5.4 Continuum Models and Micromechanics

The models mentioned in the previous section are **continuum models**. What this means is explained in what follows.

5.4.1 Stress and Scale

In the definition of the traction vector, §3.3.1, it was assumed that the ratio of force over area would reach some definite limit as the area ΔS of the surface upon which the force ΔF acts was shrunk to zero. This issue can be explored further by considering Fig. 5.4.1. Assume first that the plane upon which the force acts is fairly large; it is then shrunk and the ratio F/S tracked. A schematic of this ratio is shown in Fig. 5.4.2. At first (to the right of Fig. 5.4.2) the ratio F/S undergoes change, assuming the stress to vary within the material, as it invariably will if the material is loaded in some complex way. Eventually the plane will be so small that the ratio changes very little, perhaps with some small variability ε. If the plane is allowed to get too small, however, down below some distance h^* say and down towards the atomic level, where one might encounter "intermolecular space", there will be large changes in the ratio and the whole concept of a force acing on a single surface breaks down.

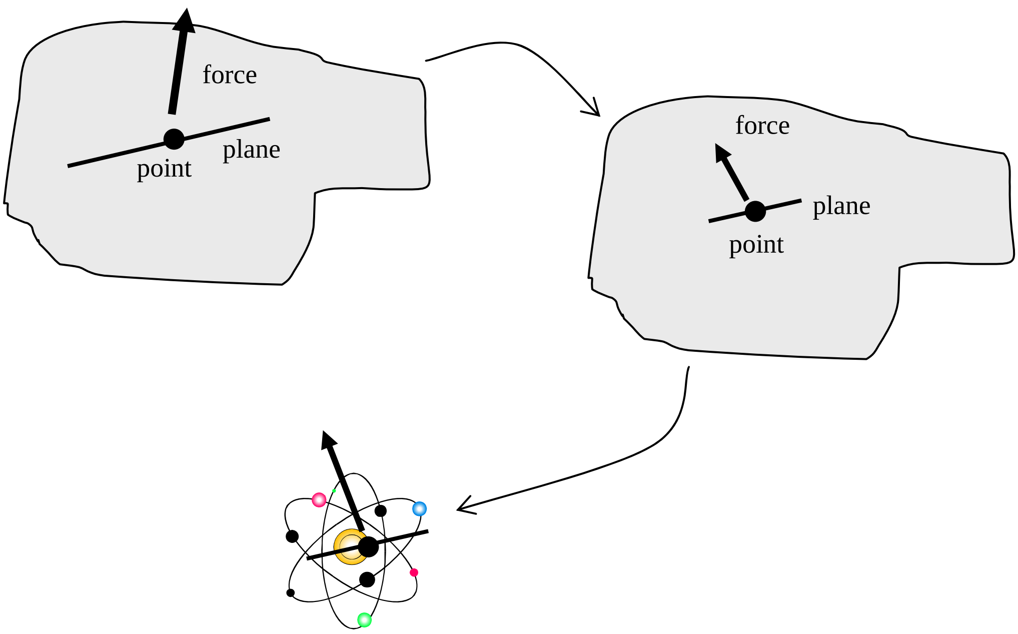

Figure 5.4.1: A force acting on an internal surface; allowing the plane on which the force acts to get progressively smaller

In a continuum model, it is assumed that the ratio F/S follows the dotted path shown in Fig. 5.4.2; a definite limit is reached as the plane shrinks to *zero size*. It should be kept in mind that the traction in a *real* material should be evaluated through

$$\mathbf{t} = \lim_{\Delta S \to (h^*)^2} \frac{\Delta F}{\Delta S} \tag{5.4.1}$$

where h^* is some minimum dimension below which there is no acceptable limit. On the other hand, it is necessary to take the limit to zero in the *mathematical* modelling of materials since that is the basis of calculus[1].

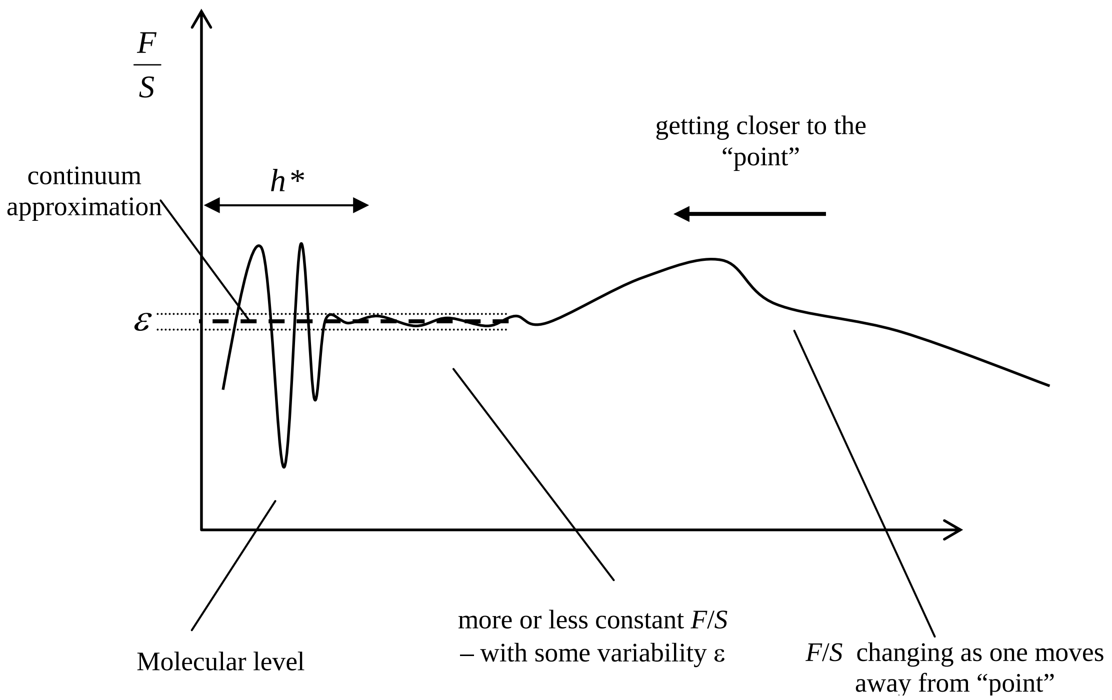

Figure 5.4.2: the change in traction as the plane upon which a force acts is reduced in size

In a continuum model, then, there is a minimum sized element one can consider, say of size $\Delta V = (h^*)^3$. When one talks about the stress on this element, the mass of this element, the density, velocity and acceleration of this element, one means the average of these quantities throughout or over the surface of the element – the discrete atomic structure within the element is ignored and is averaged out, or "smeared" out, into a **continuum element**.

The continuum element is also called a **representative volume element** (RVE), an element of material large enough for the heterogeneities to be replaced by homogenised mean values of their properties. The order of the dimensions of RVE's for some common engineering materials would be approximately (see the metal example which follows)

Metal:	0.1mm
Polymers/composites:	1mm
Wood:	10mm
Concrete:	100mm

One does not have any information about what is happening inside the continuum element – it is like a "black box". The scale of the element (and higher) is called the

[1] calculus is not used too much in this book – it is absolutely necessary and ubiquitous in more refined and advanced mechanics theories

macroscale – continuum mechanics is mechanics on the macroscale. The scale of entities within the element is termed the **microscale** – continuum models cannot give any information about what happens on the microscale.

5.4.2 Example: Metal

Metal, from a distance, appears fairly uniform. With the help of a microscope, however, it will be seen to consist of many individual grains of metal. For example, the metal shown in Fig. 5.4.3 has grains roughly 0.05mm across, and each one has very individual properties (the crystals in each grain are aligned in different directions).

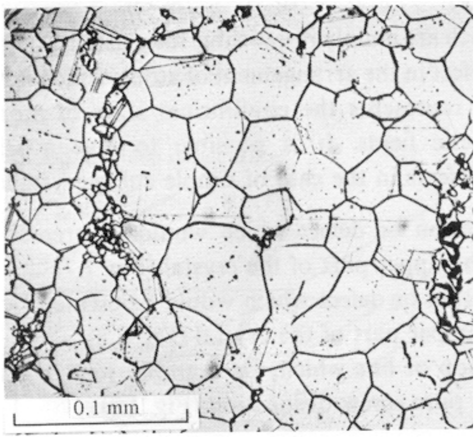

Figure 5.4.3: metal grains

If one is interested in the gross deformation of a moderately sized component of this metal, it would be sufficient to consider deformations that are averaged over volumes which are large compared to individual grains, but small compared to the whole component. A minimum dimension of, say, $h^* = 0.5\text{mm}$ for the metal of Fig. 5.4.3 would seem to suffice, and this would be the macro/micro-scale boundary, with a minimum surface area of dimension $(h^*)^2$ for the definition of stress.

When one measures physical properties of the metal "at a point", for example the density, one need only measure an average quantity over an element of the order, say, $(0.5\text{mm})^3$ or higher. It is not necessary to consider the individual grains of metal – these are inside the "black box". The model will return valuable information about the deformation of the gross material, but it will not be able to furnish any information about movement of individual grains.

It was shown how to evaluate the Young's Modulus and other properties of a metal in Section 5.2.1. The test specimens used for such tests are vastly larger than the continuum elements discussed above. Thus the test data is perfectly adequate to describe the response of the metal, on the macroscale.

What if the response of individual grains to applied loads is required? In that case a model would have to be constructed which accounted for the different mechanical properties of each grain. The metal could no longer be considered to be a uniform material, but a complex one with many individual grains, each with different properties and orientation. The macro/micro boundary could be set at about $h^* = 0.1\mu\text{m}$. There are now two problems which need to be dealt with: (1) experiments such as the tensile test would have to be conducted on specimens much smaller than the grain size in order to provide data for any mathematical model, and (2) the mathematical model will be more complex and difficult to solve.

5.4.3 Micromechanical Models

Consider the schematic of a continuum model shown in Fig. 5.4.4 below. One can determine the material's properties, such as the Young's modulus *E*, through experimentation, and the resulting mathematical continuum model can be used to make predictions about the material's response. With the improved power of computers, especially since the 1990s, it has now become possible to complement continuum models with **micromechanical models**. These models take into account more fine detail of the material's structure (for example of the individual grains of the metal discussed earlier). Usually, one will have a micromechanical model of a small (typical) RVE of material. This then provides information regarding the properties of the RVE to be included in a continuum model (rather than having a micromechanical model of the *complete* material, which is in most cases still not practical). The means by which the properties at the micro scale are averaged (for example into a "smeared out" single *E* value) and passed "up" to the continuum model is through **homogenisation theory**. Such micromechanical models can provide further insight into material behaviour than the simpler continuum model.

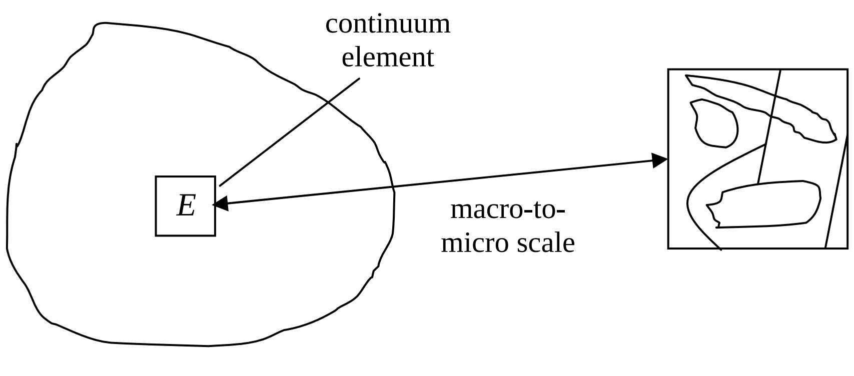

Figure 5.4.4: continuum model and micromechanical model

5.4.4 Problems

1. You want to evaluate the stiffness *E* of a metal for inclusion in a mechanics model. What *minimum* size specimen would you use in your test - $10\mu\text{m}$, 0.1mm, 5mm or 5cm?

2. Individual rice grains are separate solid particles. However, rice flowing down a chute at a food processing plant can be considered to be a fluid, and the flow of rice can be solved using the equations of mechanics. What minimum dimension h^*

should be employed for measurements in this case to ensure the validity of a continuum model of flowing rice?