5.1 Mechanics Modelling

5.1.1 The Mechanics Problem

Typical questions which mechanics attempts to answer were given in Section 1.1. In the examples given, one invariably knows (some of) the forces (or stresses) acting on the material under study, be it due to the wind, water pressure, the weight of the human body, a moving train, and so on. One also often knows something about the displacements along some portion of the material, for example it might be fixed to the ground and so the displacements there are zero. A schematic of such a generic material is shown in Fig. 5.1.1 below.

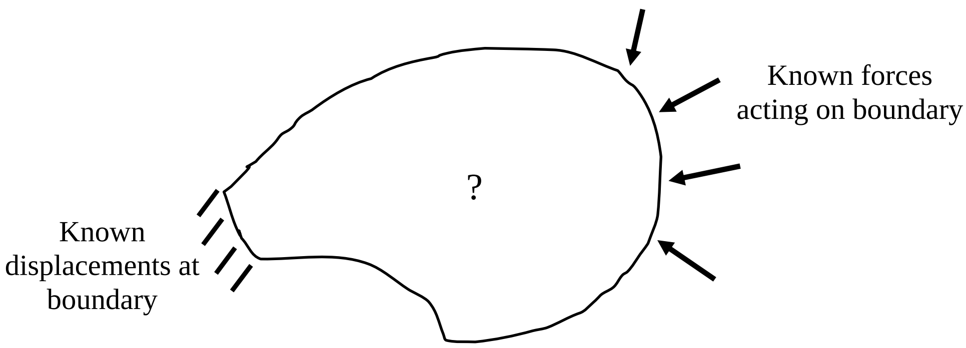

Figure 5.1.1: a material component; force and displacement are known along some portion of the boundary

The basic problem of mechanics is to determine what is happening *inside* the material. This means: what are the stresses and strains inside the material? With this information, one can answer further questions: Where are the stresses high? Where will the material first fail? What can we change to make the material function better? Where will the component move to? What is going on inside the material, at the microscopic level? Generally speaking, what is happening and what will happen?

One can relate the loads on the component to the stresses inside the body using equilibrium equations and one can relate the displacement to internal strains using kinematics relations. For example, consider again the simple rod subjected to tension forces examined in Section 3.3.1, shown again in Fig. 5.1.2. The internal normal stress σ_N on any plane oriented at an angle θ to the rod cross-section is related to the external force *F* through the equilibrium equation 3.3.1: $\sigma_N = F\cos^2\theta / A$, where *A* is the cross-sectional area. Similarly, if the ends undergo a separation/displacement of $\Delta = l - l_0$, Fig. 5.1.2b, the strain of any internal line element, at orientation θ, is $\varepsilon_N = \Delta\cos^2\theta / l_0$. However, there is no relationship between this internal stress and internal strain: for any given force, there is no way to determine the internal strain (and hence displacement of the rod); for any given displacement of the rod, there is no way to determine the internal stress (and hence force applied to the rod). The required relationship between stress and strain is discussed next.

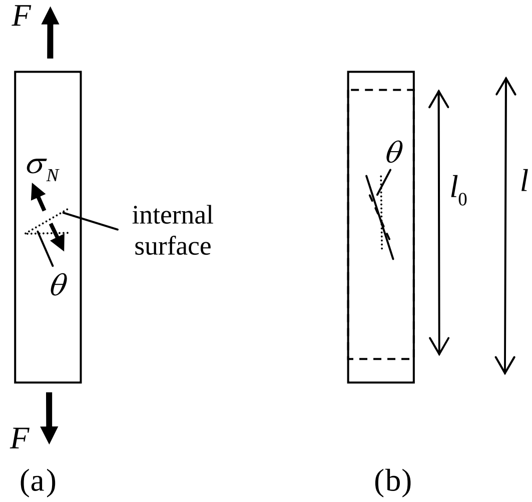

Figure 5.1.2: a slender rod; (a) internal stress due to external force, (b) internal strain due to gross displacement of rod (dotted = before straining)

5.1.2 Constitutive Equation

Stress was discussed in Chapter 3 and strain in Chapter 4. In all that discussion, no mention was made of the particular material under study, be it metallic, polymeric, biological or foodstuff (apart from the necessity that the strain be small when using the engineering strain). The concept of stress and the resulting theory of stress transformation, principal stresses and so on, are based on physical principles (Newton's Laws), which apply to *all* materials. The concept of strain is based, essentially, on geometry and trigonometry; again, it applies to all materials. However, it is the relationship *between* stress and strain which differs from material to material.

The relationship between the stress and strain for any particular material will depend on the microstructure of that material – what constitutes that material. For this reason, the stress-strain relationship is called the **constitutive relation**, or **constitutive law**. For example, metals consist of a closely packed lattice of atoms, whereas a rubber consists of a tangled mass of long-chain polymer molecules; for this reason, the strain in a metal will be different to that in rubber, when they are subjected to the same stress.

The constitutive equation allows the mechanics problem to be solved – this is shown schematically in Fig. 5.1.3.

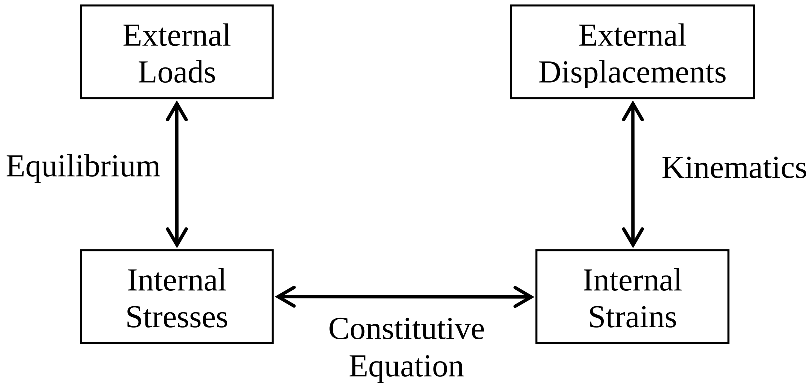

Figure 5.1.3: the role of the constitutive equation in the equations of mechanics

Example Constitutive Equations

A constitutive equation will be of the general form

$$\sigma = f(\varepsilon). \tag{5.1.1}$$

The simplest constitutive equation is a **linear elastic** relation, in which the stress is proportional to the strain:

$$\sigma \propto \varepsilon. \tag{5.1.2}$$

Although no real material satisfies precisely Eqn. 5.1.2, many do so approximately – this type of relation will be discussed in Chapters 6-8. More complex relations can involve the *rate* at which a material is strained or stressed; these types of relation will be discussed in Chapter 10.

More on constitutive equations will follow in Section 5.3.

5.1.3 Mathematical Model

Some of the questions asked earlier can be answered using experimentation. For example, one could use a car-crash test to determine the weakest points in a car. However, one cannot carry our multiple tests for each and every possible scenario – different car speeds, different obstacles into which it crashes, and so on; it would be too time-consuming and too expensive. The only practical way in which these questions can be answered is to develop a **mathematical model**. This model consists of the various equations of equilibrium and the kinematics, the constitutive relation, equations describing the shape of the material, etc. (see Fig. 5.1.3). The mathematical model will have many approximations to reality associated with it. For example, it might be assumed that the material is in the shape of a perfect sphere, when in fact it only resembles a sphere. It may be assumed that a load is applied at a "point" when in fact it is applied over a region of the material's surface. Another approximation in the mathematical model is the constitutive equation itself; the relation between stress and strain in any material can be extremely complex, and the constitutive equation can only be an approximation of the reality.

Once the mathematical model has been developed, the various equations can be solved and the model can then be used to *make a prediction*. The prediction of the model can now be tested against reality: a set of well-defined experiments can be carried out – does the material really move to where the model says it will move?

Simple models (simple constitutive relations) should be used as a first step. If the predictions of the model are wildly incorrect, the model can be adjusted (made more complicated), and the output tested again.

The equations associated with simple models can often be solved analytically, i.e. using a pen and paper. More complex models result in complex sets of equations which can only be solved approximately (though, hopefully, accurately) using a computer.