3.4 Equilibrium of Stress

Consider two perpendicular planes passing through a point *p*. The stress components acting on these planes are as shown in Fig. 3.4.1a. These stresses are usually shown together acting on a small material element of finite size, Fig. 3.4.1b. It has been seen that the stress may vary from point to point in a material but, if the element is very small, the stresses on one side can be taken to be (more or less) equal to the stresses acting on the other side. By convention, in analyses of the type which will follow, all stress components shown are *positive*.

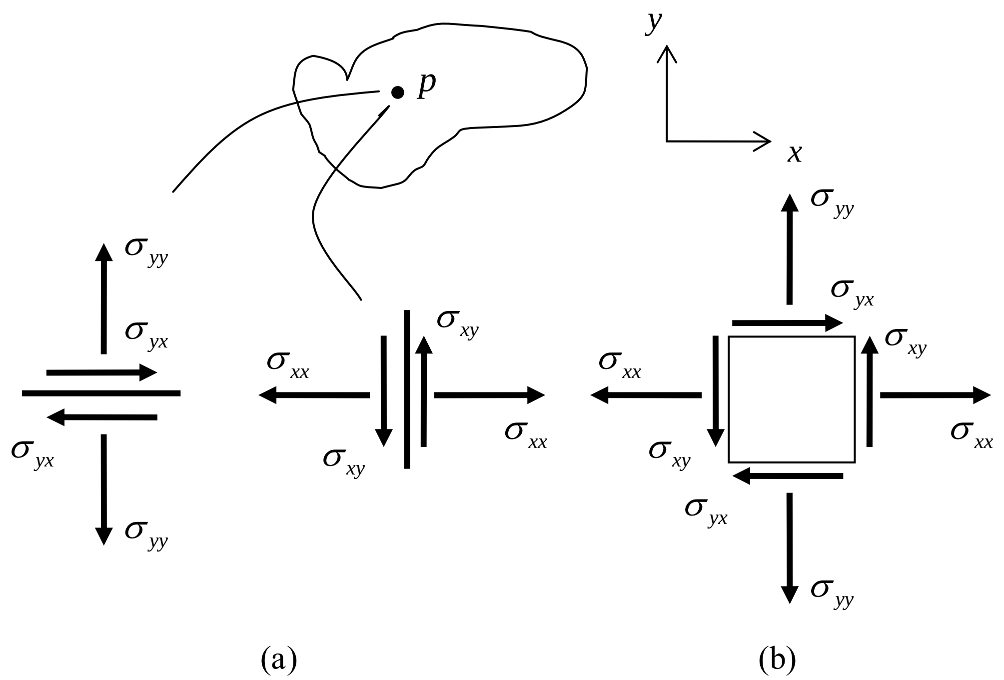

Figure 3.4.1: stress components acting on two perpendicular planes through a point; (a) two perpendicular surfaces at a point, (b) small material element at the point

The four stresses can conveniently be written in the form of a **stress matrix**:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \tag{3.4.1}$$

It will be shown below that the stress components acting on *any* other plane through *p* can be evaluated from a knowledge of only these stress components.

3.4.1 Symmetry of the Shear Stress

Consider the material element shown in Fig. 3.4.1b, reproduced in Fig. 3.4.2a below. The element has dimensions is $\Delta x \times \Delta y$ and is subjected to uniform stresses over its sides. The resultant forces of the stresses acting on each side of the element act through the side-centres, and are shown in Fig. 3.4.2b. The stresses shown are positive, but note how *positive stresses can lead to negative forces*, depending on the definition of the $x-y$ axes used. The resultant force on the complete element is seen to be zero.

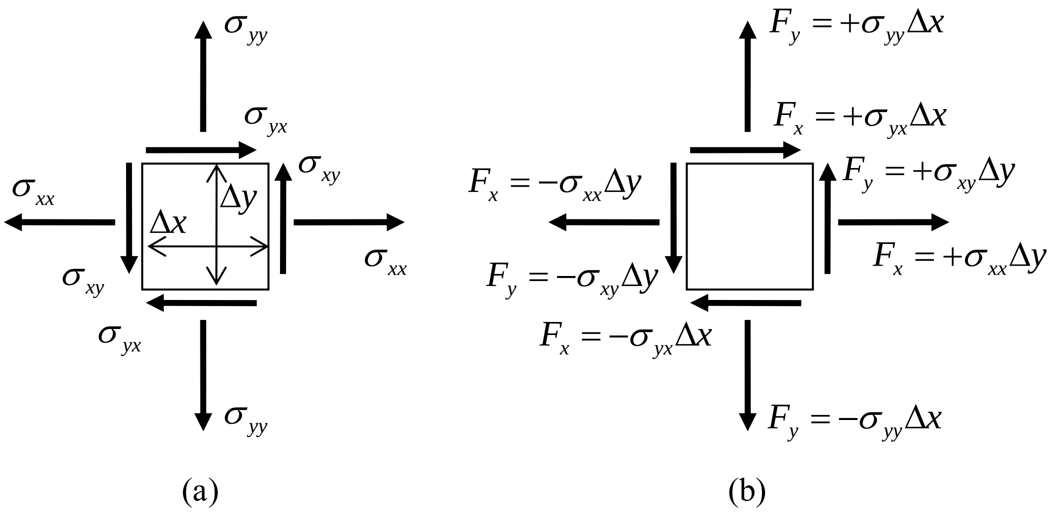

Figure 3.4.2: stress components acting on a material element; (a) stresses, (b) resultant forces on each side

By taking moments about any point in the block, one finds that {▲Problem 1}

$$\sigma_{xy} = \sigma_{yx} \tag{3.4.2}$$

Thus the shear stresses acting on the element are all equal, and for this reason the σ_{yx} stresses are usually labelled σ_{xy}, Fig. 3.4.3a, or simply labelled τ, Fig. 3.4.3b.

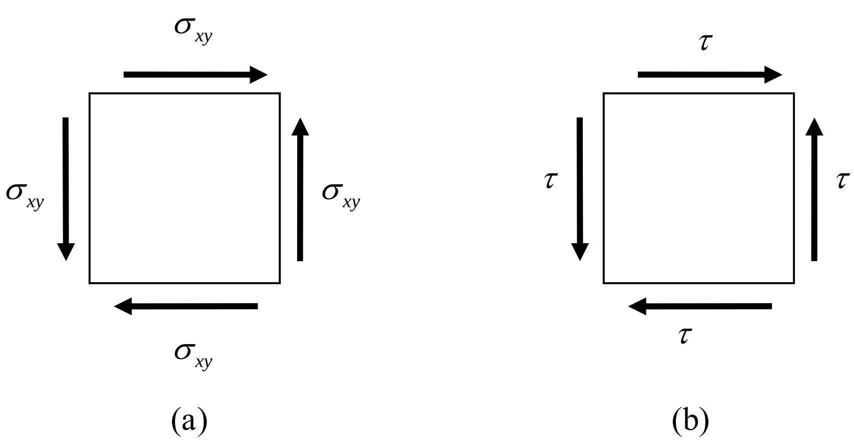

Figure 3.4.3: shear stress acting on a material element

In fact, in two-dimensional problems, the double-subscript notation is often dispensed with for simplicity, and the stress matrix can be expressed as

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{bmatrix}, \tag{3.4.3}$$

to go along with the representation shown in Fig. 3.4.4.

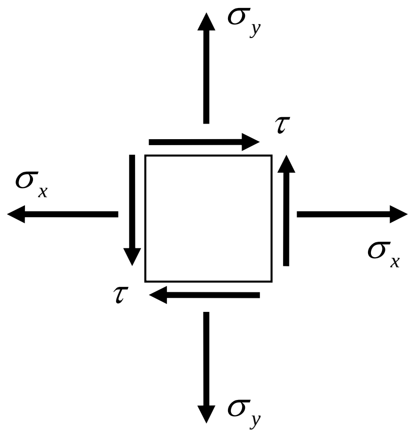

Figure 3.4.4: a simpler notation for 2D stress components (without the double subscripts)

3.4.2 Three Dimensional Stress

The three-dimensional counterpart to the two-dimensional element of Fig. 3.4.2 is shown in Fig. 3.4.5. Again, all stresses shown are positive.

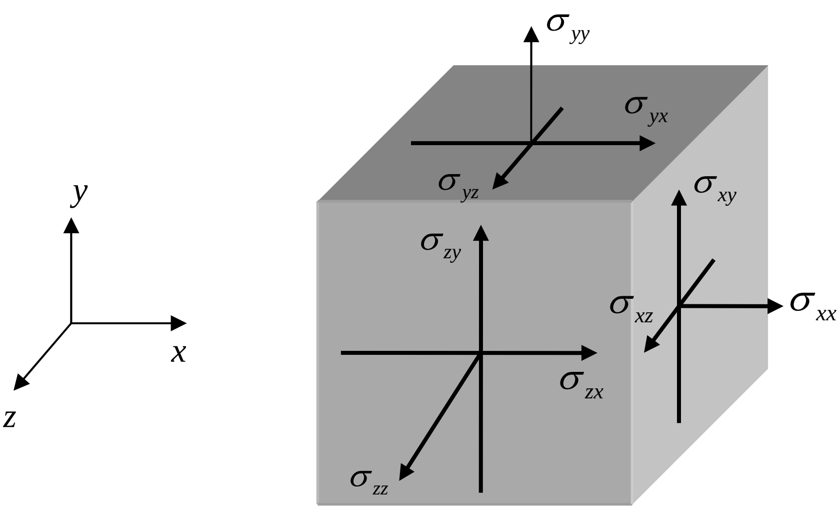

Figure 3.4.5: a three dimensional material element

Moment equilibrium in this case requires that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \tag{3.4.4}$$

The nine stress components, six of which are independent, can be written in the matrix form

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{3.4.5}$$

A vector **F** has one direction associated with it and is characterised by *three* components (F_x, F_y, F_z). The stress is a quantity which has two directions associated with it (the direction of a force and the normal to the plane on which the force acts) and is characterised by the *nine* components of Eqn. 3.4.5. Such a mathematical object is called a **tensor**. Just as the three components of a vector change with a change of coordinate axes (for example, as in Fig. 2.2.1), so the nine components of the **stress tensor** change with a change of axes. This is discussed in the next section for the two-dimensional case.

3.4.3 Stress Transformation Equations

Consider the case where the nine stress components acting on three perpendicular planes through a material particle are known. These components are σ_{xx}, σ_{xy}, etc. when using x, y, z axes, and can be represented by the cube shown in Fig. 3.4.6a. Rotate now the planes about the three axes – these new planes can be represented by the rotated cube shown in Fig. 3.4.6b; the axes normal to the planes are now labelled x', y', z' and the corresponding stress components with respect to these new axes are $\sigma'_{xx}, \sigma'_{xy}$, etc.

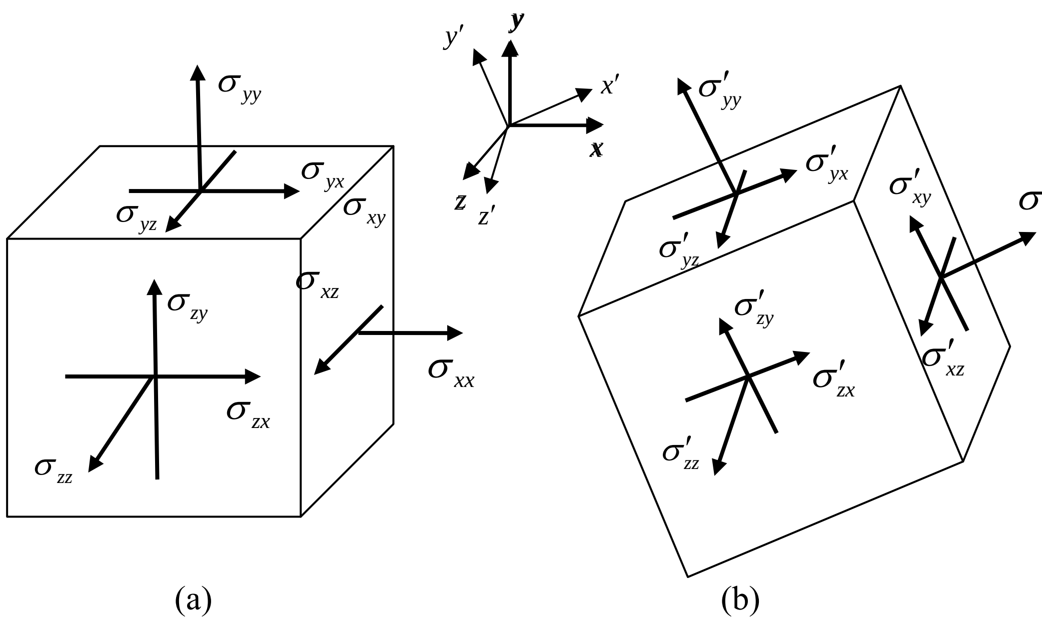

Figure 3.4.6: a three dimensional material element; (a) original element, (b) rotated element

There is a relationship between the stress components σ_{xx}, σ_{xy}, etc. and the stress components $\sigma'_{xx}, \sigma'_{xy}$, etc. The relationship can be derived using Newton's Laws. The equations describing the relationship in the fully three-dimensional case are very lengthy. Here, the relationship for the two-dimensional case will be derived – this 2D relationship will prove very useful in analysing many practical situations.

Two-dimensional Stress Transformation Equations

Assume that the stress components of Fig. 3.4.7a are known. It is required to find the stresses arising on other planes through *p*. Consider the perpendicular planes shown in Fig. 3.4.7b, obtained by rotating the original element through a positive (counterclockwise) angle θ. The new surfaces are defined by the axes $x' - y'$.

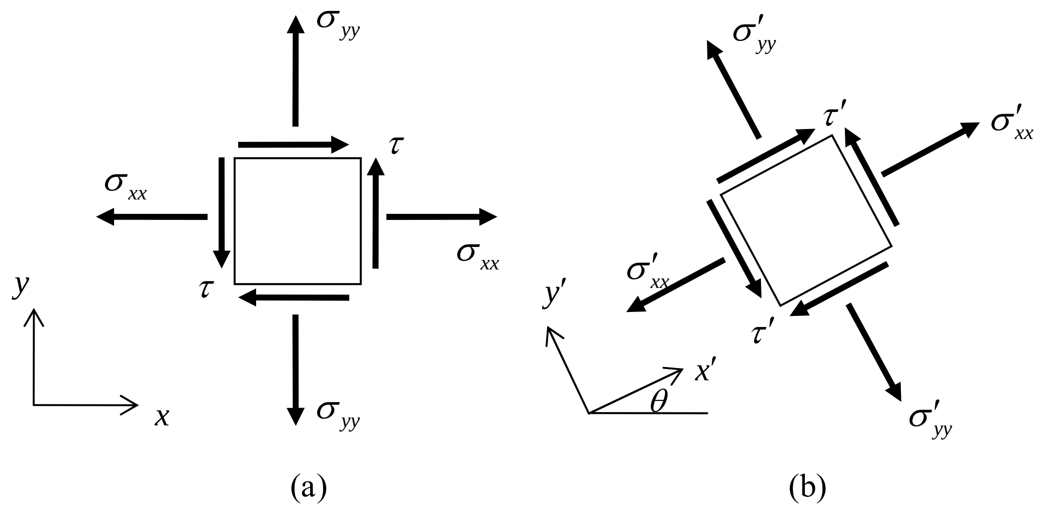

Figure 3.4.7: stress components acting on two different sets of perpendicular surfaces, i.e. in two different coordinate systems; (a) original system, (b) rotated system

To evaluate these new stress components, consider a triangular element of material at the point, Fig. 3.4.8. Carrying out force equilibrium in the direction x', one has (with unit depth into the page)

$$\sum F_{x'}: \quad \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta = 0 \qquad (3.4.6)$$

Since $|OB| = |AB|\cos\theta,\ |OA| = |AB|\sin\theta$, and dividing through by $|AB|$,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \qquad (3.4.7)$$

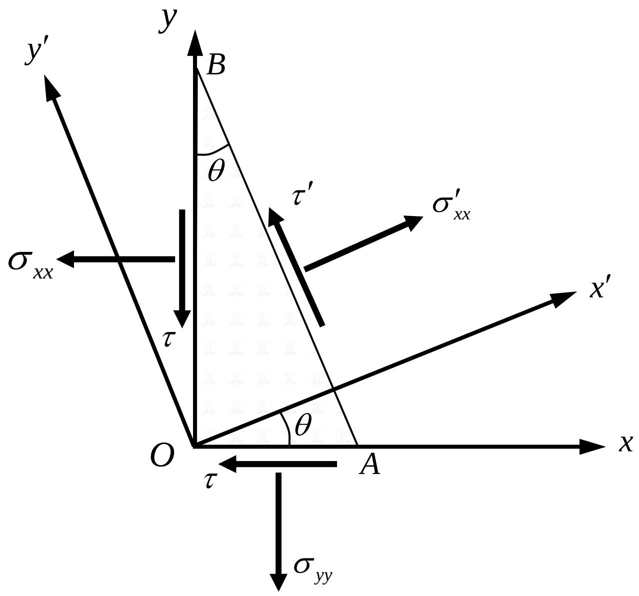

Figure 3.4.8: a free body diagram of a triangular element of material

The forces can also be resolved in the y' direction and one obtains the relation

$$\tau' = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau\cos 2\theta \tag{3.4.8}$$

Finally, consideration of the element in Fig. 3.4.9 yields two further relations, one of which is the same as Eqn. 3.4.8.

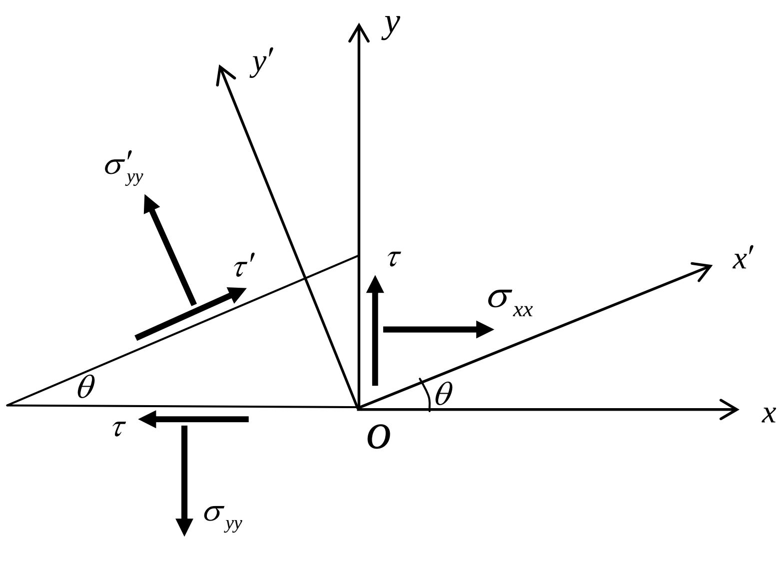

Figure 3.4.9: a free body diagram of a triangular element of material

In summary, one obtains the **stress transformation equations**:

$$\boxed{\begin{aligned} \sigma'_{xx} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\ \sigma'_{yy} &= \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\ \sigma'_{xy} &= \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy} \end{aligned}}$$

2D Stress Transformation Equations (3.4.9)

These equations have many uses, as will be seen in the next section.

In matrix form,

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (3.4.10)$$

These relations hold also in the case when there are body forces, when the material is accelerating and when there are non-uniform stress fields. (This is discussed in the next section.)

3.4.4 Problems

1. Derive Eqns. 3.4.2 by taking moments about the lower left corner of the block in Fig. 3.4.2.

2. Suppose that the stresses acting on two perpendicular planes through a point are

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

   Use the stress transformation formulae to evaluate the stresses acting on two new perpendicular planes through the point, obtained from the first set by a positive rotation of 30°. Use the conventional notation $x' - y'$ to represent the coordinate axes parallel to these new planes.