4.3 Isotropic Elastic Materials

4.3.1 Constitutive Equations for Isotropic Elastic Materials

Cauchy Stress and Isotropy

A material is said to be **isotropic** if a rotation of a particle (in the undeformed state) has no influence on the stress tensor. From §2.8.6, the condition of isotropy is then

$$\boldsymbol{\sigma}(\mathbf{F}^{\Diamond}) = \boldsymbol{\sigma}(\mathbf{F}) \quad \to \quad \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q}^{\mathrm{T}}) = \boldsymbol{\sigma}(\mathbf{F}) \tag{4.3.1}$$

where the superscript $\Diamond$ refers to deformations relative to the rotated reference configuration, Fig. 2.8.6. (This is often expressed in the equivalent form $\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{Q})$.)

A Cauchy elastic material automatically satisfies the isotropy condition when the Cauchy stress is an arbitrary tensor-valued function of the left Cauchy-Green tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{b}) \tag{4.3.2}$$

To see this, note that $\boldsymbol{\sigma}(\mathbf{b}(\mathbf{F})) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{F}^{\mathrm{T}})$, so

$$\boldsymbol{\sigma}\left(\mathbf{b}(\mathbf{F}\mathbf{Q}^{\mathrm{T}})\right) = \boldsymbol{\sigma}\left((\mathbf{F}\mathbf{Q}^{\mathrm{T}})(\mathbf{F}\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}}\right) = \boldsymbol{\sigma}\left(\mathbf{F}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{F}^{\mathrm{T}}\right) = \boldsymbol{\sigma}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right) = \boldsymbol{\sigma}\left(\mathbf{b}(\mathbf{F})\right) \tag{4.3.3}$$

Note that the condition of isotropy 4.3.1 must be satisfied for the particular rotation $\mathbf{Q} = \mathbf{R}$, so a condition to be satisfied by isotropic materials is

$$\boldsymbol{\sigma}(\mathbf{F}) = \boldsymbol{\sigma}(\mathbf{F}\mathbf{R}^{\mathrm{T}}) = \boldsymbol{\sigma}(\mathbf{v}) = \boldsymbol{\sigma}(\mathbf{b}^{1/2}) = \boldsymbol{\sigma}(\mathbf{b}) \tag{4.3.4}$$

where **v** is the left stretch tensor, and again one has 4.3.2.

In addition to satisfying the condition of isotropy, the stress must satisfy the condition of objectivity: $\boldsymbol{\sigma}^{*} = \boldsymbol{\sigma}(\mathbf{b}^{*})$, where the superscript * refers to objectivity transformations, §2.8.3. This requirement is automatically satisfied since both the Cauchy stress and the left Cauchy-Green tensors are objective spatial tensors:

$$\boxed{\mathbf{Q}\boldsymbol{\sigma}(\mathbf{b})\mathbf{Q}^{\mathrm{T}} = \boldsymbol{\sigma}(\mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}})} \qquad \textbf{Isotropy Condition for the (objective) Cauchy Stress} \tag{4.3.5}$$

PK2 Stress and Isotropy

From 3.5.7 and 2.8.58, a rigid body rotation of the reference configuration alters the PK2 stress according to

$$\mathbf{S}^{\Diamond} = J^{\Diamond}\mathbf{F}^{\Diamond-1}\boldsymbol{\sigma}^{\Diamond}\mathbf{F}^{\Diamond-\mathrm{T}} = J\left(\mathbf{F}\mathbf{Q}^{\mathrm{T}}\right)^{-1}\boldsymbol{\sigma}\left(\mathbf{F}\mathbf{Q}^{\mathrm{T}}\right)^{-\mathrm{T}} = \mathbf{Q}\mathbf{S}\mathbf{Q}^{\mathrm{T}} \tag{4.3.6}$$

On the other hand, **S** is objective when written as a function of the material tensor **C**. Then, since $\mathbf{C}^{\Diamond} = \mathbf{QCQ}^{\mathrm{T}}$ (see Eqns. 2.8.58),

$$\boxed{\mathbf{QS(C)Q}^{\mathrm{T}} = \mathbf{S}(\mathbf{QCQ}^{\mathrm{T}})}$$ **Isotropy Condition for the (objective) PK2 Stress** (4.3.7)

Isotropic Tensor Functions

The constitutive relations 4.3.5 for the Cauchy stress and 4.3.7 for the PK2 stress are very similar. In general, the second-order tensor-valued function **T** of the second-order tensor variable **B** is an **isotropic tensor functions** if

$$\boxed{\mathbf{T}\left(\mathbf{QBQ}^{\mathrm{T}}\right) = \mathbf{QT}(\mathbf{B})\mathbf{Q}^{\mathrm{T}}}$$ **Isotropic Tensor Function** (4.3.8)

for all orthogonal tensors **Q**. Thus, $\boldsymbol{\sigma}$ is an isotropic tensor function of the tensor variable **b** and $\mathbf{S}$ is an isotropic tensor function of the tensor variable $\mathbf{C}$. Isotropic functions are discussed in the Appendix to this Chapter, §4.A, and there it is shown that, for symmetric **T** and symmetric **B**, **T** must take t he form

$$\boxed{\mathbf{T}(\mathbf{B}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2}$$ **Form for a symmetric isotropic tensor function of a symmetric tensor** (4.3.9)

where $\alpha_0, \alpha_1, \alpha_2$ are scalar functions of the principal scalar invariants of **B**, 1.9.38, 1.9.46:

$$\alpha_i = \alpha_i \{\mathrm{I}_{\mathbf{B}}, \mathrm{II}_{\mathbf{B}}, \mathrm{III}_{\mathbf{B}}\} \tag{4.3.10}$$

Since the set of principal scalar invariants, the set $\{\mathrm{tr}\mathbf{S}, \mathrm{tr}\mathbf{S}^2, \mathrm{tr}\mathbf{S}^3\}$ and the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ uniquely determine one another, the coefficients α_i can be taken to be functions of any one of these three sets.

Equation 4.3.9 can be rewritten in various alternative forms using the Cayley-Hamilton theorem, 1.9.45:

$$\mathbf{B}^3 - \mathrm{I}_{\mathbf{B}}\mathbf{B}^2 + \mathrm{II}_{\mathbf{B}}\mathbf{B} - \mathrm{III}_{\mathbf{B}}\mathbf{I} = \mathbf{0}, \tag{4.3.11}$$

for example

$$\mathbf{T}(\mathbf{B}) = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1} \tag{4.3.12}$$

where

$$\beta_0 = \alpha_0 - \mathrm{II}_{\mathbf{B}}\alpha_2, \quad \beta_1 = \alpha_1 + \mathrm{I}_{\mathbf{B}}\alpha_2, \quad \beta_{-1} = \mathrm{III}_{\mathbf{B}}\alpha_2. \tag{4.3.13}$$

The Cauchy-Elastic Solid

Since for an isotropic Cauchy-elastic solid, the Cauchy stress is an isotropic tensor function of the left Cauchy-Green strain, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{b})$, Eqn. 4.3.5 (a consequence of isotropy *and* objectivity) and since $\boldsymbol{\sigma}$ and $\mathbf{b}$ are symmetric, it follows from 4.3.9 that the Cauchy stress takes the form

$$\boldsymbol{\sigma} = \alpha_0\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{I} + \alpha_1\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{b} + \alpha_2\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{b}^2 \qquad (4.3.14)$$

or, alternatively, from 4.3.12, the form

$$\boldsymbol{\sigma} = \beta_0\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{I} + \beta_1\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{b} + \beta_{-1}\left(\mathrm{I}_\mathbf{b}, \mathrm{II}_\mathbf{b}, \mathrm{III}_\mathbf{b}\right)\mathbf{b}^{-1} \qquad (4.3.15)$$

and these scalar functions are related through 4.3.13.

Similar forms hold for the PK2 stress as a function of the right Cauchy-Green strain.

4.3.2 Strain Energy and Isotropy

Consider the strain energy $W = W(\mathbf{F})$. From §2.8.4, objectivity requires that (see Eqn. 2.8.46),

$$W(\mathbf{F}) = W(\mathbf{U}) \qquad (4.3.16)$$

Also, the right stretch tensor is the square–root of the right Cauchy-Green tensor, so one can write $W = W(\mathbf{C})$, which is clearly objective, since $W(\mathbf{C}) = W(\mathbf{C}^*)$. In addition to the objectivity requirement, isotropy requires that $W(\mathbf{C}) = W(\mathbf{C}^\Diamond)$, or, 2.8.58b

$$\boxed{W(\mathbf{C}) = W(\mathbf{QCQ}^{\mathrm{T}})} \qquad (4.3.17)$$

Isotropy Condition for the (objective) Strain Energy (4.3.17)

Isotropic Scalar Functions

The strain energy of Eqn. 4.3.17 is also an isotropic function; in general, the scalar function ϕ of the second-order tensor variable **B** is an **isotropic scalar functions** if

$$\boxed{\phi\left(\mathbf{QBQ}^{\mathrm{T}}\right) = \phi(\mathbf{B})} \qquad (4.3.18)$$

Isotropic Scalar Function (4.3.18)

for all orthogonal tensors **Q**. Thus, W is an isotropic scalar function of the tensor variable $\mathbf{C}$. It is shown in the Appendix to this Chapter, §4.A, that, for symmetric **B**, ϕ must take the form

$$\boxed{\phi(\mathbf{B}) = \phi\left(\left\{\mathrm{I}_\mathbf{B}, \mathrm{II}_\mathbf{B}, \mathrm{III}_\mathbf{B}\right\}\right)} \qquad (4.3.19)$$

Form for an isotropic scalar function of a symmetric tensor (4.3.19)

Thus the strain energy for an isotropic elastic material must be a function only of the three principal scalar invariants of the right Cauchy-Green strain (or of the three principal values λ_i). Since the invariants for the right- and left-Cauchy-Green strain tensors are the same (see Eqn. 2.2.15), the strain energy can also be expressed as a function of the invariants of **b**.