3.10 Convected Coordinates

Some of the important results from sections 3.1-3.9 are now re-expressed in terms of convected coordinates. As before, any relations expressed in symbolic form hold also in the convected coordinate system.

3.10.1 The Stress Tensors

Traction and Stress Components

Consider a differential parallelepiped element in the current configuration bounded by the coordinate curves as in Fig. 3.10.1 (see Fig. 1.16.2). The bounding vectors are $d\Theta^1\mathbf{g}_1$, $d\Theta^2\mathbf{g}_2$ and $d\Theta^3\mathbf{g}_3$. The surface area $d\bar{S}_1$ of a face of the elemental parallelepiped on which Θ_1 is constant, to which $\mathbf{g}^1$ is normal, is then given by Eqn. 1.16.35,

$$d\bar{S}_1 = \sqrt{g\,g^{11}}\,d\Theta^2 d\Theta^3 \qquad (3.10.1)$$

and similarly for the other surfaces.

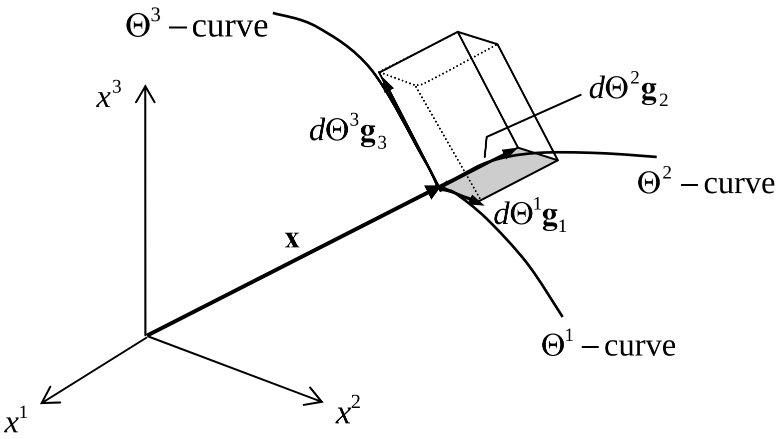

Figure 3.10.1: vector elements bounding surface elements

The **positive side** of a face is defined as that whose outward normal is in the direction of the associated contravariant base vector. The unit normal $\bar{\mathbf{n}}^i$ to a positive side is the same as the unit contravariant base vector; as in Eqn. 1.16.14,

$$\bar{\mathbf{n}}^i = \hat{\mathbf{g}}^i = \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} \qquad (\text{no sum}) \qquad (3.10.2)$$

Let the force $d\mathbf{F}^i$ acting on the surface element with normal $\bar{\mathbf{n}}^i$ be $d\bar{S}_i\mathbf{t}^{\langle i\rangle}$ (no sum over i), Fig. 3.10.2, so that $\mathbf{t}^{\langle i\rangle}$ is the traction (force per unit area).

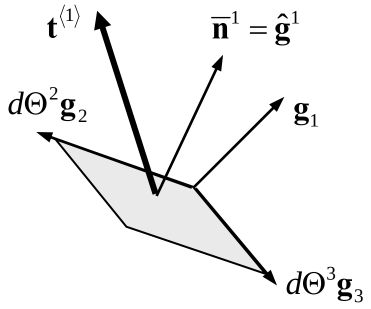

Figure 3.10.2: traction acting on a surface element

The components of $\mathbf{t}^{\langle i\rangle}$ along the unit covariant base vectors are denoted by $\sigma^{\langle ji\rangle}$:

$$\mathbf{t}^{\langle i\rangle} = \sigma^{\langle ji\rangle}\hat{\mathbf{g}}_j = \sigma^{\langle ji\rangle}\frac{1}{\sqrt{g_{jj}}}\mathbf{g}_j \tag{3.10.3}$$

with no sum over the j in the $\sqrt{g_{jj}}$ term; $\sigma^{\langle ji\rangle}$ are called the **physical stress components**, Fig. 3.10.3.

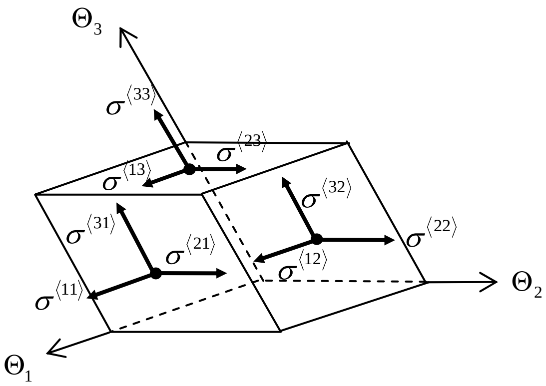

Figure 3.10.3: physical stress components

Introduce now a new vector $\mathbf{t}^i$ defined by

$$\mathbf{t}^i = \sqrt{g^{ii}}\,\mathbf{t}^{\langle i\rangle} \qquad \text{(no sum over } i\text{)} \tag{3.10.4}$$

It will be shown that this vector is contravariant, that is, transforms between coordinate systems according to 1.17.3a (and so $\mathbf{t}^{\langle i\rangle}$ does not satisfy the vector transformation rule, hence the superscript in pointed brackets). The components of $\mathbf{t}^i$ along the covariant base vectors are denoted by σ^{ji}:

$$\mathbf{t}^i = \sigma^{ji}\mathbf{g}_j \tag{3.10.5}$$

Comparing 3.10.3-5,

$$\sigma^{\langle ji \rangle} = \sqrt{\frac{g_{jj}}{g^{ii}}}\sigma^{ji} \quad \text{(no sum)} \tag{3.10.6}$$

Cauchy's Law and the Cauchy Stress Tensor

Cauchy's law can now be derived in the same way as in §3.3, by considering a small tetrahedral free-body, Fig. 3.10.4. The physical stress components $\sigma^{\langle ij \rangle}$ shown act on the negative sides of the surfaces and so act in directions opposite that of the corresponding components on the positive sides (a consequence of Cauchy's Lemma). It is required to determine the traction **t** in terms of the physical stress components and the unit normal **n** to the base area.

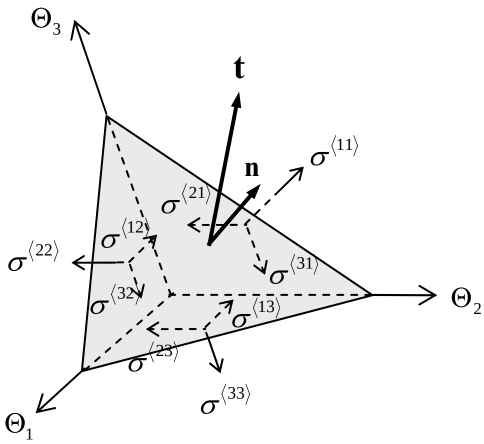

Figure 3.10.4: free body diagram of a tetrahedral portion of material

The normal to the base has components

$$\mathbf{n} = n^i \mathbf{g}_i = n_i \mathbf{g}^i \tag{3.10.7}$$

Consider the vector elements $d\mathbf{a}$ and $d\mathbf{b}$ shown in Fig. 3.10.5. Define the surface area element $d\mathbf{S}$ to be the vector with magnitude equal to twice the area of the tetrahedron base and in the direction of the normal to the base, so

$$\begin{aligned} \tfrac{1}{2} d\mathbf{S} = \tfrac{1}{2} dS\,\mathbf{n} &= \tfrac{1}{2}\left(d\mathbf{a} \times d\mathbf{b}\right) \\ &= \tfrac{1}{2}\left(\left(d\Theta^1 \mathbf{g}_1 - d\Theta^3 \mathbf{g}_3\right) \times \left(d\Theta^2 \mathbf{g}_2 - d\Theta^3 \mathbf{g}_3\right)\right) \\ &= \tfrac{1}{2}\left(d\Theta^1 d\Theta^2 \mathbf{g}_1 \times \mathbf{g}_2 + d\Theta^2 d\Theta^3 \mathbf{g}_2 \times \mathbf{g}_3 + d\Theta^3 d\Theta^1 \mathbf{g}_3 \times \mathbf{g}_1\right) \\ &= \tfrac{1}{2}\left(d\overline{\mathbf{S}}_1 + d\overline{\mathbf{S}}_2 + d\overline{\mathbf{S}}_3\right) \end{aligned} \tag{3.10.8}$$

where $d\overline{\mathbf{S}}_1$, $d\overline{\mathbf{S}}_2$, $d\overline{\mathbf{S}}_3$ are the surface element areas of the three coordinate sides of the parallelepiped of Fig. 3.10.1 (twice the area of the coordinate sides of the tetrahedron); from 3.10.2,

$$d\mathbf{S} = d\overline{\mathbf{S}}_1 + d\overline{\mathbf{S}}_2 + d\overline{\mathbf{S}}_3$$
$$dS\ \mathbf{n} = d\overline{S}_i\ \overline{\mathbf{n}}^i \qquad (3.10.9)$$
$$= d\overline{S}_i \frac{1}{\sqrt{g^{ii}}}\mathbf{g}^i$$

with no sum over the i in the $\sqrt{g^{ii}}$ term, or

$$dS\ n_i\sqrt{g^{ii}}\mathbf{g}^i = d\overline{S}_i\mathbf{g}^i \qquad (3.10.10)$$

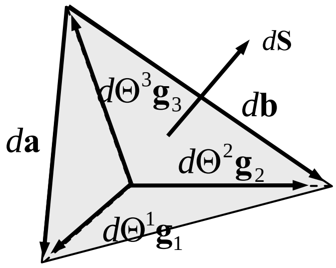

Figure 3.10.5: vector element of area for the base of the tetrahedron

The principle of linear momentum, in vector form, is then (cancelling out a factor of ½)

$$\mathbf{t}\ dS - \mathbf{t}^{\langle i\rangle} d\overline{S}_i = 0 \qquad (3.10.11)$$

From 3.10.4,

$$\mathbf{t}\ dS = \mathbf{t}^i \frac{1}{\sqrt{g^{ii}}} d\overline{S}_i = \mathbf{t}^i dSn_i \qquad (3.10.12)$$

and so

$$\mathbf{t} = \mathbf{t}^i n_i = \sigma^{ji} n_i \mathbf{g}_j \qquad (3.10.13)$$

Defining the (symmetric) Cauchy stress tensor $\boldsymbol{\sigma}$ through

$$\boxed{\boldsymbol{\sigma} = \sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j} \qquad \textbf{Cauchy Stress Tensor} \qquad (3.10.14)$$

one arrives at Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$.

The Cauchy stress is naturally a contravariant tensor because the normal vector upon which it operates to produce the traction is naturally represented in the form of a covariant vector (see 3.10.2).

Note that the stress can also be expressed in the form

$$\boldsymbol{\sigma} = \mathbf{t}^i \otimes \mathbf{g}_j \tag{3.10.15}$$

Other Stress Tensors

The PK1, PK2 and Kirchhoff stress tensors are

$$\begin{aligned} \mathbf{P} &= P^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\ \mathbf{S} &= S^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\ \boldsymbol{\tau} &= \tau^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \tag{3.10.16}$$

By definition, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$, and so $\tau^{ij} = J\sigma^{ij}$. By definition, $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, and so, from 2.9.8,

$$S^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = J\sigma^{ij}\mathbf{F}^{-1}\mathbf{g}_i \otimes \mathbf{F}^{-1}\mathbf{g}_j = J\sigma^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \tau^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \tag{3.10.17}$$

Thus, as seen already, the Kirchhoff stress is the push-forward of the PK2 stress.

Similarly, by definition $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ and so

$$\begin{aligned} P^{ij}\mathbf{G}_i \otimes \mathbf{G}_j &= J\sigma^{kj}\mathbf{g}_k \otimes \mathbf{g}_j\mathbf{F}^{-1} \\ &= J\sigma^{kj}\mathbf{g}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}\mathbf{F}\mathbf{G}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}\left(F^{i}_{\cdot m}\mathbf{G}_i \otimes \mathbf{G}^m\right)\mathbf{G}_k \otimes \mathbf{G}_j \\ &= J\sigma^{kj}F^{i}_{\cdot k}\mathbf{G}_i \otimes \mathbf{G}_j \end{aligned} \tag{3.10.18}$$

3.10.2 The Equations of Motion

The Equations of motion have been given in the symbolic form by 3.6.2 and 3.6.9. To express these in curvilinear coordinates, recall the definition of the divergence of a tensor, 1.18.28,

$$\operatorname{div}\boldsymbol{\sigma} = \frac{\partial\boldsymbol{\sigma}}{\partial\Theta^k}\mathbf{g}^k = \sigma^{ij}\,|_k\,\left(\mathbf{g}_i \otimes \mathbf{g}_j\right)\mathbf{g}^k = \sigma^{ij}\,|_j\,\mathbf{g}_i \tag{3.10.19}$$

The spatial and material descriptions of the equations of motion are then

$$\boxed{\begin{aligned} \sigma^{ij}\,|_j\,\mathbf{g}_i + b^i\mathbf{g}_i &= \rho\frac{d\left(v^i\mathbf{g}_i\right)}{dt} \\ P^{ij}\,|_j\,\mathbf{G}_i + B^i\mathbf{G}_i &= \rho_0\frac{dV^i}{dt}\mathbf{G}_i \end{aligned}} \quad \textbf{Equations of Motion} \tag{3.10.20}$$