4.1 Elastic Solids

In this section is given an overview of the common elasticity models.

4.1.1 The Linear Elastic Solid

The classical **Linear Elastic model**, or **Hookean model**, has the following linear relationship between stress and strain:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \qquad \sigma_{ij} = C_{ijmn}\varepsilon_{mn} \tag{4.1.1}$$

where $\boldsymbol{\varepsilon}$ is the small strain tensor, §2.7.

Strain Energy

In this purely mechanical theory of elastic materials, there is no dissipation of energy – all the energy of the loads is stored as elastic strain energy in the material as it deforms, and can be recovered.

For the linear elastic material, the rate of deformation is equivalent to the rate of small-strain, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, so the strain-energy function can be written as

$$dW = \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \tag{4.1.2}$$

and the total energy stored per unit volume over the complete history of straining is

$$W = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} \tag{4.1.3}$$

nnd the stress can be written as

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \tag{4.1.4}$$

Reduction in the number of Independent Elastic Constants

Since the stress and strain are symmetric, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{mn} = \varepsilon_{nm}$, the fourth-order elasticity tensor of stiffness coefficients contains the minor symmetries 1.9.65,

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{4.1.5}$$

and so the 81 coefficients reduce to 36 independent coefficients. Further, since $\mathbf{C}$ is independent of the strains,

$$W = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon} = \int \mathbf{C} : \boldsymbol{\varepsilon} d\boldsymbol{\varepsilon} = \frac{1}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} \tag{4.1.6}$$

and so

$$\mathbf{C} = \frac{\partial W}{\partial \mathbf{e} \partial \mathbf{e}} \qquad (4.1.7)$$

Now

$$\frac{\partial W}{\partial \varepsilon_{ij} \partial \varepsilon_{mn}} = \frac{\partial W}{\partial \varepsilon_{mn} \partial \varepsilon_{ij}} \qquad (4.1.8)$$

and it follows that $\mathbf{C}$ possesses the *major symmetries*:

$$C_{ijmn} = C_{mnij} \qquad (4.1.9)$$

This reduces the number of independent elastic constants from 36 to 21.

Problems involving Elastic Materials

The six constitutive equations 4.1.1, together with the equations of motion and the 6 kinematic relations relating the strains to the displacements, Eqn. 2.7.2, $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$, gives a set of 15 equations in the 15 unknowns: the six stress components, the six strain components and the three displacement components.

To maintain a linear theory, the acceleration term in the equations of motion must be linear; this is achieved by supposing the displacement gradients to be small:

$$v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_k} v_k \approx \frac{\partial u_i}{\partial t}, \qquad \frac{d^2 u_i}{dt^2} \approx \frac{\partial^2 u_i}{\partial t^2} \qquad (4.1.10)$$

When this acceleration term is included, the problem is **dynamic**. When the equations of equilibrium are used, the problem is **static**.

Compatibility

An alternative method of solution is to remove the displacements from the above system and solve only for the stresses and strains. In this case the strain-displacement relations are replaced by three compatibility equations, and there are then 12 equations in 12 unknowns. Once the system is solved, the displacements can be obtained from the strains by integration.

The Isotropic Linear Elastic Solid

When the material is isotropic, the constitutive equation holds in any coordinate system,

$$\sigma'_{ij} = C_{ijmn} \varepsilon'_{mn} \qquad (4.1.11)$$

and so the tensor of elastic constants is isotropic. The most general fourth-order isotropic tensor takes the form 1.10.7,

$$\mathbf{C} = \lambda \mathbf{I} \otimes \mathbf{I} + \mu\left(\mathbf{I} + \bar{\mathbf{I}}\right), \quad C_{ijmn} = \lambda \delta_{ij}\delta_{mn} + \mu\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) \tag{4.1.12}$$

with the fourth-order unit tensors given by 1.9.60,

$$\begin{aligned} \mathbf{I} &= \delta_{im}\delta_{jn}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n \\ \bar{\mathbf{I}} &= \delta_{in}\delta_{jm}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n \end{aligned} \tag{4.1.13}$$

which has only *two independent material constants*. Since the strain is symmetric, one has

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{C} : \boldsymbol{\varepsilon} \\ &= \left(\lambda \mathbf{I} \otimes \mathbf{I} + \mu\left(\mathbf{I} + \bar{\mathbf{I}}\right)\right) : \boldsymbol{\varepsilon} \\ &= \left(\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbf{I}\right) : \boldsymbol{\varepsilon} \end{aligned} \tag{4.1.14}$$

and the constitutive equation reduces to {▲Problem 1}

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \qquad \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{4.1.15}$$

and the two elastic constants λ, μ are called **Lamé's constants**.

Problems in Isotropic Elasticity

The 15 equations mentioned earlier can be reduced by eliminating the strains from the constitutive equation and the kinematic equation, and then substituting the resultant expression for stress into the equations of motion, giving **Navier's equations**

$$(\lambda + \mu)\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \quad (\lambda + \mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu\frac{\partial^2 u_i}{\partial x_k^2} + \rho b_i = \rho\frac{\partial^2 u}{\partial t^2} \tag{4.1.16}$$

This set of three partial differential equations is appropriate for problems involving displacement boundary conditions.

The Lame's constants and the Young's modulus and Poisson's ratio are related through

$$\begin{aligned} E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)} \\ \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \end{aligned} \tag{4.1.17}$$

The linear elastic constitutive equations in terms of the engineering constants reads

$$\begin{aligned}\varepsilon_{ij} &= -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} \\ \sigma_{ij} &= \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right]\end{aligned} \tag{4.1.18}$$

The Bulk Modulus

The tensor of elastic constants can be written in the alternative forms

$$\begin{aligned}\mathsf{C} &= 2\mu\mathsf{I} + \lambda\mathbf{I}\otimes\mathbf{I} \\ &= 2\mu\left[\mathsf{I} + \frac{\nu}{1-2\nu}\mathbf{I}\otimes\mathbf{I}\right] \\ &= \frac{E}{(1+\nu)}\left[\mathsf{I} + \frac{\nu}{1-2\nu}\mathbf{I}\otimes\mathbf{I}\right] \\ &= \kappa\mathbf{I}\otimes\mathbf{I} + 2\mu\left[\mathsf{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right]\end{aligned} \tag{4.1.19}$$

where the new constant introduced is the **bulk modulus** κ. This last expression then leads to the alternative form of the constitutive relations,

$$\boldsymbol{\sigma} = \kappa(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\,\mathrm{dev}\boldsymbol{\varepsilon} \tag{4.1.20}$$

This expression shows that the stress can be decomposed into a spherical component and a deviatoric component. For a pure volume change, $\mathrm{dev}\boldsymbol{\varepsilon} = 0$, and there are no shear stresses, $\boldsymbol{\sigma} = (\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}$; the bulk modulus is thus a measure of the resistance of the material to volume changes.

4.1.2 Geometrically Non-Linear Elastic Materials

When the strains (displacement gradients) are not small, the behaviour of the material will inevitably be non-linear. This is due to the **geometric non-linearity** of the kinematic strain-displacement relations, for example using the Green-Lagrange strains and the reference configuration,

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\mathrm{Grad}\mathbf{U} + (\mathrm{Grad}\mathbf{U})^{\mathrm{T}} + (\mathrm{Grad}\mathbf{U})^{\mathrm{T}}\,\mathrm{Grad}\mathbf{U}\right) \\ E_{ij} &= \frac{1}{2}\left\{\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right\}\end{aligned} \tag{4.1.21}$$

The Kirchhoff Material

The Kirchhoff material is an extension of the linear elastic model to the large strain range; the constitutive relation is a linear tensor relation, but non-linearities enter through $\mathbf{E}(\mathbf{u})$:

$$\mathbf{S} = \mathsf{C} : \mathbf{E}, \qquad S_{ij} = C_{ijmn} E_{mn} \tag{4.1.22}$$

where **S** is the PK2 stress tensor and **E** is the Green-Lagrange strain. Since both **S** and **E** are symmetric, the fourth-order tensor C has the minor symmetries, $C_{ijmn} = C_{jimn}$ and $C_{ijmn} = C_{ijnm}$, and so has 36 independent coefficients. Following the same arguments as before, one can define a strain energy function (per unit reference volume)

$$dW = \mathbf{S} : d\mathbf{E}, \quad dW = S_{ij} dE_{ij} \tag{4.1.23}$$

and the total energy stored per unit volume over the complete history of straining is

$$W = \int S_{ij} dE_{ij} = \frac{1}{2} C_{ijmn} E_{ij} E_{mn} = \frac{1}{2} \mathbf{E} : \mathsf{C} : \mathbf{E} \tag{4.1.24}$$

and the stress can be written as

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}}, \qquad S_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{4.1.25}$$

Again, the existence of the strain energy function implies that the matrix of elastic coefficients only has 21 independent coefficients.

When the Kirchhoff material is isotropic, the constitutive relation reduces further to

$$\mathbf{S} = \lambda (\mathrm{tr}\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E}, \qquad S_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} \tag{4.1.26}$$

As mentioned in §2.7.2, the linear elastic model can not be used when there are large rigid body rotations, even if the displacement gradients are not large. The Kirchhoff model can be used in these cases.

4.1.3 Materially Non-Linear Elastic Materials

An elastic material might also exhibit **material non-linearity** through a non-linear constitutive equation, for example the Cauchy stress might be some non-linear function of a strain measure, or of the deformation gradient:

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}(t)) \tag{4.1.27}$$

where **f** is some tensor function of the deformation gradient **F**. This constitutive equation is called the **Cauchy Elastic** material model. As can be seen, the stress is dependent on the current state only, and not on the path history, a requirement of elasticity. However, the stress in the case of a Cauchy elastic material cannot in general be written in terms of a strain-energy function. In other words, the work done might be path-dependent.

Objectivity Requirements

The notion of objectivity was introduced in §2.8. When formulating constitutive relations for materials, one must ensure that the **principle of material objectivity**(or the **principle of material frame indifference**) be satisfied. This principle states that

A constitutive law must be independent of the location of the observer (or frame of reference that is taken)

This implies that two observers, even if in relative motion with respect to each other, observe the same stress in a given body. Consider the Cauchy elastic material 4.1.27. The Cauchy stress is an objective tensor. Referring to the example of Eqns. 2.8.47-50 in §2.8.4, objectivity requires that the constitutive relation be of the form

$$\boldsymbol{\sigma} = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \tag{4.1.28}$$

The constitutive relation can also be written in terms of other stress measures. For example, using $J\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^{\mathrm{T}} = \mathbf{P}(\mathbf{R}\mathbf{U})^{\mathrm{T}} = \mathbf{P}\mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}$, one has

$$\mathbf{P} = J\mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{U}^{-1} \tag{4.1.29}$$

For the PK2 stress, one has $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, so that

$$\mathbf{S} = \det\mathbf{U}\mathbf{U}^{-1}\mathbf{f}(\mathbf{U})\mathbf{U}^{-1} \tag{4.1.30}$$

which does not depend on the rotation. This last relation can also be written in the form

$$\mathbf{S} = \det\mathbf{C}^{1/2}\mathbf{C}^{-1/2}\mathbf{f}_2\left(\mathbf{C}^{1/2}\right)\mathbf{C}^{-1/2} \equiv \mathbf{g}(\mathbf{C}) \tag{4.1.31}$$

This is clearly objective, since **S** and **C** are unaffected by an observer transformation, $\mathbf{S}^* = \mathbf{S}$ and $\mathbf{C}^* = \mathbf{C}$.

4.1.4 Hypoelastic Materials

A **hypoelastic** material is one whose constitutive law relates the rate of stress to the rate of deformation **d**. This can be written in terms of the Cauchy stress as $\dot{\boldsymbol{\sigma}} = f(\boldsymbol{\sigma}, \mathbf{d})$. Consider a simple one-dimensional linear model,

$$\dot{\sigma} = Ed \tag{4.1.32}$$

Since, in one-dimension, the stretch is $\lambda = dx/dX$, the rate of deformation is equivalent to the spatial velocity gradient l and the rate of change of a line element dx is $d(dx)/dt = l\,dx$, dividing through by dX gives $d = \dot{\lambda}/\lambda$ (see Eqn. 2.5.10), so that

$$\dot{\sigma} = E\frac{\dot{\lambda}}{\lambda} \quad \rightarrow \quad \sigma = \int d\sigma = E\int\frac{d\lambda}{\lambda} = E\ln\lambda \tag{4.1.33}$$

This shows that the stress is clearly path-independent, depending only on the current stretch. In fact, the stress can be written as the derivative of a strain energy function according to $\sigma = dW / d\lambda$, where $W = E / \lambda$.

In the three dimensional case, however, the rate of deformation can not in general be written as the rate of change of some simple function, $\mathbf{d} = d(\bullet)/dt$, and so the above calculation cannot be done, implying that the hypoelastic material cannot be written in terms of a potential function, and the work done is path-dependent. The path-dependence is, however, small when the strains are small.

4.1.5 Hyperelastic Materials

A **hyperelastic** material (or **Green elastic** material) is defined to be an elastic material for which a strain-energy function *W* exists, a scalar function of one of the strain or deformation tensors, whose derivative with respect to a strain component determines the corresponding stress component. From the above, the linear elastic model, the Kirchhoff model and the one-dimensional hypoelastic model are all examples of hyperelastic materials. The hyperelastic material is a subset of the Cauchy-elastic material. Hyperelastic material models for components under large strains will be the subject of the following sections.

4.1.6 Problems

1. Show that

$$\mathbf{I} \otimes \mathbf{I} : \boldsymbol{\varepsilon} = (\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}$$
$$(\mathbf{I} + \bar{\mathbf{I}}) : \boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}$$