4.4 Isotropic Hyperelasticity

Attention is now restricted to the case of isotropic hyperelastic materials.

4.4.1 Constitutive Equations in Material Form

Consider the general hyperelastic constitutive law 4.2.5:

$$\mathbf{S} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} \tag{4.4.1}$$

From §4.3.2, the strain energy for an isotropic hyperelastic material must be a function of the principal invariants of **C**:

$$W = W\big(\mathrm{I}_{\mathbf{C}}(\mathbf{C}), \mathrm{II}_{\mathbf{C}}(\mathbf{C}), \mathrm{III}_{\mathbf{C}}(\mathbf{C})\big) \tag{4.4.2}$$

Then, using the relations 1.11.33 for the derivatives of the invariants (taking into account that **C** is symmetric),

$$\begin{aligned}\frac{\partial W}{\partial \mathbf{C}} &= \frac{\partial W}{\partial \mathrm{I}_{\mathbf{C}}}\frac{\partial \mathrm{I}_{\mathbf{C}}}{\partial \mathbf{C}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\frac{\partial \mathrm{II}_{\mathbf{C}}}{\partial \mathbf{C}} + \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}}\frac{\partial \mathrm{III}_{\mathbf{C}}}{\partial \mathbf{C}} \\ &= \left(\frac{\partial W}{\partial \mathrm{I}_{\mathbf{C}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\mathrm{I}_{\mathbf{C}}\right)\mathbf{I} - \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\mathbf{C} + \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}}\mathrm{III}_{\mathbf{C}}\mathbf{C}^{-1}\end{aligned} \tag{4.4.3}$$

Further, the $\mathbf{C}^{-1}$ term can be replaced by a $\mathbf{C}^{2}$ term using the Cayley-Hamilton theorem. In summary then, expressing **S** in the form 4.3.9 (and 4.3.12),

$$\boxed{\begin{gathered}\mathbf{S} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}} = \begin{cases} 2\alpha_0\mathbf{I} + 2\alpha_1\mathbf{C} + 2\alpha_2\mathbf{C}^2 \\ \\ 2\beta_0\mathbf{I} + 2\beta_1\mathbf{C} + 2\beta_{-1}\mathbf{C}^{-1}\end{cases} \\ \\ \begin{aligned} \alpha_0 &= \frac{\partial W}{\partial \mathrm{I}_{\mathbf{C}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\mathrm{I}_{\mathbf{C}} + \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}}\mathrm{II}_{\mathbf{C}} & \beta_0 &= \frac{\partial W}{\partial \mathrm{I}_{\mathbf{C}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\mathrm{I}_{\mathbf{C}} \\ \alpha_1 &= -\frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}} - \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}}\mathrm{I}_{\mathbf{C}} & \beta_1 &= -\frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}} \\ \alpha_2 &= \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}} & \beta_{-1} &= \frac{\partial W}{\partial \mathrm{III}_{\mathbf{C}}}\mathrm{III}_{\mathbf{C}} \end{aligned}\end{gathered}}$$

Constitutive Equation in material description (4.4.4)

4.4.2 Constitutive Equations in Spatial Form

As mentioned at the end of §4.3.2, the invariants of **C** and **b** are the same and the strain energy can be expressed in terms of the invariants of **b**, $W = W\left(\mathrm{I}_{\mathbf{b}}(\mathbf{b}), \mathrm{II}_{\mathbf{b}}(\mathbf{b}), \mathrm{III}_{\mathbf{b}}(\mathbf{b})\right)$. Then

$$\frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} = \begin{cases} \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 \\ \\ \beta_0 \mathbf{I} + \beta_1 \mathbf{b} + \beta_{-1} \mathbf{b}^{-1} \end{cases} \tag{4.4.5}$$

and the coefficients α_i, β_i are the same as in 4.4.4

Taking the expression for $\partial W / \partial \mathbf{C}$ in 4.4.4, pre-contracting with **F**, post-contracting with $\mathbf{F}^{\mathrm{T}}$, using the definitions $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, and (pre- or post-) contracting the above similar expression for $\partial W / \partial \mathbf{b}$ in terms of the invariants of **b** with **b**, one finds that

$$\frac{\partial W}{\partial \mathbf{b}} \mathbf{b} = \mathbf{b} \frac{\partial W}{\partial \mathbf{b}} = \mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^{\mathrm{T}} \tag{4.4.6}$$

Therefore, using $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, the constitutive equation in the material formulation, $\mathbf{S} = 2\partial W(\mathbf{C}) / \partial \mathbf{C}$, can be transformed into a spatial constitutive equation:

$$\boldsymbol{\sigma} = 2J^{-1} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} \mathbf{b} = 2J^{-1} \mathbf{b} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} \tag{4.4.7}$$

It should be emphasised that this expression, unlike the relation $\mathbf{S} = 2\partial W(\mathbf{C}) / \partial \mathbf{C}$, holds only for isotropic materials.

Differentiating the strain energy with respect to **b** (exactly as in Eqn. 4.4.3) and pre- or post-contracting the result with $2J^{-1}\mathbf{b}$ then leads to

$$\boxed{\begin{gathered} \boldsymbol{\sigma} = 2J^{-1} \mathbf{b} \frac{\partial W(\mathbf{b})}{\partial \mathbf{b}} = \begin{cases} \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 \\ \\ \beta_0 \mathbf{I} + \beta_1 \mathbf{b} + \beta_{-1} \mathbf{b}^{-1} \end{cases} \\ \\ \begin{aligned} \alpha_0 &= 2J^{-1}\left[\frac{\partial W}{\partial \mathrm{III}_{\mathbf{b}}} \mathrm{III}_{\mathbf{b}}\right] & \beta_0 &= 2J^{-1}\left[\frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}} \mathrm{II}_{\mathbf{b}} + \frac{\partial W}{\partial \mathrm{III}_{\mathbf{b}}} \mathrm{III}_{\mathbf{b}}\right] \\ \alpha_1 &= 2J^{-1}\left[\frac{\partial W}{\partial \mathrm{I}_{\mathbf{b}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}} \mathrm{I}_{\mathbf{b}}\right] & \beta_1 &= 2J^{-1}\left[\frac{\partial W}{\partial \mathrm{I}_{\mathbf{b}}}\right] \\ \alpha_2 &= 2J^{-1}\left[-\frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}}\right] & \beta_{-1} &= 2J^{-1}\left[-\frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}} \mathrm{III}_{\mathbf{b}}\right] \end{aligned} \end{gathered}}$$

Constitutive Equation in spatial description (4.4.8)

Principal Directions of the Cauchy Stress and Left Cauchy-Green Tensor

Recall that the eigenvalues of the right stretch tensor **U** are the principal stretches λ_i, and that the eigenvalues of the right Cauchy-Green tensor **C** and the left Cauchy-Green tensor **b** are the squares of the principal stretches, λ_i^2 (the eigenvectors of **b** are those of **C** rotated with the rotation tensor **R**). Taking $\hat{\mathbf{n}}$ to be an eigenvector of **b** and λ^2 the corresponding eigenvalue, and using the constitutive equation $\boldsymbol{\sigma} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2$, one has

$$\boldsymbol{\sigma}\hat{\mathbf{n}} = \alpha_0 \mathbf{I}\hat{\mathbf{n}} + \alpha_1 \mathbf{b}\hat{\mathbf{n}} + \alpha_2 \mathbf{b}^2\hat{\mathbf{n}} = \left(\alpha_0 + \alpha_1\lambda^2 + \alpha_2\lambda^4\right)\hat{\mathbf{n}} \tag{4.4.9}$$

Thus $\boldsymbol{\sigma}\hat{\mathbf{n}} = \sigma\hat{\mathbf{n}}$ where σ is an eigenvalue of $\boldsymbol{\sigma}$, the principal stress, given by term inside the brackets, and so the eigenvectors, or principal directions, of the Cauchy stress and **b** coincide.

If one takes a deformation for which $b_{13} = b_{23} = 0$, it follows from 4.4.8 that, for arbitrary coefficient β_{-1}, one also has $\sigma_{13} = \sigma_{23} = 0$. The tensors **b** and $\boldsymbol{\sigma}$ are coaxial, that is, $\mathbf{b}\boldsymbol{\sigma} = \boldsymbol{\sigma}\mathbf{b}$. From this relation, one finds that one must have

$$\frac{\sigma_{11} - \sigma_{22}}{\sigma_{12}} = \frac{b_{11} - b_{22}}{b_{12}} \tag{4.4.10}$$

This relation holds for all compressible isotropic hyperelastic materials under the stress and deformation conditions stated, regardless of the particular constitutive relation.

4.4.3 Constitutive Equations in terms of the Stretches

The derivatives of the stretch with respect to **b** are given in §2.3.3. Introducing the principal stresses, and noting that $\mathbf{b}\hat{\mathbf{n}}_i = \lambda_i^2\hat{\mathbf{n}}_i$ (no sum over the *i*) it follows then that (no sum over the *i* in what follows)

$$\begin{aligned}
\sigma_i\,\hat{\mathbf{n}}_i = \boldsymbol{\sigma}\hat{\mathbf{n}}_i &= 2J^{-1}\mathbf{b}\frac{\partial W(\mathbf{b})}{\partial \mathbf{b}}\hat{\mathbf{n}}_i = 2J^{-1}\mathbf{b}\frac{\partial W(\lambda)}{\partial \lambda_k}\frac{\partial \lambda_k}{\partial \mathbf{b}}\hat{\mathbf{n}}_i \\
&= 2J^{-1}\mathbf{b}\frac{\partial W(\lambda)}{\partial \lambda_k}\frac{1}{2\lambda_k}(\hat{\mathbf{n}}_k \otimes \hat{\mathbf{n}}_k)\hat{\mathbf{n}}_i = 2J^{-1}\frac{\partial W(\lambda)}{\partial \lambda_i}\frac{1}{2\lambda_i}\mathbf{b}\hat{\mathbf{n}}_i \\
&= J^{-1}\lambda_i\frac{\partial W(\lambda)}{\partial \lambda_i}\hat{\mathbf{n}}_i
\end{aligned} \tag{4.4.11}$$

or

$$\sigma_1 = J^{-1}\lambda_1\frac{\partial W(\lambda)}{\partial \lambda_1}, \quad \sigma_2 = J^{-1}\lambda_2\frac{\partial W(\lambda)}{\partial \lambda_2}, \quad \sigma_3 = J^{-1}\lambda_3\frac{\partial W(\lambda)}{\partial \lambda_3} \tag{4.4.12}$$

Similarly, for the Piola-Kirchhoff stresses, one has

$$P_i = \frac{\partial W}{\partial \lambda_i}, \qquad S_i = \frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i} \tag{4.4.13}$$

Example (Uniaxial Stretch)

Consider a specimen of isotropic hyperelastic material under a uniaxial tension σ_{11}, with

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{4.4.14}$$

Since $\boldsymbol{\sigma}$ and **b** are coaxial (see 4.4.9), the principal stresses are, from 4.4.8,

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta_1 \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} + \beta_{-1} \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix} \tag{4.4.15}$$

Subtracting the second of these equations from the first leads to

$$\sigma_{11} = \left(\lambda_1^2 - \lambda_2^2\right)\left(\beta_1 - \beta_{-1}\frac{1}{\lambda_1^2\lambda_2^2}\right) \tag{4.4.16}$$

Subtracting the third from the second leads to

$$0 = \left(\lambda_2^2 - \lambda_3^2\right)\left(\beta_1 - \beta_{-1}\frac{1}{\lambda_2^2\lambda_3^2}\right) \tag{4.4.17}$$

which implies that $\lambda_2 = \lambda_3$ (the terms inside the second bracket can also be zero, but not if $\beta_1 > 0$, $\beta_{-1} \le 0$, as is usually the case).

The material parameters $\beta_0, \beta_1, \beta_{-1}$ can be obtained from such a test by measuring σ_{11}, λ_1 and λ_2. On the other hand, if the constitutive relation, that is, the parameters, are known, then one can obtain the stretches by specifying the uniaxial stress.

■

4.4.4 Incompressible Materials

When the material is incompressible, then

$$J = \det \mathbf{F} = \mathrm{III}_{\mathbf{C}} = \mathrm{III}_{\mathbf{b}} = 1 \tag{4.4.18}$$

and the strain energy function is only dependent on two invariants:

$$W = W(\mathrm{I}_{\mathbf{C}}, \mathrm{II}_{\mathbf{C}}) = W(\mathrm{I}_{\mathbf{b}}, \mathrm{II}_{\mathbf{b}}), \tag{4.4.19}$$

and the constitutive equations are the same as those given earlier for the compressible material, with $\mathrm{III}_{\mathbf{b}} = 1, J^{-1} = 1$. However, the derivative $\partial W / \partial \mathrm{III}_{\mathbf{C}}$ is an unknown and in particular it is unknown at $\mathrm{III}_{\mathbf{C}} = 1$.

Returning to the more general constitutive equations for an incompressible material, 4.2.36, 4.2.39, one has

$$\mathbf{S} = 2\frac{\partial W(\mathrm{I}_{\mathbf{C}}, \mathrm{II}_{\mathbf{C}})}{\partial \mathbf{C}} - p\mathbf{C}^{-1}, \qquad \boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W(\mathrm{I}_{\mathbf{C}}, \mathrm{II}_{\mathbf{C}})}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}} - p\mathbf{I} \tag{4.4.20}$$

Carrying out the differentiation as before, one now has, directly from these equations,

$$\boxed{\begin{gathered} \mathbf{S} = \beta_0 \mathbf{I} + \beta_1 \mathbf{C} - p\mathbf{C}^{-1} \\ \beta_0 = 2\left[\frac{\partial W}{\partial \mathrm{I}_{\mathbf{C}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}}\mathrm{I}_{\mathbf{C}}\right] \qquad \beta_1 = -2\frac{\partial W}{\partial \mathrm{II}_{\mathbf{C}}} \end{gathered}}$$

Constitutive Equation for an Isotropic Incompressible Hyperelastic Material (Material Form) (4.4.21)

and

$$\boxed{\begin{gathered} \boldsymbol{\sigma} = \alpha_1 \mathbf{b} + \alpha_{-1}\mathbf{b}^{-1} - p\mathbf{I} \\ \alpha_1 = 2\frac{\partial W}{\partial \mathrm{I}_{\mathbf{b}}} \qquad \alpha_{-1} = -2\frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}} \end{gathered}}$$

Constitutive Equation for an Isotropic Incompressible Hyperelastic Material (Spatial Form) (4.4.22)

Comparing these equations with the more general compressible equations: in the **S** equation, 4.4.4, the hydrostatic pressure is equivalent to $-2\partial W / \partial \mathrm{III}_{\mathbf{C}}$ (with $\mathrm{III}_{\mathbf{C}} = 1$); in the $\boldsymbol{\sigma}$ equation, 4.4.8, it is $p = -2[\partial W / \partial \mathrm{III}_{\mathbf{b}} + \mathrm{II}_{\mathbf{b}} \partial W / \partial \mathrm{II}_{\mathbf{b}}]$.

Similarly, the equations can be written in terms of the principal stretches and principal stresses (the incompressibility constraint is now $\lambda_1 \lambda_2 \lambda_3 = 1$):

$$\boxed{\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad P_i = \frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_i} p, \quad S_i = \frac{1}{\lambda_i}\frac{\partial W}{\partial \lambda_i} - \frac{1}{\lambda_i^2} p} \tag{4.4.23}$$

Example (Strip Biaxial Tension)

Consider an incompressible isotropic hyperelastic material which is stretched in the $\mathbf{e}_1$ direction but its dimensions in the $\mathbf{e}_2$ direction is held constant, with

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} \tag{4.4.24}$$

With $\sigma_3 = 0$,one finds that

$$p = 2\frac{\partial W}{\partial \mathrm{I}_\mathbf{b}}\frac{1}{\lambda^2} - 2\frac{\partial W}{\partial \mathrm{II}_\mathbf{b}}\lambda^2 \tag{4.4.25}$$

and the stresses are

$$\sigma_i = 2\left[\frac{\partial W}{\partial \mathrm{I}_\mathbf{b}}\left(\lambda_i^2 - \frac{1}{\lambda^2}\right) + \frac{\partial W}{\partial \mathrm{II}_\mathbf{b}}\left(\lambda^2 - \frac{1}{\lambda_i^2}\right)\right], \qquad i = 1, 2 \tag{4.4.26}$$

■

Example (Biaxial Stretch)

Consider the biaxial stretch of a thin sheet of incompressible isotropic hyperelastic material, with

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1/\lambda_1\lambda_2 \end{bmatrix} \tag{4.4.27}$$

From 4.4.18:

$$\sigma_i = \alpha_1\lambda_i^2 + \alpha_{-1}\lambda_i^{-2} - p, \quad i = 1, 2, 3, \quad \left(\alpha_1 = 2\frac{\partial W}{\partial \mathrm{I}_\mathbf{b}}, \alpha_{-1} = -2\frac{\partial W}{\partial \mathrm{II}_\mathbf{b}}\right) \tag{4.4.28}$$

Taking the stretches to be in-plane, with the larger faces stress-free, one has $\sigma_3 = 0$ so that, with $\lambda_3 = 1/\lambda_1\lambda_2$, one can solve for p,

$$p = \alpha_1\frac{1}{\lambda_1^2\lambda_2^2} + \alpha_{-1}\lambda_1^2\lambda_2^2$$

and the stresses in the sheet are

$$\sigma_1 = \left(\alpha_1 - \alpha_{-1}\lambda_2^2\right)\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right), \quad \sigma_2 = \left(\alpha_1 - \alpha_{-1}\lambda_1^2\right)\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right) \tag{4.4.29}$$

If the strain energy is given in terms of the stretches, one would use 4.4.23 rather than 4.4.22.

The PK1 stresses, $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, are, directly from these equations,

$$P_1 = \frac{\sigma_1}{\lambda_1} = \left(\alpha_1 - \alpha_{-1}\lambda_2^2\right)\left(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2}\right), \quad P_2 = \frac{\sigma_2}{\lambda_2} = \left(\alpha_1 - \alpha_{-1}\lambda_1^2\right)\left(\lambda_2 - \frac{1}{\lambda_1^2\lambda_2^3}\right) \tag{4.4.30}$$

The total forces acting on the lateral surfaces are then, from 3.5.4,

$$F_1 = P_1 A_0 = A_0\left(\alpha_1 - \alpha_{-1}\lambda_2^2\right)\left(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2}\right), \quad F_2 = P_2 A_0 = A_0\left(\alpha_1 - \alpha_{-1}\lambda_1^2\right)\left(\lambda_2 - \frac{1}{\lambda_1^2\lambda_2^3}\right) \tag{4.4.31}$$

where A_0 is the area in the reference configuration.

The material parameters α_1, α_{-1} can be determined from such a test by measuring the applied forces. This is helped by controlling the stretches λ_1 and λ_2 in such a way that the invariant $\mathrm{I}_{\mathbf{b}}$ is held constant, or the invariant $\mathrm{II}_{\mathbf{b}}$ is held constant.

When the two applied forces are equal, $F_1 = F_2$, one finds that

$$\left(\lambda_1 - \lambda_2\right)\left(\left(1 + \lambda_1^3\lambda_2^3\right)\left(1 + \frac{\alpha_{-1}}{\alpha_1}\lambda_1\lambda_2\right) - \frac{\alpha_{-1}}{\alpha_1}\left(\lambda_1 + \lambda_2\right)^2\right) = 0 \tag{4.4.32}$$

The term inside the first brackets gives the expected symmetric solution $\lambda_1 = \lambda_2$. However, the term inside the second brackets can also be zero for $\lambda_1 \neq \lambda_2$, but this will only occur for fairly large stretches for typical values of the material parameters. The symmetric equal-stretch solution holds up until this happens. For example, for the typical value $-\alpha_{-1}/\alpha_1 = 0.1$, the second term becomes zero when $\lambda_1 = \lambda_2 \approx 3.17$. The symmetric solution is unstable, in the sense that the forces must be precisely equal for it to hold. If there is even the slightest imbalance between F_1 and F_2, then $\lambda_1 \approx \lambda_2$ up to about a value of 3.17 but then the near-square will begin to deform into a rectangle, with the response now keeping close to the asymmetric solution. The long side of the rectangle will be along the direction of the slightly larger force and will keep lengthening until the stretch in the other direction returns back to $\lambda = 1$.

This large strain problem differs from the problems of linear elasticity, which have unique solutions.

■

Example (Simple Shear)

The case of simple shear was discussed in detail in §2.2.6. In the equations following 2.2.40, let

$$\cos 2\beta = \sin\theta, \quad \sin 2\beta = \cos\theta, \quad \tan 2\beta = \frac{1}{\tan\theta} = \frac{2}{k} \tag{4.4.33}$$

Then the principal material directions can be expressed as

$$\hat{\mathbf{N}}_1 = \sin\beta \mathbf{E}_1 + \cos\beta \mathbf{E}_2, \quad \hat{\mathbf{N}}_2 = -\cos\beta \mathbf{E}_1 + \sin\beta \mathbf{E}_2 \tag{4.4.34}$$

With $\mathbf{n} = \mathbf{RN}$, the principal spatial directions are (see figure 4.4.1 below)

$$\hat{\mathbf{n}}_1 = \cos\beta \mathbf{E}_1 + \sin\beta \mathbf{E}_2, \quad \hat{\mathbf{n}}_2 = -\sin\beta \mathbf{E}_1 + \cos\beta \mathbf{E}_2 \tag{4.4.35}$$

The principal stretches are

$$\lambda_1 = \frac{1+\cos 2\beta}{\sin 2\beta}, \quad \lambda_2 = \frac{1-\cos 2\beta}{\sin 2\beta} \quad \text{and} \quad 1 \tag{4.4.36}$$

and the principal stresses are, from 4.4.18,

$$\begin{aligned} \sigma_1 &= \alpha_1 \lambda_1^2 + \alpha_{-1}\frac{1}{\lambda_1^2} - p \\ \sigma_2 &= 2\alpha_1 \lambda_2^2 + \alpha_{-1}\frac{1}{\lambda_2^2} - p \qquad \left(\alpha_1 = 2\frac{\partial W}{\partial \mathrm{I}_\mathbf{b}}, \alpha_{-1} = -2\frac{\partial W}{\partial \mathrm{II}_\mathbf{b}}\right) \\ \sigma_3 &= \alpha_1 + \alpha_{-1} - p \end{aligned} \tag{4.4.37}$$

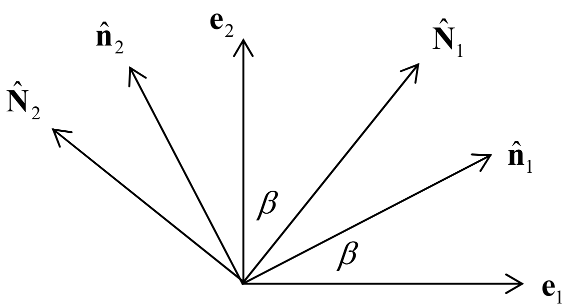

Figure 4.4.1: principal directions for pure shear

The strain energy here is a function of the invariants of **b**, which are

$$\mathrm{I}_\mathbf{b} = 3 + k^2, \quad \mathrm{II}_\mathbf{b} = 3 + k^2 \quad (\mathrm{III}_\mathbf{b} = 1) \tag{4.4.38}$$

In the original Cartesian coordinates, the stresses are

$$\begin{aligned}\sigma_{11} &= \sigma_1 \cos^2 \beta + \sigma_2 \sin^2 \beta \\ \sigma_{22} &= \sigma_1 \sin^2 \beta + \sigma_2 \cos^2 \beta \\ \sigma_{12} &= (\sigma_1 - \sigma_2) \sin\beta \cos\beta\end{aligned} \tag{4.4.39}$$

It follows that (this is a universal relation, independent of the constitutive relation)

$$\sigma_{11} - \sigma_{22} = (2/\tan 2\beta)\sigma_{12} = k\sigma_{12} \tag{4.4.40}$$

The fact that and $\sigma_{11} \neq \sigma_{22}$ is called the **Poynting effect**. The stress are now

$$\begin{aligned}\sigma_{11} &= \alpha_1\left(1+k^2\right) + \alpha_{-1} - p \\ \sigma_{22} &= \alpha_1 + \alpha_{-1}\left(1+k^2\right) - p \\ \sigma_{12} &= \left(\alpha_1 - \alpha_{-1}\right)k \\ \sigma_{33} &= \alpha_1 + \alpha_{-1} - p\end{aligned} \tag{4.4.41}$$

The proportionality factor between the shear stress and the shear strain measure k is called the shear modulus:

$$\mu = 2\left(\frac{\partial W}{\partial \mathrm{I}_{\mathbf{b}}} + \frac{\partial W}{\partial \mathrm{II}_{\mathbf{b}}}\right) \tag{4.4.42}$$

In the case of plane stress, the hydrostatic pressure p can be evaluated, and one finds that

$$\sigma_{11} = \alpha_1 k^2, \qquad \sigma_{22} = \alpha_{-1} k^2, \qquad \sigma_{12} = \mu k \tag{4.4.43}$$

The stresses acting on the deformed material are shown in Fig. 4.4.2. The unit normal $\mathbf{e}$ to the sloping surface (on the right hand side) is

$$\mathbf{e} = \frac{1}{\sqrt{1+k^2}}\left(\mathbf{e}_1 - k\mathbf{e}_2\right) \tag{4.4.44}$$

and, from Cauchy's law, the traction acting on that surface is

$$\mathbf{t} = \frac{1}{\sqrt{1+k^2}}\begin{bmatrix}\sigma_{11} - k\sigma_{12} \\ \sigma_{12} - k\sigma_{22} \\ 0\end{bmatrix} \tag{4.4.45}$$

The normal and shear traction are then (using the Poynting effect)

$$\begin{aligned}t_N &= \frac{1}{1+k^2}\left[\left(\sigma_{11} - k\sigma_{12}\right) - k\left(\sigma_{12} - k\sigma_{22}\right)\right] = \sigma_{11} - \frac{k\left(2+k^2\right)}{1+k^2}\sigma_{12} = \sigma_{22} - \frac{k}{1+k^2}\sigma_{12} \\ t_S &= \frac{1}{1+k^2}\left[k\left(\sigma_{11} - k\sigma_{12}\right) + \left(\sigma_{12} - k\sigma_{22}\right)\right] = \frac{1}{1+k^2}\sigma_{12}\end{aligned}$$

(4.4.46)

and tend to σ_{11} and σ_{12} as $k \to 0$.

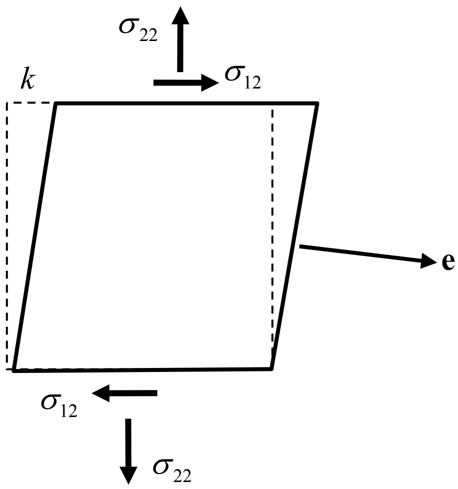

Figure 4.4.2: stresses acting on the material in simple shear

It is clear then that the pure shear of material discussed here can only occur when normal stresses are applied to the faces. There is of course no volume change for an incompressible material; for a compressible material, however, there will in general be a volume change unless normal stresses are applied, a phenomenon known as the **Kelvin effect**.

■

See also the shear problem using convected coordinates discussed in the Appendix to this Chapter, §4.A.3.

4.4.5 Series Expansion of the Strain Energy Function

It is sometimes helpful when seeking specific constitutive laws for isotropic hyperelastic materials to expand the strain energy function into an infinite power series of the form

$$W = W(\mathrm{I}_\mathbf{C}, \mathrm{II}_\mathbf{C}, \mathrm{III}_\mathbf{C}) = \sum_{p,q,r=0}^{\infty} \alpha_{pqr} (\mathrm{I}_\mathbf{C} - 3)^p (\mathrm{II}_\mathbf{C} - 3)^q (\mathrm{III}_\mathbf{C} - 1)^r \qquad (4.4.47)$$

In the undeformed configuration, $\mathrm{I}_\mathbf{C} = \mathrm{II}_\mathbf{C} = 3$, $\mathrm{III}_\mathbf{C} = 1$, and the strain energy is zero. For incompressible materials, this reads as

$$W = \sum_{p,q=0}^{\infty} \alpha_{pq} (\mathrm{I}_\mathbf{C} - 3)^p (\mathrm{II}_\mathbf{C} - 3)^q \qquad (4.4.48)$$

The series can be written in terms of the principal stretches by noting that

$$\begin{aligned} \mathrm{I}_{\mathbf{C}} &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ \mathrm{II}_{\mathbf{C}} &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ \mathrm{III}_{\mathbf{C}} &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned} \tag{4.4.49}$$

Next are considered some particular forms of the strain energy function.

4.4.6 Incompressible Material Models

First, models of isotropic hyperelastic incompressible materials are examined.

The Ogden Model (Incompressible)

The strain energy for the **Ogden model** is defined to be

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{n} \frac{\mu_i}{\alpha_i} \left\{ \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right\} \tag{4.4.50}$$

where the μ_i, α_i are experimentally determined material constants. Note that this model does not include terms involving the products of different stretches, which appear in the most general power series 4.4.48. However, the Ogden model allows for fractional powers of α_i. Only three terms in the series are usually needed to correlate the model with experimental data of incompressible rubber. Typical values are

$$\begin{aligned} \alpha_1 &= 1.3 \qquad & \mu_1 &= 6300 \text{ kPa} \\ \alpha_1 &= 5.0 \qquad & \mu_2 &= 1.2 \text{ kPa} \\ \alpha_2 &= -2.0 \qquad & \mu_3 &= -10 \text{ kPa} \end{aligned}$$

The Mooney-Rivlin Model (Incompressible)

The **Mooney-Rivlin** model, applicable to incompressible materials, is

$$W = c_1(\mathrm{I}_{\mathbf{C}} - 3) + c_2(\mathrm{II}_{\mathbf{C}} - 3) \tag{4.4.51}$$

And, from 4.4.22, the stresses are then

$$\boldsymbol{\sigma} = 2c_1\mathbf{b} - 2c_2\mathbf{b}^{-1} - p\mathbf{I} \tag{4.4.52}$$

From the simple shear example discussed in §4.4.4, the shear modulus for the Mooney-Rivlin material is $2(c_1 + c_2)$, implying that the shear stress to shear strain measure k is linear.

The Mooney-Rivlin model is often used to model rubber-like materials and the numerical value of c_2 is usually much smaller than c_1.

Note that the Mooney-Rivlin model can be viewed as a special case of the Ogden model, since setting $n = 2, \alpha_1 = 2, \alpha_2 = -2$ in that 4.4.50, and using the incompressibility constraint $\lambda_1^2 \lambda_2^2 \lambda_3^2 = 1$ gives

$$\begin{aligned} W &= \frac{\mu_1}{2}\left\{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right\} - \frac{\mu_2}{2}\left\{\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right\} \\ &= \frac{\mu_1}{2}\left\{\mathrm{I}_\mathbf{C} - 3\right\} - \frac{\mu_2}{2}\left\{\mathrm{II}_\mathbf{C} - 3\right\} \end{aligned} \tag{4.4.53}$$

The Neo-Hookean Model (Incompressible)

The **Neo-Hookean** is a special case of the Mooney-Rivlin model, wherein $c_2 = 0$, giving

$$W = c_1\left(\mathrm{I}_\mathbf{C} - 3\right) \tag{4.4.54}$$

It is useful as a model of non-linear elasticity within the small-strain range.

The Varga Model (Incompressible)

The **Varga** model is another special case of the Ogden model:

$$W = c_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) \tag{4.4.55}$$

The Yeoh Model (Incompressible)

The **Yeoh** model was motivated by the problem of predicting the behaviour of rubbers containing reinforcing fillers such as carbon black, for which the Mooney-Rivlin model does not succeed too well. The strain energy here takes the form

$$W = c_1(\mathrm{I}_\mathbf{C} - 3) + c_2(\mathrm{I}_\mathbf{C} - 3)^2 + c_3(\mathrm{I}_\mathbf{C} - 3)^3 \tag{4.4.56}$$

As usual, $W(\mathbf{I}) = 0$ in the reference configuration and the strain energy is a convex function, increasing monotonically with $\mathrm{I}_\mathbf{C}$ (and attains a minimum in the reference configuration, so that $\partial W / \partial \mathrm{I}_\mathbf{C} = 0$ must contain no real roots, which places restrictions on the values that the constants c_1, c_2, c_3 can take).

The Arruda-Boyce Model (Incompressible)

The Arruda-Boyce model is based on a proposed molecular network structure to a material, and the strain energy is

$$W = \mu\left[\frac{1}{2}\left(\mathrm{I}_\mathbf{C} - 1\right) + \frac{1}{20n}\left(\mathrm{I}_\mathbf{C}^2 - 9\right) + \frac{11}{1050n^2}\left(\mathrm{I}_\mathbf{C}^3 - 27\right) + \cdots\right] \tag{4.4.57}$$

The two parameters are μ, the shear modulus, and n, the number of segments in a polymer chain.

4.4.7 Compressible Material Models

The Mooney-Rivlin Model (Compressible)

The compressible Mooney-Rivlin model reads as

$$W = c(J-1)^2 - 2(c_1 + c_2)\ln J + c_1\left(\mathrm{I}_{\mathbf{b}} - 3\right) + c_2\left(\mathrm{II}_{\mathbf{b}} - 3\right) \tag{4.4.58}$$

The stresses are then, from 4.4.8, with $J = \sqrt{\mathrm{III}_{\mathbf{b}}}$,

$$\boldsymbol{\sigma} = \frac{2}{\sqrt{\mathrm{III}_{\mathbf{b}}}}\left\{\left[c\left(\mathrm{III}_{\mathbf{b}} - \sqrt{\mathrm{III}_{\mathbf{b}}}\right) - 4(c_1 + c_2)\right]\mathbf{I} + \left[c_1 + c_2\left(\mathrm{tr}\mathbf{b}\right)\right]\mathbf{b} - c_2\mathbf{b}^2\right\} \tag{4.4.59}$$

The Neo-Hookean Model (Compressible)

Neo-Hookean models are characterised by their dependence on the first (as in the incompressible case) and third invariants, but not on the second invariant. There are quite a number of Neo-Hookean models available, for example (the last term here is the incompressible Neo-Hookean term)

$$W = \frac{\lambda}{2}(\ln J)^2 - \mu \ln J + \frac{1}{2}\mu\left(\mathrm{I}_{\mathbf{b}} - 3\right) \tag{4.4.60}$$

With corresponding stresses

$$\begin{aligned}\boldsymbol{\sigma} &= 2J^{-1}\frac{\partial W}{\partial \mathrm{III}_{\mathbf{b}}}\mathrm{III}_{\mathbf{b}}\mathbf{I} + 2J^{-1}\frac{\partial W}{\partial \mathrm{I}_{\mathbf{b}}}\mathbf{b} \\ &= J^{-1}\left[\lambda \ln J\mathbf{I} + \mu(\mathbf{b} - \mathbf{I})\right]\end{aligned} \tag{4.4.61}$$

The λ, μ of this compressible Neo-Hookean model are the classical Lamé constants, which can be seen by particularising the constitutive equation to the small strain range.

Another Neo-Hookean model is

$$W = \frac{1}{2}\kappa\left(\ln J\right)^2 + \frac{1}{2}\mu\left(J^{-2/3}\mathrm{I}_{\mathbf{b}} - 3\right) \tag{4.4.62}$$

with corresponding stresses

$$\boldsymbol{\sigma} = J^{-1}\left(\kappa \ln J + \mu J^{-2/3}\left(\mathbf{b} - \frac{1}{3}(\mathrm{tr}\mathbf{b})\mathbf{I}\right)\right)$$
$$= J^{-1}\left(\kappa \ln J + \mu J^{-2/3}\mathrm{dev}\mathbf{b}\right) \tag{4.4.63}$$

The Blatz and Ko Model (Compressible)

Blatz and Ko proposed a compressible model based on a combination of theoretical arguments and experimental work on foamed materials:

$$W = f\frac{\mu}{2}\left[\left(\mathrm{I}_{\mathbf{b}} - 3\right) + \frac{1}{\beta}\left(\mathrm{III}_{\mathbf{b}}^{-\beta} - 1\right)\right] + \left(1 - f\right)\frac{\mu}{2}\left[\left(\frac{\mathrm{II}_{\mathbf{b}}}{\mathrm{III}_{\mathbf{b}}} - 3\right) + \frac{1}{\beta}\left(\mathrm{III}_{\mathbf{b}}^{+\beta} - 1\right)\right] \tag{4.4.64}$$

where

$$\beta = \frac{\nu}{1 - 2\nu} \tag{4.4.65}$$

and μ, ν denote the shear modulus and Poisson's ratio, and $f \in [0,1]$ is an interpolation parameter. When the material is incompressible, $\mathrm{III}_{\mathbf{b}} = 1$, and the model reduces to the Mooney-Rivlin model. Also, by letting $f = 1$, and replacing $\mathrm{III}_{\mathbf{b}}$ with J^2, the Blatz and Ko model reduces to the Neo-Hookean model

$$W = \frac{\mu}{2}\left(\mathrm{I}_{\mathbf{b}} - 3\right) + \frac{\mu}{2}\frac{1}{\beta}\left(J^{-2\beta} - 1\right) \tag{4.4.66}$$

4.4.8 Problems

1. Evaluate the PK2 stress for the Mooney-Rivlin material (incompressible)