7.2.4, 7.26 and 7.2.9, the angle of twist at the end of the member – or the twist at one end relative to that at the other end – is

$$\boxed{\phi = \frac{TL}{GJ}} \qquad (7.2.10)$$

Example

Consider the problem shown in Fig.7.2.4, two torsion members of lengths L_1, L_2, diameters d_1, d_2 and shear moduli G_1, G_2, built-in at *A* and subjected to torques T_B and T_C. Equilibrium of moments can be used to determine the unknown torques acting in each member:

$$-T_1 + T_B + T_C = 0, \qquad -T_2 + T_C = 0 \tag{7.2.11}$$

so that $T_1 = T_B + T_C$ and $T_2 = T_C$.

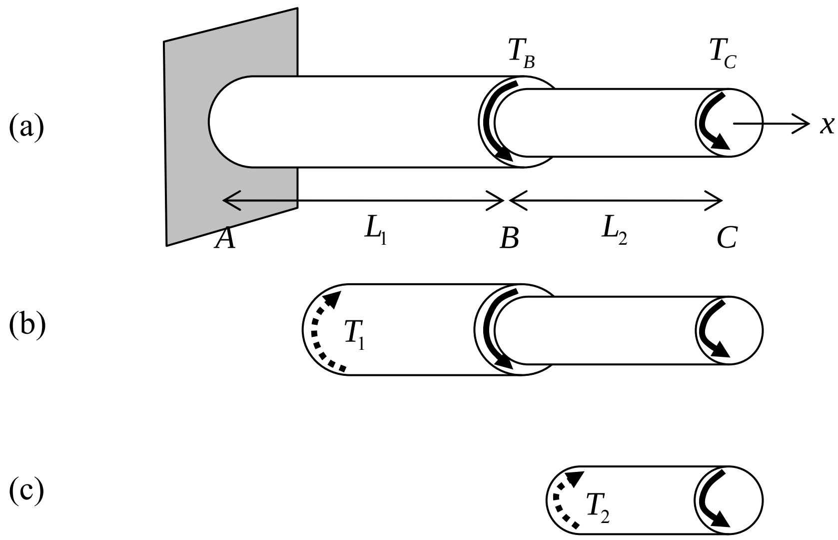

Figure 7.2.4: A structure consisting of two torsion members; (a) subjected to torques T_B and T_C, (b,c) free-body diagrams

The shear stresses in each member are therefore

$$\tau_1 = \frac{r(T_B + T_C)}{J_1}, \qquad \tau_2 = \frac{rT_C}{J_2} \tag{7.2.12}$$

where $J_1 = \pi d_1^4 / 32$ and $J_2 = \pi d_2^4 / 32$.

From Eqn. 7.2.10, the angle of twist at *B* is given by $\phi_B = T_1 L_1 / G_1 J_1$. The angle of twist at *C* is then

$$\phi_C = \frac{T_2 L_2}{G_2 J_2} - \phi_B \tag{7.2.13}$$

■

Statically indeterminate problems can be solved using methods analogous to those used in the section 7.1 for uniaxial members.

Example

Consider the structure in Fig. 7.2.5, similar to that in Fig. 7.2.4 only now both ends are built-in and there is only a single applied torque, T_B.

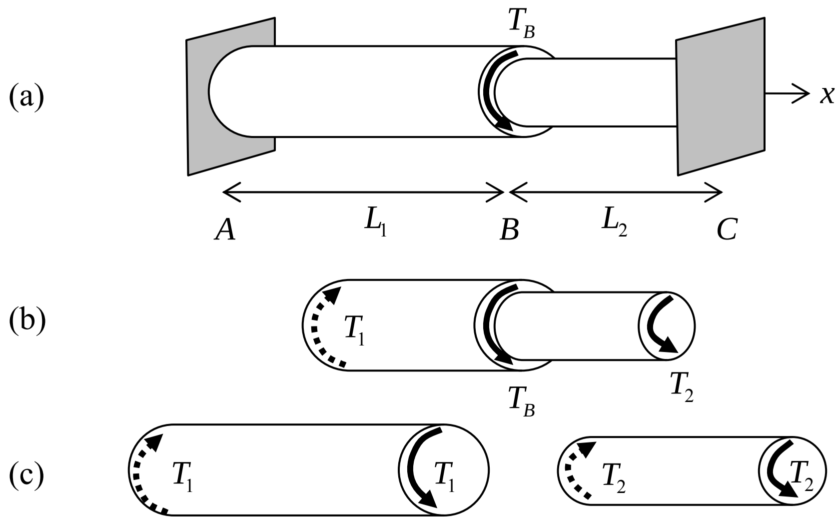

Figure 7.2.5: A structure consisting of two torsion members; (a) subjected to a Torque T_B, (b) free-body diagram, (c) separate elements

Referring to the free-body diagram of Fig. 7.2.5b, there is only one equation of equilibrium with which to determine the two unknown member torques:

$$-T_1 + T_B + T_2 = 0 \tag{7.2.14}$$

and so the deformation of the structure needs to be considered. A systematic way of dealing with this situation is to consider each element separately, as in Fig. 7.2.5c. The twist in each element is

$$\phi_1 = \frac{T_1 L_1}{G_1 J_1}, \quad \phi_2 = \frac{T_2 L_2}{G_2 J_2} \tag{7.2.15}$$

The total twist is zero and so $\phi_1 + \phi_2 = 0$ which, with Eqn. 7.2.14, can be solved to obtain

$$T_1 = +\frac{L_2 G_1 J_1}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B, \quad T_2 = -\frac{L_1 G_2 J_2}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B \tag{7.2.16}$$

The rotation at *B* can now be determined, $\phi_B = \phi_1 = -\phi_2$.

■

7.2.2 Stress Distribution in Torsion Members

The shear stress in Eqn. 7.2.9 is acting over a cross-section of a torsion member. From the symmetry of the stress, it follows that shear stresses act also along the length of the member, as illustrated to the left of Fig. 7.2.6. Shear stresses do not act *on* the surface of the element shown, as it is a free surface.

Any element of material not aligned with the axis of the cylinder will undergo a complex stress state, as shown to the right of Fig. 7.2.6. The stresses acting on an element are given by the stress transformation equations, Eqns. 3.4.9:

$$\sigma'_{xx} = +\sin 2\theta\tau, \qquad \sigma'_{yy} = -\sin 2\theta\tau, \qquad \sigma'_{xy} = +\cos 2\theta\tau \tag{7.2.17}$$

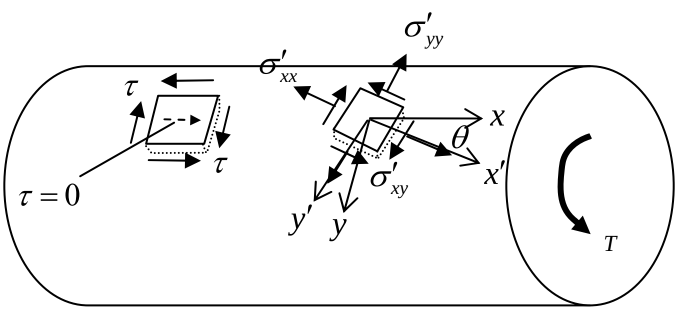

Figure 7.2.6: Stress distribution in a torsion member

From Eqns. 3.5.4-5, the maximum normal (principal) stresses arise on planes at $\theta = \pm 45^o$ and are $\sigma_1 = +\tau$ and $\sigma_2 = -\tau$. Thus the maximum tensile stress in the member occurs at 45^o to the axis and arises at the surface. The maximum shear stress is simply τ, with $\theta = 0$.

7.2.3 Problems

1. A shaft of length L and built-in at both ends is subjected to two external torques, T at A and $2T$ at B, as shown below. The shaft is of diameter d and shear modulus G. Determine the maximum (absolute value of) shear stress in the shaft and determine the angle of twist at B.