4.A Appendix to Chapter 4

4.A.1 Isotropic Functions

The scalar-, vector- and tensor-valued functions ϕ, **a** and **T** of the scalar variable ϕ, vector variable **v** and second-order tensor variable **B** are **isotropic functions** if

$$
\begin{array}{lll}
 & \phi(\mathbf{Qv}) = \phi(\mathbf{v}) & \phi(\mathbf{QBQ}^{\mathrm{T}}) = \phi(\mathbf{B}) \\
\mathbf{a}(\phi) = \mathbf{Qa}(\phi) & \mathbf{a}(\mathbf{Qv}) = \mathbf{Qa}(\mathbf{v}) & \mathbf{a}(\mathbf{QBQ}^{\mathrm{T}}) = \mathbf{Qa}(\mathbf{B}) \\
\mathbf{T}(\phi) = \mathbf{QT}(\phi)\mathbf{Q}^{\mathrm{T}} & \mathbf{T}(\mathbf{Qv}) = \mathbf{QT}(\mathbf{v})\mathbf{Q}^{\mathrm{T}} & \mathbf{T}(\mathbf{QBQ}^{\mathrm{T}}) = \mathbf{QT}(\mathbf{B})\mathbf{Q}^{\mathrm{T}}
\end{array}
$$

Isotropic Functions (4.A.1)

for all orthogonal tensors **Q**.

Isotropic functions are also called **isotropic invariants**. Here follow some examples.

Examples (of Isotropic Functions)

1. The scalar-valued function of a second order tensor $\phi(\mathbf{T}) = \det \mathbf{T}$ is an isotropic function since

$$\phi(\mathbf{QTQ}^{\mathrm{T}}) = \det(\mathbf{QTQ}^{\mathrm{T}}) = \det \mathbf{T}$$

2. The scalar-valued function of two second order tensors $\phi(\mathbf{A},\mathbf{B}) = \mathrm{tr}(\mathbf{AB})$ is an isotropic function of its two tensor variables since

$$\phi(\mathbf{QAQ}^{\mathrm{T}},\mathbf{QBQ}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{QAQ}^{\mathrm{T}}\mathbf{QBQ}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{QABQ}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{AB})$$

   More generally, the function $\mathrm{tr}(\mathbf{A}^m\mathbf{B}^m)$, *m* an integer, is isotropic

3. The vector-valued function **a** of a vector **v** and a second order tensor **T**, $\mathbf{a}(\mathbf{v},\mathbf{T}) = \mathbf{Av}$ is an isotropic function since

$$\mathbf{a}(\mathbf{Qv},\mathbf{QT}^m\mathbf{Q}^{\mathrm{T}}) = \mathbf{QT}^m\mathbf{Q}^{\mathrm{T}}\mathbf{Qv} = \mathbf{QTv} = \mathbf{Qa}(\mathbf{v},\mathbf{T})$$

   Indeed the function $\mathbf{a}(\mathbf{v},\mathbf{T}) = \mathbf{T}^m\mathbf{v}$, *m* an integer is an isotropic function.

4. The tensor-valued function of a second order tensor: $\mathbf{A}(\mathbf{T}) = \mathbf{T}^2$ is an isotropic function since

$$\mathbf{A}(\mathbf{QTQ}^{\mathrm{T}}) = (\mathbf{QTQ}^{\mathrm{T}})^2 = (\mathbf{QTQ}^{\mathrm{T}})(\mathbf{QTQ}^{\mathrm{T}}) = \mathbf{QT}^2\mathbf{Q}^{\mathrm{T}} = \mathbf{QA}(\mathbf{T})\mathbf{Q}^{\mathrm{T}}$$

   Indeed the function $\mathbf{A}(\mathbf{T}) = \mathbf{T}^m$, *m* an integer is an isotropic function.

■

Restrictions on the form that isotropic functions can take is next examined.

Isotropic Scalar-valued Functions

Consider first an isotropic scalar-valued function of a vector **u**, $\phi(\mathbf{u})$, so that $\phi(\mathbf{u}) = \phi(\mathbf{Qu})$. Since only the magnitude of **u** is invariant under an orthogonal tensor transformation, it follows that ϕ depends on **u** only through $|\mathbf{u}| = \mathbf{u}\cdot\mathbf{u}$, so $\phi \equiv \phi(\mathbf{u}\cdot\mathbf{u})$. Here, $\mathbf{u}\cdot\mathbf{u}$ is called the **integrity basis** of ϕ.

Similarly, an isotropic scalar-valued function of two arguments is defined through

$$\phi(\mathbf{u},\mathbf{v}) = \phi(\mathbf{Qu},\mathbf{Qv}) \tag{4.A.2}$$

for every orthogonal **Q**, and its integrity basis consists of the three scalar invariants

$$\mathbf{u}\cdot\mathbf{u}, \quad \mathbf{u}\cdot\mathbf{v}, \quad \mathbf{v}\cdot\mathbf{v} \tag{4.A.3}$$

since only the lengths of the two vectors and the angle between them are preserved under a rotation.

Consider next a scalar-valued isotropic function ϕ of a *symmetric* second-order tensor **S**. Since **S** is symnmeytric, it has the spectral decomposition representation $\mathbf{S} = \sum \lambda_i \mathbf{s}_i \otimes \mathbf{s}_i$, where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues and $\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ are the eigenvectors of **S**. Since **S** is isotropic,

$$\phi(\mathbf{S}) = \phi\left(\mathbf{QSQ}^{\mathrm{T}}\right) = \phi\left(\mathbf{Q}\left(\sum \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i\right)\mathbf{Q}^{\mathrm{T}}\right) = \phi\left(\sum \lambda_i \mathbf{Q}\hat{\mathbf{n}}_i \otimes \mathbf{Q}\hat{\mathbf{n}}_i\right) \tag{4.A.4}$$

Thus ϕ is independent of the orientation of the principal directions of **S** and so must depend only on the three principal values,

$$\phi(\mathbf{S}) = f(\lambda_1, \lambda_2, \lambda_3) \tag{4.A.5}$$

Note also that *f* must be a symmetric function of the eigenvalues. For example, take **Q** to be a positive rotation about $\hat{\mathbf{n}}_3$. Then $\mathbf{Q}\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$, $\mathbf{Q}\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$ and $\mathbf{Q}\hat{\mathbf{n}}_3 = \hat{\mathbf{n}}_3$, so

$$\phi(\mathbf{S}) = \phi\left(\mathbf{QSQ}^{\mathrm{T}}\right) = \phi(\lambda_2 \hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 + \lambda_1 \hat{\mathbf{n}}_2 \otimes \hat{\mathbf{n}}_2 + \lambda_3 \hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) = f(\lambda_2, \lambda_1, \lambda_3) \tag{4.A.6}$$

and, similarly, the subscripts on any pair of eigenvalues in 4.A.5 can be interchanged.

Since the set $\{\mathrm{tr}\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{S}^3\}$, the set of three principal scalar invariants $\{\mathrm{I}_\mathbf{S}, \mathrm{II}_\mathbf{S}, \mathrm{III}_\mathbf{S}\}$ and the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ uniquely determine one another, any of these sets can be regarded as the integrity basis of $\phi(\mathbf{S})$.

Some important isotropic scalar-valued functions and their integrity bases are listed in Table 4.A.1 below. The integrity basis consists of that entry together with appropriate entries from higher up in the Table, for example the integrity basis for a tensor **A** and two vectors **u** and **v** is

$$\mathbf{u}\cdot\mathbf{u},\quad \mathbf{v}\cdot\mathbf{v},\quad \mathbf{u}\cdot\mathbf{v},\quad \mathrm{tr}\mathbf{A},\quad \mathrm{tr}\mathbf{A}^2,\quad \mathrm{tr}\mathbf{A}^3$$
$$\mathbf{uAu},\quad \mathbf{uA}^2\mathbf{u},\quad \mathbf{vAv},\quad \mathbf{vA}^2\mathbf{v},\quad \mathbf{uAv},\quad \mathbf{uA}^2\mathbf{v}$$

	Isotropic Function		Integrity Basis
Scalar-valued functions	$\phi(\mathbf{u})$	$=\phi(\mathbf{Qu})$	$\mathbf{u}\cdot\mathbf{u}$
	$\phi(\mathbf{u},\mathbf{v})$	$=\phi(\mathbf{Qu},\mathbf{Qv})$	$\mathbf{u}\cdot\mathbf{v}$
	$\phi(\mathbf{A})$	$=\phi(\mathbf{QAQ}^{\mathrm{T}})$	$\mathrm{tr}\mathbf{A},\ \mathrm{tr}\mathbf{A}^2,\ \mathrm{tr}\mathbf{A}^3$
$\mathbf{A},\mathbf{B},\mathbf{C}$ are	$\phi(\mathbf{u},\mathbf{A})$	$=\phi(\mathbf{Qu},\mathbf{QAQ}^{\mathrm{T}})$	$\mathbf{uAu},\ \mathbf{uA}^2\mathbf{u}$
symmetric tensors	$\phi(\mathbf{A},\mathbf{B})$	$=\phi(\mathbf{QAQ}^{\mathrm{T}},\mathbf{QBQ}^{\mathrm{T}})$	$\mathrm{tr}\mathbf{AB},\ \mathrm{tr}\mathbf{A}^2\mathbf{B},\ \mathrm{tr}\mathbf{AB}^2,\ \mathrm{tr}\mathbf{A}^2\mathbf{B}^2$
	$\phi(\mathbf{u},\mathbf{A},\mathbf{v})$	$=\phi(\mathbf{Qu},\mathbf{QAQ}^{\mathrm{T}},\mathbf{Qv})$	$\mathbf{uAv},\ \mathbf{uA}^2\mathbf{v}$
	$\phi(\mathbf{A},\mathbf{B},\mathbf{C})$		$\mathrm{tr}\mathbf{ABC}$
	Four or more tensors		redundant

Table 4.A.1: Isotropic Scalar Functions and Integrity Bases

Isotropic Vector-valued Functions

Next, consider a vector-valued isotropic function **a** of a vector **v**, so $\mathbf{Qa}(\mathbf{v})=\mathbf{a}(\mathbf{Qv})$. To find the dependence of **a** on **v**, consider the scalar-valued function ϕ given by (note that ϕ here is linear in its first argument, **u**):

$$\phi(\mathbf{u},\mathbf{v})=\mathbf{u}\cdot\mathbf{a}(\mathbf{v}) \tag{4.A.7}$$

It follows that

$$\phi(\mathbf{Qu},\mathbf{Qv})=\mathbf{Qu}\cdot\mathbf{a}(\mathbf{Qv})=\mathbf{Qu}\cdot\mathbf{Qa}(\mathbf{v})=\mathbf{u}\cdot\mathbf{a}(\mathbf{v})=\phi(\mathbf{u},\mathbf{v}) \tag{4.A.8}$$

and so ϕ is an isotropic function of its two vector arguments and must depend only on the three invariants 4.A.3, and so takes the general form

$$\phi(\mathbf{u}\cdot\mathbf{u},\mathbf{u}\cdot\mathbf{v},\mathbf{v}\cdot\mathbf{v})=\mathbf{u}\cdot\alpha(\mathbf{v}\cdot\mathbf{v})\mathbf{v} \tag{4.A.9}$$

Finally, **a** must take the form

$$\mathbf{a}(\mathbf{v})=\alpha\mathbf{v} \tag{4.A.10}$$

where the coefficient α is a function of the scalar invariant of **v**, i.e. $\mathbf{v}\cdot\mathbf{v}$.

Note that the only isotropic vector function **a** of a tensor **B** is the null vector $\mathbf{a} = \mathbf{o}$.

Another important isotropic vector-valued functions is that of a vector and symmetric tensor. This and their integrity bases are listed in Table 4.A.2 below.

	Isotropic Function		Integrity Basis
Vector-valued functions	$\mathbf{a}(\mathbf{v})$	$\mathbf{a}(\mathbf{Qv}) = \mathbf{Qa}(\mathbf{v})$	$\mathbf{v}$
	$\mathbf{a}(\mathbf{T})$	$\mathbf{a}(\mathbf{QTQ}^{\mathrm{T}}) = \mathbf{Qa}(\mathbf{T})$	$\mathbf{o}$
$\mathbf{S}$ is a symmetric tensor	$\mathbf{a}(\mathbf{v},\mathbf{S})$	$\mathbf{a}(\mathbf{Qv},\mathbf{QSQ}^{\mathrm{T}}) = \mathbf{Qa}(\mathbf{v},\mathbf{S})$	$\mathbf{Sv},\ \mathbf{S}^2\mathbf{v}$

Table 4.A.2: Isotropic Vector Functions and Integrity Bases

Isotropic Tensor-valued Functions

Consider next a second-order tensor-valued function **T** of a tensor **B**. To find how **T** depends on **B**, this time consider the scalar-valued function ϕ given by (again, note that by definition ϕ is linear in its first argument, **A**)

$$\phi(\mathbf{A},\mathbf{B}) = \mathrm{tr}[\mathbf{AT}(\mathbf{B})] \qquad (4.A.11)$$

It follows that

$$\begin{aligned} \phi(\mathbf{QAQ}^{\mathrm{T}},\mathbf{QBQ}^{\mathrm{T}}) &= \mathrm{tr}[\mathbf{QAQ}^{\mathrm{T}}\mathbf{T}(\mathbf{QBQ}^{\mathrm{T}})] \\ &= \mathrm{tr}[\mathbf{QAQ}^{\mathrm{T}}\mathbf{QT}(\mathbf{B})\mathbf{Q}^{\mathrm{T}}] \\ &= \mathrm{tr}[\mathbf{QAT}(\mathbf{B})\mathbf{Q}^{\mathrm{T}}] \\ &= \mathrm{tr}[\mathbf{AT}(\mathbf{B})] \\ &= \phi(\mathbf{A},\mathbf{B}) \end{aligned} \qquad (4.A.12)$$

Thus ϕ is an isotropic function of its two tensor arguments and so, if **A** and **B** are symmetric, is a function of the ten invariants listed in Table 4.A.1. Since ϕ is linear in **A**, it can only depend on six of these ten invariants, namely $\mathrm{tr}\mathbf{A},\ \mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{B}^3$, $\mathrm{tr}\mathbf{AB},\ \mathrm{tr}\mathbf{AB}^2$, and so takes the form

$$\phi = \mathrm{tr}[\mathbf{AT}(\mathbf{B})] = \mathrm{tr}[\mathbf{A}(\alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2)] \qquad (4.A.13)$$

and so **T** takes the form

$$\boxed{\mathbf{T}(\mathbf{B}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2}$$

Form for a symmetric isotropic tensor function of a symmetric tensor (4.A.14)

where $\alpha_0, \alpha_1, \alpha_2$ are scalar functions of the invariants of **B**. Equation 4.A.13 can be rewritten in various alternative forms using the Cayley-Hamilton theorem, 1.9.45.

Some important symmetric isotropic tensor-valued functions are listed in Table 4.A.3 below.

	Isotropic Function		Integrity Basis
Tensor-valued functions $\mathbf{T}, \mathbf{A}, \mathbf{B}$ are symmetric tensors	$\mathbf{T}(\mathbf{v})$	$\mathbf{T}(\mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{T}(\mathbf{v})\mathbf{Q}^{\mathrm{T}}$	$\mathbf{I},\ \mathbf{v} \otimes \mathbf{v}$
	$\mathbf{T}(\mathbf{A})$	$\mathbf{T}(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{T}(\mathbf{A})\mathbf{Q}^{\mathrm{T}}$	$\mathbf{I},\ \mathbf{A},\ \mathbf{A}^2$
	$\mathbf{T}(\mathbf{u}, \mathbf{v})$	$\mathbf{T}(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{T}(\mathbf{u}, \mathbf{v})\mathbf{Q}^{\mathrm{T}}$	$\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}$
	$\mathbf{T}(\mathbf{u}, \mathbf{A})$	$\mathbf{T}(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{T}(\mathbf{u}, \mathbf{A})\mathbf{Q}^{\mathrm{T}}$	$\mathbf{u} \otimes \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{u} \otimes \mathbf{u},\ \ \mathbf{A}\mathbf{u} \otimes \mathbf{A}\mathbf{u}$
	$\mathbf{T}(\mathbf{u}, \mathbf{S}, \mathbf{v})$	$\mathbf{T}(\mathbf{Q}\mathbf{u}, \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{v})$ $= \mathbf{Q}\mathbf{T}(\mathbf{u}, \mathbf{A}, \mathbf{v})\mathbf{Q}^{\mathrm{T}}$	$\mathbf{u} \otimes \mathbf{A}\mathbf{v} + \mathbf{A}\mathbf{v} \otimes \mathbf{u}$ $\mathbf{v} \otimes \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{u} \otimes \mathbf{v}$
	$\mathbf{T}(\mathbf{A}, \mathbf{B})$		$\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A},\ \ \mathbf{A}\mathbf{B}\mathbf{A},\ \ \mathbf{B}\mathbf{A}\mathbf{B}$

Table 4.A.3: Isotropic (Symmetric) Tensor Functions and Integrity Bases

Some Results for Isotropic Functions

Here follow some other important results regarding isotropic functions.

1. The principal values of an isotropic tensor function **T** of a tensor **B** are scalar invariants of **B**.

   To show this, let $t_i(\mathbf{B})$ be the principal values of $\mathbf{T}(\mathbf{B})$ and let $t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})$ be the principal values of $\mathbf{T}(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})$. Then

$$\det(\mathbf{T}(\mathbf{B}) - t_i(\mathbf{B})\mathbf{I}) = 0, \quad \det(\mathbf{T}(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}) - t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})\mathbf{I}) = 0$$

   Because of the isotropy, and using the relation 1.9.13a, $\det(\mathbf{A}\mathbf{B}) = \det\mathbf{A}\det\mathbf{B}$, the second of these can be written as

$$\begin{aligned}\det(\mathbf{Q}\mathbf{T}(\mathbf{B})\mathbf{Q}^{\mathrm{T}} - t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})\mathbf{I}) &= \det(\mathbf{Q}\mathbf{T}(\mathbf{B})\mathbf{Q}^{\mathrm{T}} - t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})\mathbf{Q}\mathbf{I}\mathbf{Q}^{\mathrm{T}}) \\ &= \det(\mathbf{T}(\mathbf{B}) - t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})\mathbf{I})\end{aligned}$$

   This holds for all orthogonal **Q** and hence

$$t_i(\mathbf{B}) = t_i(\mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}) \qquad (4.A.15)$$

   which is the definition of an isotropic scalar invariant of **B**.

2. An isotropic tensor function **T** of a tensor **B** is coaxial with **B**.

   This follows directly from 4.A.14, since $\mathbf{B}^n$ has the same principal directions as **B**.

3. Let **T** be a symmetric isotropic tensor function of the symmetric tensor **B**; if in addition the function **T** is a *linear* function of **B**, then it has the representation

$$\mathbf{T}(\mathbf{B}) = \alpha(\operatorname{tr}\mathbf{B})\mathbf{I} + \beta\mathbf{B} \tag{4.A.16}$$

where α, β are arbitrary constants (independent of **B**).

This follows directly from 4.A.14, noting that only the first invariant, $\operatorname{tr}\mathbf{B}$, is linear in **B**. It will be noted that this is the form of the (isotropic) linear elastic material model, 4.1.15.

4. Let C be a fourth-order isotropic function, that is

$$C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \tag{4.A.17}$$

with the minor symmetries 1.9.65, $C_{ijkl} = C_{jikl} = C_{ijlk}$. Then it has the representation

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{4.A.18}$$

In terms of the identity tensors of §1.9.16 (compare with Eqn. 1.10.7),

$$\mathsf{C} = \lambda\mathbf{I}\otimes\mathbf{I} + \mu\left(\mathsf{I} + \bar{\mathsf{I}}\right) \tag{4.A.19}$$

To show this, consider a symmetric second-order tensor **S** and define $\mathbf{A} = \mathsf{C} : \mathbf{S}$. Then the index notation for $\mathbf{A}(\mathbf{S})$ is $C_{ijkl}S_{kl}$, and **A** is clearly symmetric. Then

$$\begin{aligned}
\mathbf{A}(\mathbf{QSQ}^{\mathrm{T}}): \quad C_{ijkl}\left(Q_{km}S_{mn}Q_{ln}\right) &= Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs}Q_{km}S_{mn}Q_{ln} \\
&= Q_{ip}Q_{jq}\delta_{rm}\delta_{sn}C_{pqrs}S_{mn} \\
&= Q_{ip}Q_{jq}C_{pqmn}S_{mn} \\
\mathbf{QA}(\mathbf{S})\mathbf{Q}^{\mathrm{T}}: \quad Q_{im}C_{mnkl}S_{kl}Q_{jn} &
\end{aligned} \tag{4.A.20}$$

from which it can be seen that **A** is a symmetric isotropic tensor function of the tensor variable **S**. Further, **A** is linear in **S**, and for **S** symmetric, it follows that **A** takes the representation 4.A.16,

$$\mathbf{A}(\mathbf{S}) = \lambda(\operatorname{tr}\mathbf{S})\mathbf{I} + 2\mu\mathbf{S} \tag{4.A.21}$$

In component form, this is

$$\begin{aligned}
A_{ij} &= \lambda\delta_{ij}S_{kk} + 2\mu S_{ij} \\
&= \lambda\delta_{ij}S_{kk} + \mu\left(S_{ij} + S_{ji}\right) \\
&= \lambda\delta_{ij}\delta_{ikl}S_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)S_{kl}
\end{aligned} \tag{4.A.22}$$

from which 4.A.18 follows.

4.A.2 The Symmetry Group

The nonempty set *G* with a binary operation, that is, to each pair of elements $a, b \in G$ there is assigned an element $ab \in G$, is called a **group** if the following axioms hold:

1. **associative law**: $(ab)c = a(bc)$ for any $a, b, c \in G$
2. **identity element**: there exists an element $e \in G$, called the identity element, such that $ae = ea = a$
3. **inverse**: for each $a \in G$, there exists an element $a^{-1} \in G$, called the inverse of *a*, such that $aa^{-1} = a^{-1}a = e$

Consider the set of tensors **G** of 4.3.2. Since for two tensors $\mathbf{G}_1$ and $\mathbf{G}_2$ in *G*,

$$\boldsymbol{\sigma}(\mathbf{F}\mathbf{G}_1\mathbf{G}_2) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{G}_1) = \boldsymbol{\sigma}(\mathbf{F}) \tag{4.A.23}$$

$\mathbf{G}_1\mathbf{G}_2 \in G$. The associative law clearly holds, the identity element is **I** and the inverse of **G** is $\mathbf{G}^{-1}$. Thus the set of tensors **G** forms a group.

4.A.3 Shear of an Isotropic Square Block

Consider a combined stretch and simple shear of an isotropic hyperelastic material, Fig. 4.A.1. Relative to the Cartesian coordinate system

$$x_1 = \lambda_1 X_1 + k\lambda_2 X_2, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \tag{4.A.24}$$

Then

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & k\lambda_2 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{4.A.25}$$

and so can be considered to be a homogeneous stretch followed by a simple shear. The left Cauchy-Green strain and inverse are

$$\mathbf{b} = \begin{bmatrix} \lambda_1^2 + k^2\lambda_2^2 & k\lambda_2^2 & 0 \\ k\lambda_2^2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \mathbf{b}^{-1} = \begin{bmatrix} 1/\lambda_1^2 & -k/\lambda_1^2 & 0 \\ -k/\lambda_1^2 & 1/\lambda_2^2 + k^2/\lambda_1^2 & 0 \\ 0 & 0 & 1/\lambda_3^2 \end{bmatrix} \tag{4.A.26}$$

The compressible and incompressible isotropic relations are (4.4.8 and 4.4.22 respectively)

$$\begin{aligned}\boldsymbol{\sigma}^{(c)} &= \beta_0\mathbf{I} + \beta_1\mathbf{b} + \beta_{-1}\mathbf{b}^{-1} \\ \boldsymbol{\sigma}^{(i)} &= -p\mathbf{I} + \alpha_1\mathbf{b} + \alpha_{-1}\mathbf{b}^{-1}\end{aligned} \tag{4.A.27}$$

Substituting in the Cauchy-Green strains, one finds that $\sigma_{13} = \sigma_{23} = 0$ and

$$\sigma_{12}^{(c)} = k\left(\beta_1\lambda_2^2 - \beta_{-1}\frac{1}{\lambda_1^2}\right), \qquad \sigma_{12}^{(i)} = k\left(\alpha_1\lambda_2^2 - \alpha_{-1}\frac{1}{\lambda_1^2}\right) \tag{4.A.28}$$

Using this relation, it can then be seen that

$$\sigma_{11} - \sigma_{22} = \frac{\lambda_1^2 - \lambda_2^2 + k^2\lambda_2^2}{k\lambda_2^2}\sigma_{12} \tag{4.A.29}$$

which holds for both compressible and incompressible materials, and is the universal relation analogous to 4.4.40. Here, however, the stretches can be chosen so as to make the normal stress-difference zero.

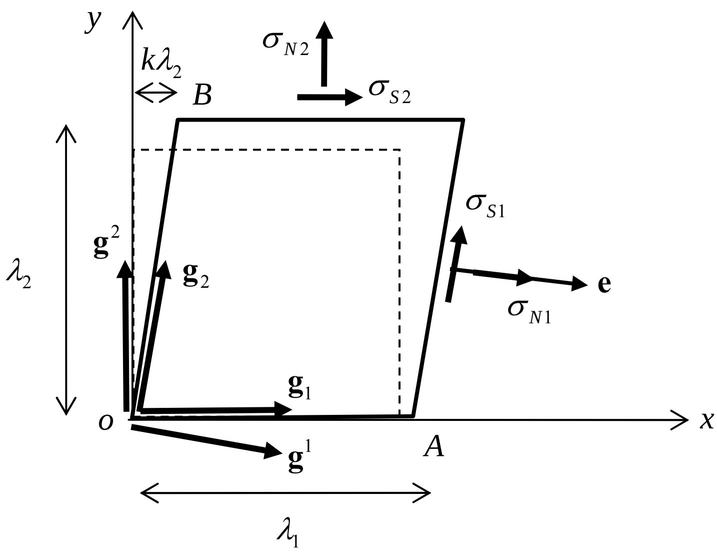

Figure 4.A.1: block under stretch and simple shear

Introduce now base vectors $\mathbf{g}_1, \mathbf{g}_2$ along the edges of the deformed block, with corresponding contravariant base vectors $\mathbf{g}^1$ and $\mathbf{g}^2$, Fig. 4.A.1, so that

$$\begin{aligned}&\mathbf{g}_1 = \mathbf{e}_1, \quad \mathbf{g}_2 = k\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{g}_3 = \mathbf{e}_3 \\ &\mathbf{g}^1 = \mathbf{e}_1 - k\mathbf{e}_2, \quad \mathbf{g}^2 = \mathbf{e}_2, \quad \mathbf{g}^3 = \mathbf{e}_3\end{aligned} \tag{4.A.30}$$

The metric coefficients are

$$g_{ij} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g = 1 \tag{4.A.31}$$

From 3.9.2, the unit normals to the block surfaces are (see 4.4.44)

$$\overline{\mathbf{n}}^1 = \hat{\mathbf{g}}^1 = \frac{\mathbf{g}^1}{\sqrt{g^{11}}} = \frac{\mathbf{e}_1 - k\mathbf{e}_2}{\sqrt{1+k^2}}, \quad \overline{\mathbf{n}}^2 = \hat{\mathbf{g}}^2 = \frac{\mathbf{g}^2}{\sqrt{g^{22}}} = \mathbf{e}_2 \tag{4.A.32}$$

The stress components with respect to the curvilinear system can be obtained from the transformation rule in §1.13.1:

$$\left[\overline{\sigma}^{ij}\right] = \left[A_m^{\cdot i}\right]^T \left[\sigma^{mn}\right]\left[A_n^{\cdot j}\right], \quad A_i^{\cdot j} = \mathbf{e}_i \cdot \mathbf{g}^j = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.A.33}$$

leading to

$$\left[\overline{\sigma}^{ij}\right] = \begin{bmatrix} \sigma_{11} - 2k\sigma_{12} + k^2\sigma_{22} & \sigma_{12} - k\sigma_{22} & 0 \\ \sigma_{21} - k\sigma_{22} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \tag{4.A.34}$$

The normal and shear stresses acting on the surfaces of the block are (see Fig. 4.A.1) are

$$\sigma_{N2} = \sigma_{22}, \quad \sigma_{S2} = \sigma_{12}, \quad \sigma_{N1} = \overline{\sigma}^{11}, \quad \sigma_{S1} = \overline{\sigma}^{12} \tag{4.A.35}$$

In order that the normal stresses acting on the block are zero, then, one requires

$$\sigma_{22} = 0, \qquad \sigma_{11} - 2k\sigma_{12} = 0 \tag{4.A.36}$$

From 4.A.29, this means that

$$\lambda_1^2 = \left(1 + k^2\right)\lambda_2^2 \tag{4.A.37}$$

A physical interpretation of this results is that the lengths of the sides of the deformed block are equal, $|oA| = |oB|$ in Fig. 4.A.1. In this case, 4.A.34 reduces to, using 4.A.28,

$$\begin{aligned} \begin{Bmatrix} \left[\overline{\sigma}^{ij}\right]^{(c)} \\ \left[\overline{\sigma}^{ij}\right]^{(i)} \end{Bmatrix} &= \begin{Bmatrix} \sigma_{12}^{(c)} \\ \sigma_{12}^{(i)} \end{Bmatrix} \left(\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1\right) \\ &= k\left[\begin{Bmatrix} \beta_1 \\ \alpha_1 \end{Bmatrix} \frac{\lambda_1^2}{1+k^2} - \begin{Bmatrix} \beta_{-1} \\ \alpha_{-1} \end{Bmatrix} \frac{1}{\lambda_1^2}\right] \left(\mathbf{g}_1 \otimes \mathbf{g}_2 + \mathbf{g}_2 \otimes \mathbf{g}_1\right) \end{aligned} \tag{4.A.38}$$

Thus a state of pure shear is achieved, with only shear stresses acting on the faces, and a square block deforms into a rhombic block.

Consider now the (incompressible) Neo-Hookean model, Eqn. 4.4.54, for which

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2c_1\mathbf{b} \tag{4.A.39}$$

The stress components are then

$$\bar{\sigma}^{ij} = -pg^{ij} + 2c_1 b^{ij} \tag{4.A.39}$$

The metric components g^{ij} are given by 4.A.31. The contravariant components of the left Cauchy-Green strain can be obtained from coordinate transformation equations similar to 4.A.33 (with $b_{ij} = b^{ij}$ in the Cartesian system), leading to

$$\left[b^{ij}\right] = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 + k^2\lambda_2^2 & k\lambda_2^2 & 0 \\ k\lambda_2^2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{4.A.40}$$

with $\lambda_1\lambda_2\lambda_3 = 1$. Then, with the stress taking the representation 4.A.38, with $\alpha_1 = 2c_1$, $\alpha_{-1} = 0$,

$$\begin{bmatrix} 0 & 2kc_1\lambda_1^2/\left(1+k^2\right) & 0 \\ 2kc_1\lambda_1^2/\left(1+k^2\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -p\begin{bmatrix} 1+k^2 & -k & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2c_1\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{4.A.41}$$

Solving leads to

$$\begin{gathered} p = 2c_1\left(1+k^2\right)^{-1/3} \\ \lambda_1 = \left(1+k^2\right)^{1/3}, \quad \lambda_2 = \lambda_3 = \left(1+k^2\right)^{-1/6} \end{gathered} \tag{4.A.42}$$

The solution shows that $\lambda_1 > 1$ and $\lambda_2 < 1$ and so the block deforms as in Fig. 4.A.2.

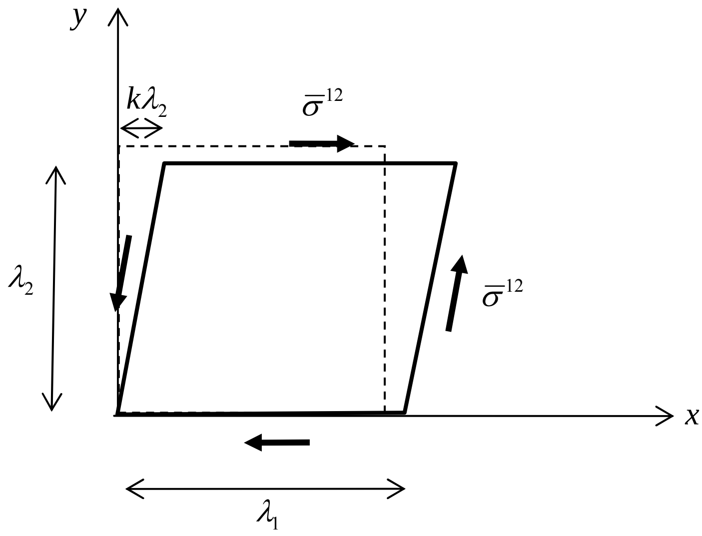

Figure 4.A.2: simple shear of a Neo-Hookean block

Note that, in contrast to the decomposition