4.1 Strain

If an object is placed on a table and then the table is moved, each material particle moves in space. The particles undergo a **displacement**. The particles have moved in space as a **rigid body**. The material remains unstressed. On the other hand, when a material is acted upon by a set of forces, it *changes size and/or shape*, it **deforms**. This deformation is described using the concept of **strain**. The study of this movement and deformation, without reference to the forces or anything else which might "cause" it, is called **kinematics**.

4.1.1 One Dimensional Strain

The Engineering Strain

Consider a slender rod, fixed at one end and stretched, as illustrated in Fig. 4.1.1; the original position of the rod is shown dotted.

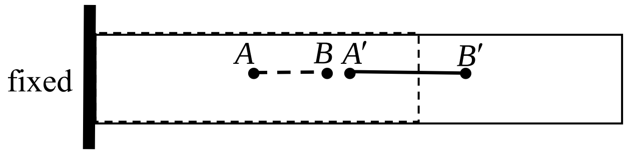


Figure 4.1.1: the strain at a point *A* in a stretched slender rod; *AB* is a line element in the unstretched rod, $A'B'$ is the same line element in the stretched rod

There are a number of different ways in which this stretching/deformation can be described (see later). Here, what is perhaps the simplest measure, the **engineering strain**, will be used. To determine the strain at point *A*, Fig. 4.1.1, consider a small line element *AB* emanating from *A* in the unstretched rod. The points *A* and *B* move to A' and B' when the rod has been stretched. The (engineering) strain ε at *A* is then defined as[1]

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} \tag{4.1.1}$$

The strain at other points in the rod can be evaluated in the same way.

If a line element is stretched to twice its original length, the strain is 1. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.5. The strain is often expressed as a percentage; a 100% strain is a strain of 1, a 200% strain is a strain of 2, etc. Most engineering materials, such as metals and concrete, undergo extremely small strains in practical applications, in the range 10^{-6} to 10^{-2}; rubbery materials can easily undergo large strains of 100%.

[1] this is the strain at point *A*. The strain at *B* is evidently the same – one can consider the line element *AB* to emanate fom point *B* (it does not matter whether the line element emanates out from the point to the "left" or to the "right")

Consider now two adjacent line elements *AE* and *EB* (not necessarily of equal length), which move to $A'E'$ and $E'B'$, Fig. 4.1.2. If the rod is stretching **uniformly**, that is, if all line elements are stretching in the same proportion along the length of the rod, then $|A'E'|/|AE| = |E'B'|/|EB|$, and $\varepsilon^{(A)} = \varepsilon^{(E)}$; the strain is the same at all points along the rod.

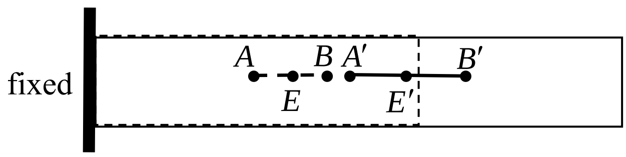


Figure 4.1.2: the strain at a point *A* and the strain at point *E* in a stretched rod

In this case, one could equally choose the line element *AB* or the element *AE* in the calculation of the strain at *A*, since

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} = \frac{|A'E'| - |AE|}{|AE|}$$

In other words it does not matter what the length of the line element chosen for the calculation of the strain at *A* is. In fact, if the length of the rod before stretching is L_0 and after stretching it is L, Fig. 4.1.3, the strain everywhere is (this is equivalent to choosing a “line element” extending the full length of the rod)

$$\varepsilon = \frac{L - L_0}{L_0} \tag{4.1.2}$$

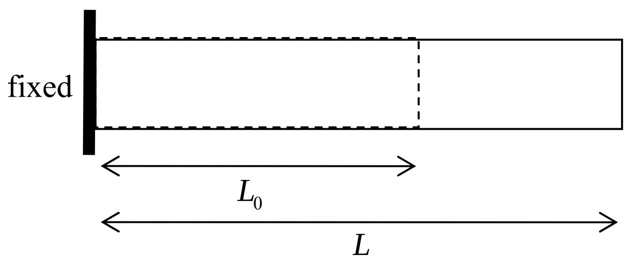


Figure 4.1.3: a stretched slender rod

On the other hand, when the strain is *not* uniform, for example $|A'E'|/|AE| \neq |E'B'|/|EB|$, then the length of the line element does matter. In this case, to be precise, the line element *AB* in the definition of strain in Eqn. 4.1.1 should be “infinitely small”; the smaller the line element, the more accurate will be the evaluation of the strain. The strains considered in this book will be mainly uniform.

Displacement, Strain and Rigid Body Motions

To highlight the difference between displacement and strain, and their relationship, consider again the stretched rod of Fig 4.1.1. Fig 4.1.4 shows the same rod: the two points A and B undergo displacements $u^{(A)} = |AA'|, u^{(B)} = |BB'|$. The strain at A, Eqn 4.1.1, can be re-expressed in terms of these displacements:

$$\varepsilon^{(A)} = \frac{u^{(B)} - u^{(A)}}{|AB|} \tag{4.1.3}$$

In words, the strain is a measure of the *change* in displacement as one moves along the rod.

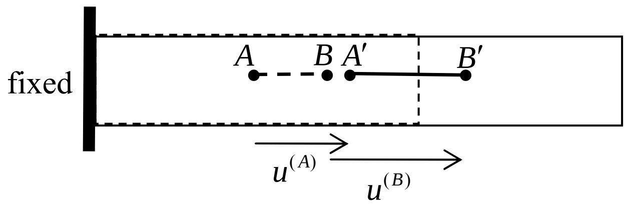


Figure 4.1.4: displacements in a stretched rod

Consider a line element emanating from the left-hand fixed end of the rod. The displacement at the fixed end is zero. However, the strain at the fixed end is *not* zero, since the line element there will change in length. This is a case where the displacement is zero but the strain is not zero.

Consider next the case where the rod is not fixed and simply moves/translates in space, without any stretching, Fig. 4.1.5. This is a case where the displacements are all non-zero (and in this case everywhere the same) but the strain is everywhere zero. This is in fact a feature of a good measure of strain: it should be zero for any rigid body motion; the strain should only measure the deformation.

Figure 4.1.5: a rigid body translation of a rod

Note that if one knows the strain at all points in the rod, one cannot be sure of the rod's exact position in space – again, this is because strain does not include information about possible rigid body motion. To know the precise position of the rod, one must also have some information about the displacements.

The True Strain

As mentioned, there are many ways in which deformation can be measured. Many different strains measures are in use apart from the engineering strain, for example the Green-Lagrange strain and the Euler-Almansi strain: referring again to Fig. 4.1.1, these are

$$\text{Green-Lagrange } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|AB|^2}, \qquad \text{Euler-Alamnsi } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|A'B'|^2} \tag{4.1.4}$$

Many of these strain measures are used in more advanced theories of material behaviour, particularly when the deformations are very large. Apart from the engineering strain, just one other measure will be discussed in any detail here: the **true strain** (or **logarithmic strain**), since it is often used in describing material testing (see Chapter 5).

The true strain may be defined as follows: define a small increment in strain to be the change in length divided by the *current* length: $d\varepsilon_t = dL / L$. As the rod of Fig. 4.1.1 stretches (uniformly), this current length continually changes, and the total strain thus defined is the accumulation of these increments:

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right). \tag{4.1.5}$$

If a line element is stretched to twice its original length, the (true) strain is 0.69. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.69. The fact that a stretching and a contraction of the material by the same factor results in strains which differ only in sign is one of the reasons for the usefulness of the true strain measure.

Another reason for its usefulness is the fact that the true strain is additive. For example, if a line element stretches in two steps from lengths L_1 to L_2 to L_3, the total true strain is

$$\varepsilon_t = \ln\left(\frac{L_3}{L_2}\right) + \ln\left(\frac{L_2}{L_1}\right) = \ln\left(\frac{L_3}{L_1}\right),$$

which is the same as if the stretching had occurred in one step. This is not true of the engineering strain.

The true strain and engineering strain are related through (see Eqn. 4.1.2, 4.1.5)

$$\varepsilon_t = \ln(1+\varepsilon) \tag{4.1.6}$$

One important consequence of this relationship is that the smaller the deformation, the less the difference between the two strains. This can be seen in Table 4.1 below, which shows the values of the engineering and true strains for a line element of initial length 1mm, at different stretched lengths. (In fact, using a Taylor series expansion, $\varepsilon_t = \ln(1+\varepsilon) \approx \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots$, for small ε.) Almost all strain measures in use are similar in this way: they are defined such that they are more or less equal when the deformation is small. Put another way, when the deformations are small, it does not really matter which strain measure is used, since they are all essentially the same – in that case it is often sensible to use the simplest measure.

L_0 (mm)	L (mm)	ε	ε_t
1	2	1	0.693
1	1.5	0.5	0.405
1	1.4	0.4	0.336
1	1.3	0.3	0.262
1	1.2	0.2	0.182
1	1.1	0.1	0.095
1	1.01	0.01	0.00995
1	1.001	0.001	0.000995

Table 4.1: true strain and engineering strain at different stretches

If one defines strain to be "change in length over length", then the true strain would be more "correct" than the engineering strain. On the other hand, if the strain is considered to have a variety of (related) definitions, such as Eqns 4.1.2, 4.1.4-5, then no strain measure is really more "correct" than any other; the usefulness of a strain measure will depend on the application and the problem at hand.

4.1.2 Two Dimensional Strain

The two dimensional case is similar to the one dimensional case, in that material deformation can be described by imagining the material to be a collection of small line elements. As the material is deformed, the line elements stretch, or get shorter, only now they can also rotate in space relative to each other. This movement of line elements is encompassed in the idea of strain: the "strain at a point" is all the stretching, contracting and rotating of *all* line elements emanating from that point, with all the line elements together making up the continuous material, as illustrated in Fig. 4.1.6.

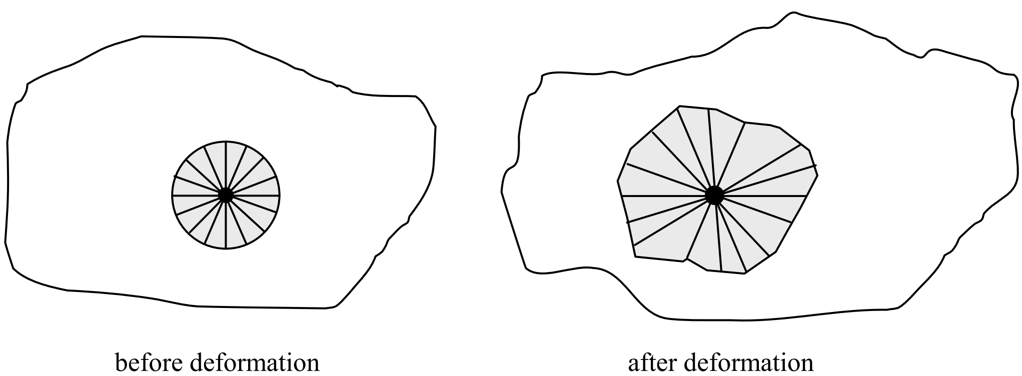


Figure 4.1.6: a deforming material element; original state of line elements and their final position after straining

It turns out that the strain at a point is completely characterised by the movement of *any two mutually perpendicular line-segments*. If it is known how these perpendicular line-segments are stretching, contracting and rotating, it will be possible to determine how any other line element at the point is behaving, by using a **strain transformation rule** (see later). This is analogous to the way the stress at a point is characterised by the stress acting on perpendicular planes through a point, and the stress components on other planes can be obtained using the stress transformation formulae.

So, for the two-dimensional case, consider two perpendicular line-elements emanating from a point. When the material that contains the point is deformed, two things (can) happen:

(1) the line segments will *change length* and
(2) the *angle* between the line-segments *changes*.

The change in length of line-elements is called **normal strain** and the change in angle between initially perpendicular line-segments is called **shear strain**.

As mentioned earlier, a number of different definitions of strain are in use; here, the following, most commonly used, definition will be employed, which will be called the **exact strain**:

Normal strain in direction x: (denoted by ε_{xx})
change in length (per unit length) of a line element originally lying in the x−direction

Normal strain in direction y: (denoted by ε_{yy})
change in length (per unit length) of a line element originally lying in the y−direction

Shear strain: (denoted by ε_{xy})
(half) the change in the original right angle between the two perpendicular line elements

Referring to Fig. 4.1.7, the (exact) strains are

$$\varepsilon_{xx} = \frac{A'B' - AB}{AB}, \quad \varepsilon_{yy} = \frac{A'C' - AC}{AC}, \quad \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda). \qquad (4.1.7)$$

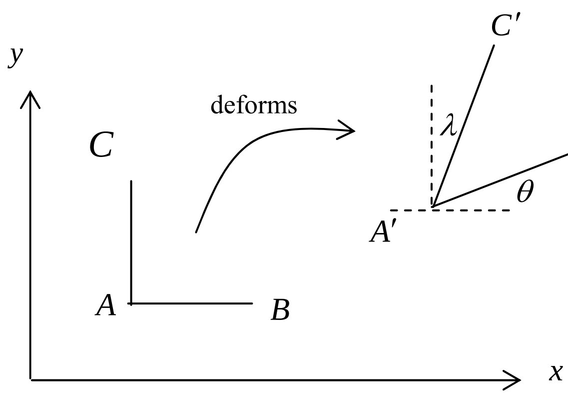


Figure 4.1.7: strain at a point *A*

These 2D strains can be represented in the matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.1.8}$$

As with the stress, the strain matrix is symmetric, with, by definition, $\varepsilon_{xy} = \varepsilon_{yx}$.
Note that the point *A* in Fig. 4.1.7 has also undergone a displacement **u**(*A*). This displacement has two components, u_x and u_y, as shown in Fig. 4.1.8 (and similarly for the points *B* and *C*).

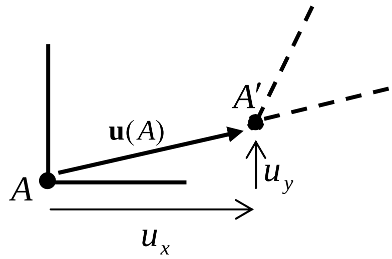


Figure 4.1.8: displacement of a point *A*

The line elements not only change length and the angle between them changes – they can also move in space as rigid-bodies. Thus, for example, the normal and shear strain in the three examples shown in Fig. 4.1.9 are the same, even though the displacements occurring in each case are different – *strain is independent of rigid body motions*.

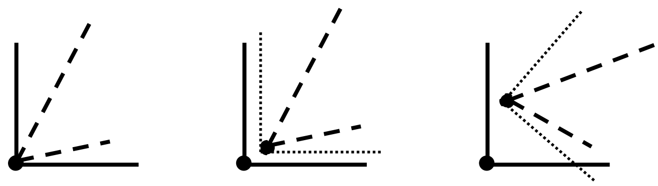


Figure 4.1.9: rigid body motions

The Engineering Strain

Suppose now that the deformation is very small, so that, in Fig. 4.1.10, $A'B' \approx A'B^*$; here $A'B^*$ is the projection of $A'B'$ in the $x-$direction. In that case,

$$\varepsilon_{xx} \approx \frac{A'B^* - AB}{AB}. \tag{4.1.9}$$

Similarly, one can make the approximations

$$\varepsilon_{yy} \approx \frac{A'C^* - AC}{AC}, \quad \varepsilon_{xy} \approx \frac{1}{2}\left(\frac{B^*B'}{AB} + \frac{C^*C'}{AC}\right), \tag{4.1.10}$$

the expression for shear strain following from the fact that, for a *small angle*, the angle (measured in radians) is approximately equal to the tan of the angle.

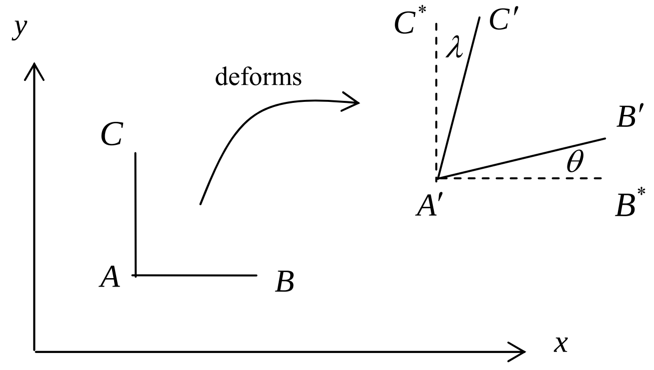


Figure 4.1.10: small deformation

This approximation for the normal strains is called the **engineering strain** or **small strain** or **infinitesimal strain** and is valid when the *deformations are small*. The advantage of the small strain approximation is that the mathematics is simplified greatly.

Example

Two perpendicular lines are etched onto the fuselage of an aircraft. During testing in a wind tunnel, the perpendicular lines deform as in Fig. 4.1.10. The coordinates of the line end-points (referring to Fig. 4.1.10) are:

$$
\begin{array}{ll}
C:(0.0000, 1.0000) & C':(0.0025, 1.0030) \\
A:(0.0000, 0.0000) & A':(0.0000, 0.0000) \\
B:(1.0000, 0.0000) & B':(1.0045, 0.0020)
\end{array}
$$

The exact strains are, from Eqn. 4.1.9, (to 8 decimal places)

$$
\varepsilon_{xx} = \frac{\sqrt{|A'B^*|^2 + |B^*B'|^2}}{|AB|} - 1 = 0.00450199
$$

$$
\varepsilon_{yy} = \frac{\sqrt{|A'C^*|^2 + |C^*C'|^2}}{|AC|} - 1 = 0.00300312
$$

$$
\varepsilon_{xy} = \frac{1}{2}\left(\arctan\left(\frac{|B^*B'|}{|A'B^*|} \right) + \arctan\left(\frac{|C^*C'|}{|A'C^*|} \right) \right) = 0.00224178
$$

The engineering strains are, from Eqns. 4.1.10-11,

$$
\varepsilon_{xx} = \frac{|A'B^*|}{|AB|} - 1 = 0.0045, \quad \varepsilon_{yy} = \frac{|A'C^*|}{|AC|} - 1 = 0.003, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{|B^*B'|}{|AB|} + \frac{|C^*C'|}{|AC|} \right) = 0.00225
$$

As can be seen, for the small deformations which occurred, the errors in making the small-strain approximation are extremely small, less than 0.11% for all three strains.

■

Small strain is useful in characterising the small deformations that take place in, for example, (1) engineering materials such as concrete, metals, stiff plastics and so on, (2) linear viscoelastic materials such as many polymeric materials (see Chapter 10), (3) some porous media such as soils and clays at moderate loads, (4) almost any material if the loading is not too high.

Small strain is inadequate for describing large deformations that occur, for example, in many rubbery materials, soft tissues, engineering materials at large loads, etc. In these cases the more precise definition 4.1.7 (or a variant of it) is required. That said, the engineering strain and the concepts associated with it are an excellent introduction to the more involved large deformation strain measures.

In one dimension, there is no distinction between the exact strain and the engineering strain – they are the same. Differences arise between the two in the two-dimensional case when the material shears (as in the example above), or rotates as a rigid body (as will be discussed further below).

Engineering Shear Strain and Tensorial Shear Strain

The definition of shear strain introduced above is the **tensorial shear strain** ε_{xy}. The **engineering shear strain**[2] γ_{xy} is defined as twice this angle, i.e. as $\theta + \lambda$, and is often used in Strength of Materials and elementary Solid Mechanics analyses.

4.1.3 Sign Convention for Strain

A positive normal strain means that the line element is lengthening. A negative normal strain means the line element is shortening.

For shear strain, one has the following convention: when the two perpendicular line elements are both directed in the positive directions (say x and y), or both directed in the negative directions, then a positive shear strain corresponds to a *decrease* in right angle. Conversely, if one line segment is directed in a positive direction whilst the other is directed in a negative direction, then a positive shear strain corresponds to an *increase* in angle. The four possible cases of shear strain are shown in Fig. 4.1.11a (all four shear strains are positive). A box undergoing a positive shear and a negative shear are also shown, in Figs. 4.1.11b,c.

[2] not to be confused with the term *engineering strain*, i.e. *small strain*, used throughout this Chapter

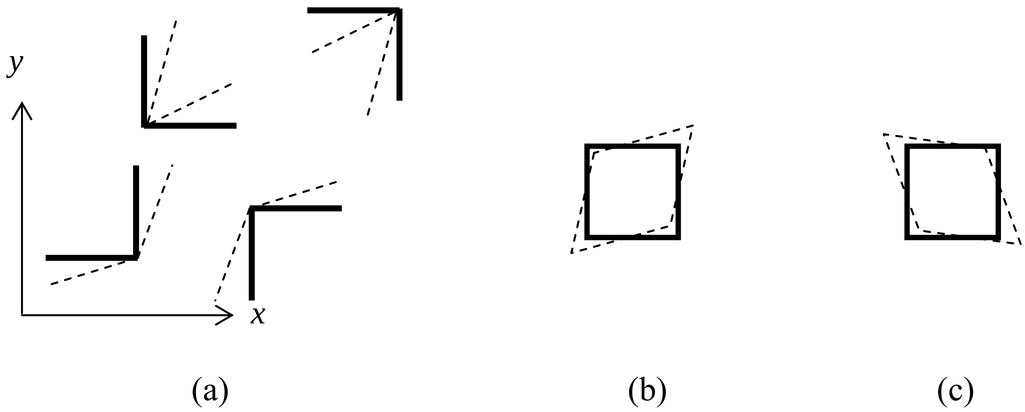


Figure 4.1.11: sign convention for shear strain; (a) line elements undergoing positive shear, (b) a box undergoing positive shear, (c) a box undergoing negative shear

4.1.4 Geometrical Interpretation of the Engineering Strain

Consider a small "box" element and suppose it to be so small that the strain is constant/uniform throughout - one says that the strain is **homogeneous**. This implies that straight lines remain straight after straining and parallel lines remain parallel. A few simple deformations are examined below and these are related to the strains.

A positive normal strain $\varepsilon_{xx} > 0$ is shown in Fig. 4.1.12a. Here the undeformed box element (dashed) has elongated. As mentioned already, knowledge of the strain alone is not enough to determine the position of the strained element, since it is free to move in space as a rigid body. The displacement over some part of the box is usually specified, for example the left hand end has been fixed in Fig. 4.1.12b. A negative normal strain acts in Fig. 4.1.12c.

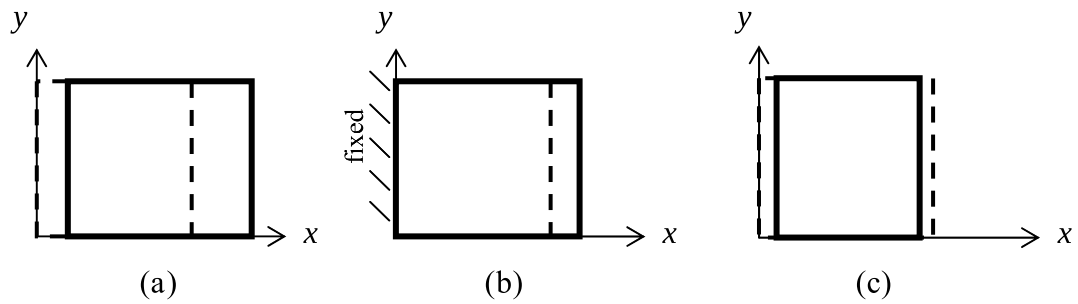


Figure 4.1.12: normal strain; (a) positive normal strain, (b) positive normal strain with the left-hand end fixed in space, (c) negative normal strain

A case known as **simple shear** is shown in Fig. 4.1.13a, and that of **pure shear** is shown in Fig. 4.1.13b. In both illustrations, $\varepsilon_{xy} > 0$. A pure (rigid body) **rotation** is shown in Fig. 4.1.13c (zero strain).

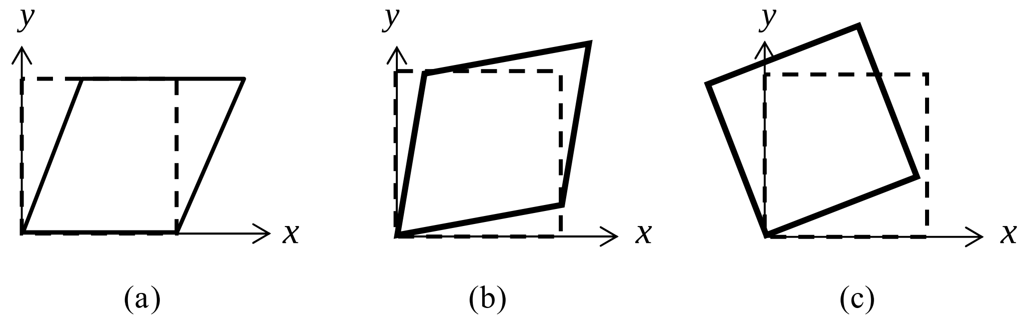


Figure 4.1.13: (a) simple shear, (b) pure shear, (c) pure rotation

Any shear strain can be decomposed into a pure shear and a pure rotation, as illustrated in Fig. 4.1.14.

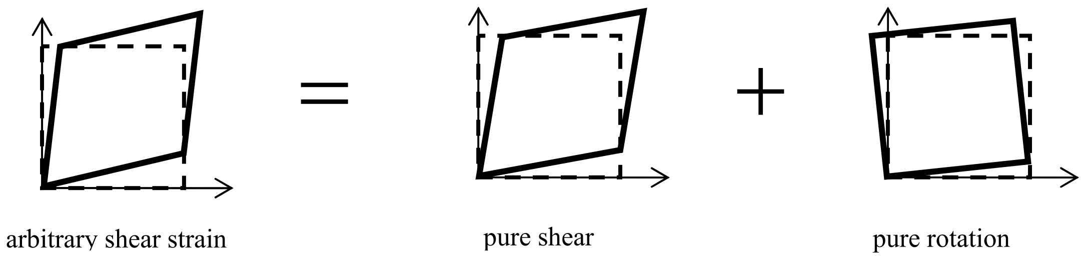


Figure 4.1.14: shear strain decomposed into a pure shear and a pure rotation

4.1.5 Large Rotations and the Small Strain

The example in section 4.1.2 above illustrated that the small strain approximation is good, provided the deformations are small. However, this is provided also that any *rigid body rotations are small*. To illustrate this, consider a square material element (with sides of unit length) which undergoes a pure rigid body rotation of θ, Fig. 4.1.15. The exact strains 4.1.7 remain zero. The small shear strain remains zero also. However, the small normal strains are seen to be $\varepsilon_{xx} = \varepsilon_{yy} = \cos\theta - 1$. Using a Taylor series expansion, this is equal to $\varepsilon_{xx} = \varepsilon_{yy} \approx -\theta^2 / 2 + \theta^4 / 24 - \cdots$. Thus, when θ is small, the rotation-induced strains are of the magnitude/order θ^2. If θ is of the same order as the strains themselves, i.e. in the range $10^{-6} - 10^{-2}$, then θ^2 will be very much smaller than θ and the rotation-induced strains will not introduce any inaccuracy; the small strains will be a good approximation to the actual strains. If, however, the rotation is large, then the engineering normal strains will be wildly inaccurate. For example, when $\theta = 45^{\circ}$, the rotation-induced normal strains are ≈ -0.3, and will likely be larger than the actual strains occurring in the material.

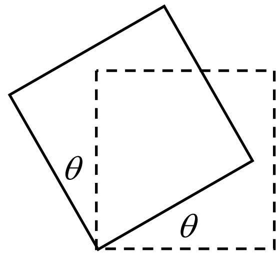


Figure 4.1.15: an element undergoing a rigid body rotation

As an example, consider a cantilevered beam which undergoes large bending, Fig. 4.1.16. The shaded element shown might well undergo small normal and shear strains. However, because of the large rotation of the element, additional spurious engineering normal strains are induced. Use of the precise definition, Eqn. 4.1.7, is required in cases such as this.

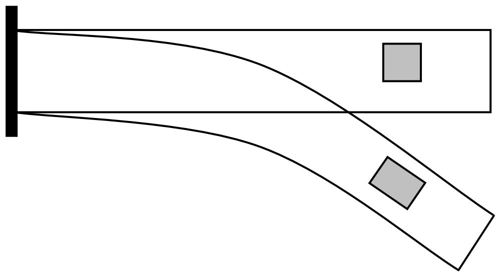


Figure 4.1.16: Large rotations of an element in a bent beam

4.1.6 Three Dimensional Strain

The above can be generalized to three dimensions. In the general case, there are three normal strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, and three shear strains, $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$. The ε_{zz} strain corresponds to a change in length of a line element initially lying along the *z* axis. The ε_{yz} strain corresponds to half the change in the originally right angle of two perpendicular line elements aligned with the *y* and *z* axes, and similarly for the ε_{zx} strain. Straining in the $y-z$ plane ($\varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}$) is illustrated in Fig. 4.1.17 below.

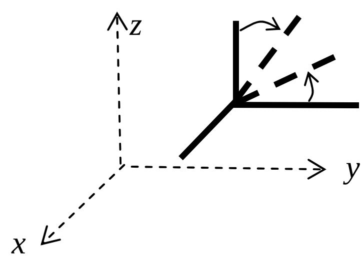


Figure 4.1.17: strains occurring in the $y-z$ plane

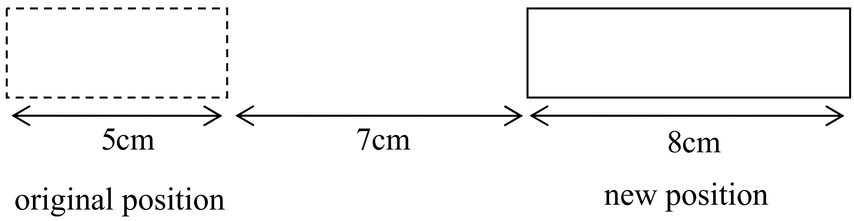
The 3D strains can be represented in the (symmetric) matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{4.1.11}$$

As with the stress (see Eqn. 3.4.5), there are nine components in 3D, with 6 of them being independent.

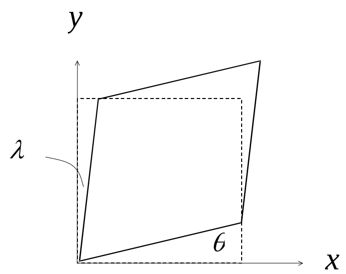
4.1.7 Problems

1. Consider a rod which moves and deforms (uniformly) as shown below.
   (a) What is the displacement of the left-hand end of the rod?
   (b) What is the engineering strain at the left-hand end of the rod



2. A slender rod of initial length 2cm is extended (uniformly) to a length 4cm. It is then compressed to a length of 3cm.
   (a) Calculate the engineering strain and the true strain for the extension
   (b) Calculate the engineering strain and the true strain for the compression
   (c) Calculate the engineering strain and the true strain for one step, i.e. an extension from 2cm to 3cm.
   (d) From your calculations in (a,b,c), which of the strain measures is additive?

3. An element undergoes a homogeneous strain, as shown. There is no normal strain in the element. The angles are given by $\lambda = 0.001$ and $\theta = 0.002$ radians. What is the (tensorial) shear strain in the element?

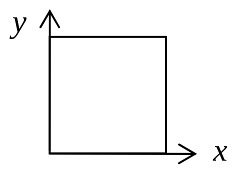


4. In a fixed $x - y$ reference system established for the test of a large component, three points A, B and C on the component have the following coordinates before and after loading (see Figure 4.1.10):

$$C:(0.0000,1.5000) \qquad C':(-0.0025,1.5030)$$
$$A:(0.0000,0.0000) \qquad A':(0.0000,0.0000)$$
$$B:(2.0000,0.0000) \qquad B':(2.0045,0.0000)$$

Determine the actual strains and the small strains (at/near point *A*). What is the error in the small strain compared to the actual strains?

5. Sketch the deformed shape for the material shown below under the following strains (*A*, *B* constant):
   (i) $\varepsilon_{xx} = A > 0$ (taking $\varepsilon_{yy} = \varepsilon_{xy} = 0$) – assume that the right-hand edge is fixed
   (ii) $\varepsilon_{yy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{xy} = 0$) – assume that the lower edge is fixed
   (iii) $\varepsilon_{xy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{yy} = 0$) – assume that the left-hand edge is fixed



6. The element shown below undergoes the change in position and dimensions shown (dashed square = undeformed). What are the three engineering strains $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$?