3.6 The Equations of Motion and Symmetry of Stress

In Part II, §1.1, the Equations of Motion were derived using Newton's Law applied to a differential material element. Here, they are derived using the principle of linear momentum.

3.6.1 The Equations of Motion (Spatial Form)

Application of Cauchy's law $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ and the divergence theorem 1.14.21 to 3.2.7 leads directly to the global form of the equations of motion

$$\int_v [\operatorname{div}\boldsymbol{\sigma} + \mathbf{b}]dv = \int_v \rho\,\dot{\mathbf{v}}\,dv, \qquad \int_v \left[\frac{\partial \sigma_{ij}}{\partial x_j} + b_i\right]dv = \int_v \rho\,\dot{v}_i\,dv \tag{3.6.1}$$

The corresponding local form is then

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\frac{d\mathbf{v}}{dt}, \qquad \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho\frac{dv_i}{dt}} \quad \textbf{Equations of Motion} \tag{3.6.2}$$

The term on the right is called the inertial, or kinetic, term, representing the change in momentum. The material time derivative of the spatial velocity field is

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{v} \quad \text{so} \quad \frac{dv_1}{dt} = \frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial x_1}v_1 + \frac{\partial v_1}{\partial x_2}v_2 + \frac{\partial v_1}{\partial x_3}v_3\right), \text{ etc.}$$

and it can be seen that the equations of motion are non-linear in the velocities.

Equations of Equilibrium

When the acceleration is zero, the equations reduce to the equations of equilibrium,

$$\boxed{\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = 0} \quad \textbf{Equations of Equilibrium} \tag{3.6.3}$$

Flows

A **flow** is a set of quantities associated with the system of forces **t** and **b**, for example the quantities $\mathbf{v}, \boldsymbol{\sigma}, \rho$. A flow is **steady** if the associated spatial quantities are independent of time. A **potential flow** is one for which the velocity field can be written as the gradient of a scalar function, $\mathbf{v} = \operatorname{grad}\phi$. An **irrotational flow** is one for which $\operatorname{curl}\mathbf{v} = 0$.

3.6.2 The Equations of Motion (Material Form)

In the spatial form, the linear momentum of a mass element is $\rho \mathbf{v} dv$. In the material form it is $\rho_0 \mathbf{V} dV$. Here, **V** is the same velocity as **v**, only it is now expressed in terms of the material coordinates **X**, and $\rho dv = \rho_0 dV$. The linear momentum of a collection of material particles occupying the volume *v* in the current configuration can thus be expressed in terms of an integral over the corresponding volume *V* in the reference configuration:

$$\boxed{\mathbf{L}(t) = \int_V \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)dV} \quad \textbf{Linear Momentum (Material Form)} \quad (3.6.4)$$

and the principle of linear momentum is now, using 3.1.31,

$$\frac{d}{dt}\int_V \rho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \equiv \mathbf{F}(t) \qquad (3.6.5)$$

The external forces **F** to be considered are those acting on the *current* configuration. Suppose that the surface force acting on a surface element ds in the current configuration is $d\mathbf{f}_{\text{surf}} = \mathbf{t}ds = \mathbf{T}dS$, where **t** and **T** are, respectively, the Cauchy traction vector and the PK1 traction vector (Eqns. 3.5.3-4). Also, just as the PK1 stress measures the actual force in the current configuration, but per unit surface area in the reference configuration, one can introduce the **reference body force B**: this is the actual body force acting in the current configuration, per unit volume in the reference configuration. Thus if the body force acting on a volume element dv in the current configuration is $d\mathbf{f}_{\text{body}}$, then

$$d\mathbf{f}_{\text{body}} = \mathbf{b}dv = \mathbf{B}dV \qquad (3.6.6)$$

The resultant force acting on the body is then

$$\mathbf{F}(t) = \int_S \mathbf{T}dS + \int_V \mathbf{B}dV, \qquad F_i = \int_S T_i\, dS + \int_V B_i\, dV \qquad (3.6.7)$$

Using Cauchy's law, $\mathbf{T} = \mathbf{PN}$, where **P** is the PK1 stress, and the divergence theorem 1.12.21, 3.6.5 and 3.6.7 lead to

$$\int_V [\text{Div}\mathbf{P} + \mathbf{B}]dV = \int_V \rho_0 \frac{d\mathbf{V}}{dt} dV \qquad (3.6.8)$$

and the corresponding local form is

$$\boxed{\text{Div}\,\mathbf{P} + \mathbf{B} = \rho_0 \frac{d\mathbf{V}}{dt}, \qquad \frac{\partial P_{ij}}{\partial X_j} + B_i = \rho_0 \frac{dV_i}{dt}}$$

Equations of Motion (Material Form) (3.6.9)

Derivation from the Spatial Form

The equations of motion can also be derived directly from the spatial equations. In order to do this, one must first show that $\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right)$ is zero. One finds that (using the divergence theorem, Nanson's formula 2.2.59 and the fact that $\mathrm{div}\mathbf{I}=0$)

$$\begin{aligned}
&\int_V \mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right) dV = \int_S J\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS = \int_s \mathbf{n}ds = \int_s \mathbf{I}\mathbf{n}ds = \int_v \mathrm{div}\mathbf{I}\,dv = 0 \\
&\int_V \frac{\partial\left(JF_{ji}^{-1}\right)}{\partial X_j} dV = \int_S JF_{ji}^{-1}N_i dS = \int_s n_i ds = \int_s \delta_{ij} n_j ds = \int_v \frac{\partial \delta_{ij}}{\partial x_i} dv = 0
\end{aligned} \tag{3.6.10}$$

This result is known as the **Piola identity**. Thus, with the PK1 stress related to the Cauchy stress through 3.5.8, $\mathbf{P}=J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$, and using identity 1.14.16c,

$$\begin{aligned}
\mathrm{Div}\mathbf{P} &= \mathrm{Div}\left(\boldsymbol{\sigma}\left(J\mathbf{F}^{-\mathrm{T}}\right)\right) \\
&= \boldsymbol{\sigma}\mathrm{Div}\left(J\mathbf{F}^{-\mathrm{T}}\right) + \mathrm{Grad}\boldsymbol{\sigma} : \left(J\mathbf{F}^{-\mathrm{T}}\right) \\
&= J\mathrm{Grad}\boldsymbol{\sigma} : \mathbf{F}^{-\mathrm{T}}
\end{aligned} \tag{3.6.11}$$

From 2.2.8c,

$$\mathrm{Div}\mathbf{P} = J\mathrm{div}\boldsymbol{\sigma} \tag{3.6.12}$$

Then, with $dv = JdV$ and 3.6.6, the equations of motion in the spatial form can now be transformed according to

$$\int_v \left[\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}\right] dv = \int_v \rho\dot{\mathbf{v}}dv \quad \rightarrow \quad \int_V \left[\mathrm{Div}\mathbf{P} + \mathbf{B}\right] dV = \int_V \rho_0 \dot{\mathbf{V}} dV$$

as before.

3.6.3 Symmetry of the Cauchy Stress

It will now be shown that the principle of angular momentum leads to the requirement that the Cauchy stress tensor is symmetric. Applying Cauchy's law to 3.2.11,

$$\begin{aligned}
&\int_s \mathbf{r}\times(\boldsymbol{\sigma}\mathbf{n})ds + \int_v \mathbf{r}\times\mathbf{b}dv = \frac{d}{dt}\int_v \mathbf{r}\times\rho\mathbf{v}dv \\
&\int_s \varepsilon_{ijk} x_j \sigma_{kl} n_l dS + \int_v \varepsilon_{ijk} x_j b_k\, dv = \frac{d}{dt}\int_v \varepsilon_{ijk} x_j \rho v_k\, dv
\end{aligned} \tag{3.6.13}$$

The surface integral can be converted into a volume integral using the divergence theorem. Using the index notation, and concentrating on the integrand of the resulting volume integral, one has, using 1.3.14 (the permutation symbol is a constant here, $\partial\varepsilon_{ijk}/\partial x_l = 0$),

$$\varepsilon_{ijk} \frac{\partial(x_j \sigma_{kl})}{\partial x_l} = \varepsilon_{ijk} \left\{ x_j \frac{\partial \sigma_{kl}}{\partial x_l} + \sigma_{kl} \delta_{jl} \right\} = \varepsilon_{ijk} \left\{ x_j \frac{\partial \sigma_{kl}}{\partial x_l} + \sigma_{kj} \right\} \equiv \mathbf{r} \times \operatorname{div}\boldsymbol{\sigma} + \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} \qquad (3.6.14)$$

where E is the third-order permutation tensor, Eqn. 1.9.6, $\mathsf{E} = \varepsilon_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$. Thus, with the Reynold's transport identity 3.1.31,

$$\int_v \left\{ \mathbf{r} \times \operatorname{div}\boldsymbol{\sigma} + \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} \right\} dv + \int_v \mathbf{r} \times \mathbf{b}\, dv = \int_v \rho \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) dv \qquad (3.6.15)$$

The material derivative of this cross product is

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{v} = \mathbf{r} \times \frac{d\mathbf{v}}{dt} + \mathbf{v} \times \mathbf{v} = \mathbf{r} \times \frac{d\mathbf{v}}{dt} \qquad (3.6.16)$$

and so

$$\int_v \mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} dv + \int_v \mathbf{r} \times \left\{ \operatorname{div}\boldsymbol{\sigma} + \mathbf{b} - \rho \frac{d\mathbf{v}}{dt} \right\} dv = 0 \qquad (3.6.17)$$

From the equations of motion 2.6.2, the term inside the brackets is zero, so that

$$\mathsf{E} : \boldsymbol{\sigma}^{\mathrm{T}} = 0, \qquad \varepsilon_{ijk} \sigma_{kj} = 0 \qquad (3.6.18)$$

It follows, from expansion of this relation, that the matrix of stress components must be symmetric:

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}, \qquad \sigma_{ij} = \sigma_{ji}} \qquad \textbf{Symmetry of Stress} \qquad (3.6.19)$$

3.6.4 Consequences in the Material Form

Here, the consequences of 3.6.19 on the PK1 and PK2 stresses is examined. Using the result $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ and 3.5.8, $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}$,

$$J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}} = \left(J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}\right)^{\mathrm{T}} = J^{-1}\mathbf{F}\mathbf{P}^{\mathrm{T}} \qquad (3.6.20)$$

so that

$$\mathbf{P}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{P}^{\mathrm{T}}, \qquad P_{ik}F_{jk} = F_{ik}P_{jk} \qquad (3.6.21)$$

These equations are trivial when $i = j$, not providing any constraint on **P**. On the other hand, when $i \neq j$ one has the three equations

$$
\begin{aligned}
P_{11}F_{21} + P_{12}F_{22} + P_{13}F_{23} &= F_{11}P_{21} + F_{12}P_{22} + F_{13}P_{23} \\
P_{11}F_{31} + P_{12}F_{32} + P_{13}F_{33} &= F_{11}P_{31} + F_{12}P_{32} + F_{13}P_{33} \\
P_{21}F_{31} + P_{22}F_{32} + P_{23}F_{33} &= F_{21}P_{31} + F_{22}P_{32} + F_{23}P_{33}
\end{aligned}
\qquad (3.6.22)
$$

Thus angular momentum considerations imposes these three constraints on the PK1 stress (as they imposed the three constraints $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$ on the Cauchy stress).

It has already been seen that a consequence of the symmetry of the Cauchy stress is the symmetry of the PK2 stress **S**; thus, formally, the symmetry of **S** is the result of the angular momentum principle.