3.5 Stress Measures for Large Deformations

Thus far, the surface forces acting within a material have been described in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$. The Cauchy stress is also called the **true stress**, to distinguish it from other stress tensors, some of which will be discussed below. It is called the *true* stress because it is a true measure of the force per unit area in the current, deformed, configuration. When the deformations are small, there is no distinction to be made between this deformed configuration and some reference, or undeformed, configuration, and the Cauchy stress is the sensible way of describing the action of surface forces. When the deformations are large, however, one needs to refer to some reference configuration. In this case, there are a number of different possible ways of defining the action of surface forces; some of these stress measures often do not have as clear a physical meaning as the Cauchy stress, but are useful nonetheless.

3.5.1 The First Piola – Kirchhoff Stress Tensor

Consider two configurations of a material, the reference and current configurations. Consider now a vector element of surface in the reference configuration, $\mathbf{N}dS$, where dS is the area of the element and $\mathbf{N}$ is the unit normal. After deformation, the material particles making up this area element now occupy the element defined by $\mathbf{n}ds$, where ds is the area and $\mathbf{n}$ is the normal in the current configuration. Suppose that a force $d\mathbf{f}$ acts on the surface element (in the current configuration). Then by definition of the Cauchy stress

$$d\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}\,ds \tag{3.5.1}$$

The **first Piola-Kirchhoff stress** tensor $\mathbf{P}$ (which will be called the **PK1 stress** for brevity) is defined by

$$d\mathbf{f} = \mathbf{P}\mathbf{N}\,dS \tag{3.5.2}$$

The PK1 stress relates the force acting in the *current* configuration to the surface element in the *reference* configuration. Since it relates to both configurations, it is a two-point tensor.

The (Cauchy) traction vector was defined as

$$\mathbf{t} = \frac{d\mathbf{f}}{ds}, \qquad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \tag{3.5.3}$$

Similarly, one can introduce a **PK1 traction vector T** such that

$$\mathbf{T} = \frac{d\mathbf{f}}{dS}, \qquad \mathbf{T} = \mathbf{P}\mathbf{N} \tag{3.5.4}$$

Whereas the Cauchy traction is the actual physical force per area on the element in the current configuration, the PK1 traction is a fictitious quantity – the force acting on an element in the current configuration divided by the area of the corresponding element in the reference configuration. Note that, since $d\mathbf{f} = \mathbf{t}ds = \mathbf{T}dS$, it follows that $\mathbf{T}$ and $\mathbf{t}$ act in the same direction (but have different magnitudes), Fig. 3.5.1.

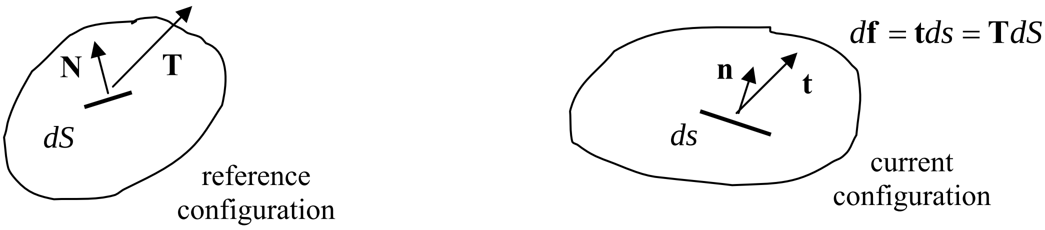

Figure 3.5.1: Traction vectors

Uniaxial Tension

Consider a uniaxial tensile test whereby a specimen is stretched uniformly by a constant force $\mathbf{f}$, Fig. 3.5.2. The initial cross-sectional area of the specimen is A_0 and the cross-sectional area of the specimen at time t is $A(t)$. The Cauchy (true) stress is

$$\boldsymbol{\sigma}(t) = \frac{\mathbf{f}}{A(t)} \tag{3.5.5}$$

and the PK1 stress is

$$\mathbf{P} = \frac{\mathbf{f}}{A_0} \tag{3.5.6}$$

This stress measure, force over area of the undeformed specimen, as used in the uniaxial tensile test, is also called the **engineering stress**.

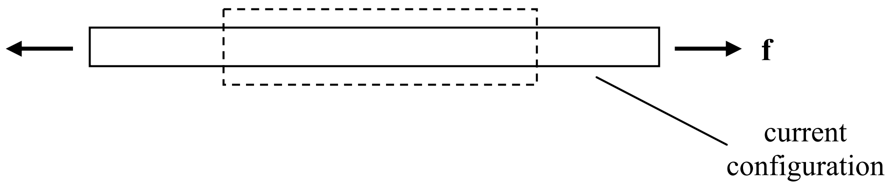

Figure 3.5.2: Uniaxial tension of a bar

The Nominal Stress

The PK1 stress tensor is also called the **nominal stress tensor**. Note that many authors use a different definition for the nominal stress, namely $\mathbf{T} = \mathbf{N}\mathbf{P}$, and then define the PK1 stress to be the transpose of this $\mathbf{P}$. Thus all authors use the same definition for the PK1 stress, but a slightly different definition for the nominal stress.

Relation between the Cauchy and PK1 Stresses

From the above definitions,

$$\boldsymbol{\sigma}\mathbf{n}ds = \mathbf{P}\mathbf{N}dS \tag{3.5.7}$$

Using Nanson's formula, 2.2.59, $\mathbf{n}ds = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS$,

$$\boxed{\begin{aligned}\mathbf{P} &= J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} \\ \boldsymbol{\sigma} &= J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}}\end{aligned}} \qquad \textbf{PK1 stress} \tag{3.5.8}$$

The Cauchy stress is symmetric, but the deformation gradient is not. Hence the PK1 stress tensor is *not symmetric*, and this restricts its use as an alternative stress measure to the Cauchy stress measure. In fact, this lack of symmetry and lack of a clear physical meaning makes it uncommon for the PK1 stress to be used in the modeling of materials. It is, however, useful in the description of the momentum balance laws in the material description, where **P** plays an analogous role to that played by the Cauchy stress $\boldsymbol{\sigma}$ in the equations of motion (see later).

3.5.2 The Second Piola – Kirchhoff Stress Tensor

The **second Piola – Kirchhoff stress** tensor, or the **PK2 stress**, **S**, is defined by

$$\boxed{\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}} \qquad \textbf{PK2 stress} \tag{3.5.9}$$

Even though the PK2 does not admit a physical interpretation (except in the simplest of cases, but see the interpretation below), there are three good reasons for using it as a measure of the forces acting in a material. First, one can see that

$$\left(\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)^{\mathrm{T}} = \left(\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$$

and since the Cauchy stress is symmetric, so is the PK2 stress:

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.5.10}$$

A second reason for using the PK2 stress is that, together with the Euler-Lagrange strain **E**, it gives the power of a deforming material (see later). Third, it is parameterized by material coordinates only, that is, it is a material tensor field, in the same way as the Cauchy stress is a spatial tensor field.

Note that the PK1 and PK2 stresses are related through

$$\mathbf{P} = \mathbf{F}\mathbf{S}, \qquad \mathbf{S} = \mathbf{F}^{-1}\mathbf{P} \tag{3.5.11}$$

The PK2 stress can be interpreted as follows: take the force vector in the current configuration $d\mathbf{f}$ and locate a corresponding vector in the undeformed configuration according to $d\bar{\mathbf{f}} = \mathbf{F}^{-1} d\mathbf{f}$. The PK2 stress tensor is this fictitious force divided by the corresponding area element in the reference configuration: $d\bar{\mathbf{f}} = \mathbf{SN}dS$, and 3.5.9 follows from 3.5.2, 3.5.8:

$$d\mathbf{f} = \mathbf{P}\,\mathbf{N}dS = J\boldsymbol{\sigma}\,\mathbf{F}^{-\mathrm{T}}\,\mathbf{N}dS$$

3.5.3 Alternative Stress Tensors

Some other useful stress measures are described here.

The Kirchhoff Stress

The **Kirchhoff stress tensor** $\boldsymbol{\tau}$ is defined as

$$\boxed{\boldsymbol{\tau} = J\boldsymbol{\sigma}} \qquad \textbf{Kirchhoff Stress} \qquad (3.5.12)$$

It is a spatial tensor field parameterized by spatial coordinates. One reason for its use is that, in many equations, the Cauchy stress appears together with the Jacobian and the use of $\boldsymbol{\tau}$ simplifies formulae.

Note that the Kirchhoff stress is the push forward of the PK2 stress; from 2.12.9b, 2.12.11b,

$$\begin{aligned} \boldsymbol{\tau} &= \chi_*(\mathbf{S})^{\#} = \mathbf{FSF}^{\mathrm{T}} \\ \mathbf{S} &= \chi_*^{-1}(\boldsymbol{\tau})^{\#} = \mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}} \end{aligned} \qquad (3.5.13)$$

The Corotational Cauchy Stress

The **corotational stress** $\hat{\boldsymbol{\sigma}}$ is defined as

$$\boxed{\hat{\boldsymbol{\sigma}} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R}} \qquad \textbf{Corotational Stress} \qquad (3.5.14)$$

where **R** is the orthogonal rotation tensor. Whereas the Cauchy stress is related to the PK2 stress through $\boldsymbol{\sigma} = J^{-1}\mathbf{FSF}^{\mathrm{T}}$, the corotational stress is related to the PK2 stress through (with **F** replaced by the right (symmetric) stretch tensor **U**):

$$\hat{\boldsymbol{\sigma}} = J^{-1}\mathbf{USU}^{\mathrm{T}} = J^{-1}\mathbf{U}\left(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)\mathbf{U} = \left(\mathbf{UF}^{-1}\right)\boldsymbol{\sigma}\left(\mathbf{F}^{-\mathrm{T}}\mathbf{U}\right) = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} \qquad (3.5.15)$$

The corotational stress is defined on the intermediate configuration of Fig. 2.10.8. It can be regarded as the push forward of the PK2 stress from the reference configuration through the stretch **U**, scaled by J^{-1} (Eqn. 2.12.28b):

$$\hat{\boldsymbol{\sigma}} = J^{-1}\chi_*(\mathbf{S})^{\#}{}_{\mathrm{U(G)}} = J^{-1}S^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = J^{-1}S^{ij}\left(\mathbf{UG}_i \otimes \mathbf{UG}_j\right) = J^{-1}\mathbf{USU}^{\mathrm{T}} = J^{-1}\mathbf{USU} \quad (3.5.16)$$

or as the pull-back of the Cauchy stress with respect to **R** (Eqn. 2.12.27f):

$$\hat{\boldsymbol{\sigma}} = \chi_*^{-1}(\boldsymbol{\sigma})^{\#}{}_{\mathbf{R}(\mathbf{g})} = \sigma^{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j = \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} \tag{3.5.17}$$

The Biot Stress

The **Biot** (or **Jaumann**) **stress tensor** $\mathbf{T}_{\mathrm{B}}$ is defined as

$$\boxed{\mathbf{T}_{\mathrm{B}} = \mathbf{R}^{\mathrm{T}}\mathbf{P} = \mathbf{U}\mathbf{S}} \qquad \textbf{Biot Stress} \tag{3.5.18}$$

From 3.5.11, it is similar to the PK1 stress, only with **F** replaced by **U**.

Example

Consider a **pre-stressed** thin plate with $\sigma_{11} = \sigma_1^0$, $\sigma_{22} = \sigma_2^0$, that is, it has a non-zero stress although no forces are acting[1], Fig. 3.5.3. In this initial state, $\mathbf{F} = \mathbf{I}$ and, considering a two-dimensional state of stress,

$$\boldsymbol{\sigma} = \mathbf{P} = \mathbf{S} = \hat{\boldsymbol{\sigma}} = \boldsymbol{\tau} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix} \sigma_1^0 & 0 \\ 0 & \sigma_2^0 \end{bmatrix}$$

The material is now rotated *as a rigid body* 45° counterclockwise – the stress-state is "frozen" within the material and rotates with it. Then

$$\mathbf{F} = \mathbf{R} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The stress components with respect to the rotated x_i^* axes shown in Fig. 3.5.3b are $\sigma_{11}^* = \sigma_1^0$, etc.; the components with respect to the spatial axes x_i can be found from the stress transformation rule $[\boldsymbol{\sigma}] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}^*][\mathbf{Q}] = [\mathbf{R}][\boldsymbol{\sigma}^*][\mathbf{R}^{\mathrm{T}}]$, and so

$$\boldsymbol{\sigma} = \begin{bmatrix} \frac{1}{2}(\sigma_1^0 + \sigma_2^0) & \frac{1}{2}(\sigma_1^0 - \sigma_2^0) \\ \frac{1}{2}(\sigma_1^0 - \sigma_2^0) & \frac{1}{2}(\sigma_1^0 + \sigma_2^0) \end{bmatrix}$$

Note that the Cauchy stress changes with this rigid body rotation. Further, with $J = 1$,

$$\boldsymbol{\tau} = \boldsymbol{\sigma}, \quad \mathbf{P} = \begin{bmatrix} \sigma_1^0/\sqrt{2} & -\sigma_2^0/\sqrt{2} \\ \sigma_1^0/\sqrt{2} & \sigma_2^0/\sqrt{2} \end{bmatrix}, \quad \mathbf{S} = \hat{\boldsymbol{\sigma}} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix} \sigma_1^0 & 0 \\ 0 & \sigma_2^0 \end{bmatrix}$$

Note that the PK1 stress is not symmetric. Now attach axes x^* to the material and rotate these axes with the specimen as it rotates, as in Fig. 3.5.3b. The components with respect

[1] for example a piece of metal can be deformed; when the *load is removed* it is often pre-stressed – there is a non-zero state of stress in the material

to these rotated axes give the corotational stress; the corotational stress is the stress in a body, taking out the stress changes caused by rigid body rotations – one says that the corotational stress (and PK2 stress) "rotate" with the body.

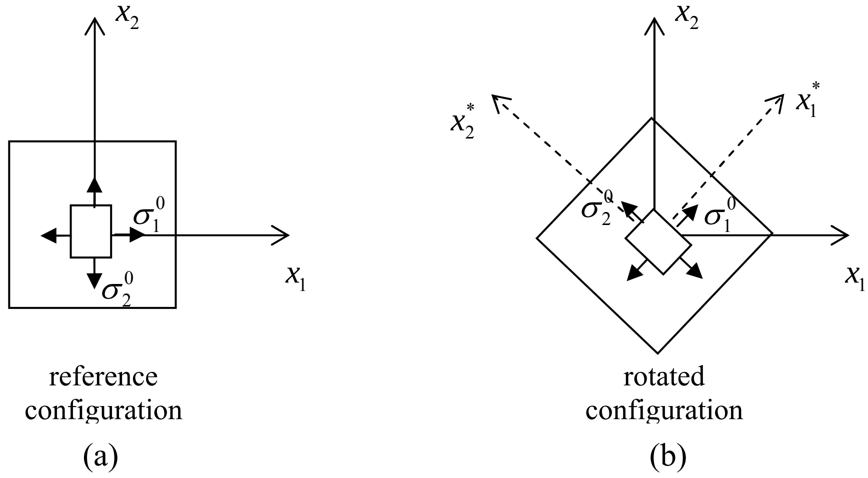

Figure 3.5.3: Pre-stressed material; (a) original position, (b) rotated configuration

■

3.5.4 Small deformations

From §2.7, when the deformations are small, neglecting terms involving products of displacement gradients,

$$\mathbf{F} = \mathbf{I} + \mathrm{grad}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2 = \mathbf{I} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \qquad (3.5.19)$$

Here, $\mathrm{O}(\mathrm{grad}\mathbf{u})$ means terms of the order of displacement gradients (and higher) have been neglected and $\mathrm{O}(\mathrm{grad}\mathbf{u})^2$ means terms of the order of products of displacement gradients (and higher) have been neglected. Also,

$$\begin{aligned} J &= \det\mathbf{F} \\ &= \det\left(\mathbf{I} + \mathrm{grad}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2\right) = 1 + \mathrm{div}\mathbf{u} + \mathrm{O}(\mathrm{grad}\mathbf{u})^2 = 1 + \mathrm{O}(\mathrm{grad}\mathbf{u}) \end{aligned} \qquad (3.5.20)$$

From 3.5.8 and 3.5.9, using 3.5.19-20, one has

$$\begin{aligned} J\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^{\mathrm{T}} \quad &\to \quad \boldsymbol{\sigma} + \mathrm{O}(\mathrm{grad}\mathbf{u}) = \mathbf{P} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \\ J\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}} \quad &\to \quad \boldsymbol{\sigma} + \mathrm{O}(\mathrm{grad}\mathbf{u}) = \mathbf{S} + \mathrm{O}(\mathrm{grad}\mathbf{u}) \end{aligned} \qquad (3.5.21)$$

In the linear theory then, with $\mathrm{O}(\mathrm{grad}\mathbf{u}) \to 0$, the stress measures encountered in this section are all equivalent.

3.5.5 Objective Stress Tensors

In order to ascertain the objectivity of the stress tensors, first note that, *by definition*, force is an objective vector, and therefore so also is the traction vector. Similarly for the normal vector. The normal and traction vectors transform under an observer transformation according to 2.8.10, $\mathbf{n}^* = \mathbf{Qn}$ and $\mathbf{t}^* = \mathbf{Qt}$. Then

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \quad \rightarrow \quad \mathbf{Q}^{\mathrm{T}}\mathbf{t}^* = \boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\mathbf{n}^* \quad \rightarrow \quad \mathbf{t}^* = \left(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\right)\mathbf{n}^* \tag{3.5.22}$$

and so $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$; according to 2.8.12, the Cauchy stress is objective. The PK2 stress **S** is objective, since it is a material tensor unaffected by an observer transformation. For the PK1 stress, using 2.8.23,

$$\mathbf{P}^* = J^*\boldsymbol{\sigma}^*\left(\mathbf{F}^*\right)^{-\mathrm{T}} = J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}\left(\mathbf{QF}\right)^{-\mathrm{T}} = \mathbf{Q}\left(J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right) \tag{3.5.23}$$

and so, according to 2.8.16, **P** is objective (transforming like a vector, being a two-point tensor).

3.5.6 Objective Stress Rates

One needs to incorporate stress rates in models of materials where the response depends on the rate of stressing, for example with viscoelastic materials. As discussed in §2.8.5, the rates of objective tensors are not necessarily objective. As discussed in §2.12.3, the Lie derivative of a spatial second order tensor is objective. For the Cauchy stress, there are a number of different objective rates one can use, based on the Lie derivative (see Eqns. 2.8.35-36, 2.12.41, 2.12.44):

$$\begin{array}{lll} \textbf{Cotter-Rivlin stress rate} & \dot{\boldsymbol{\sigma}} + \mathbf{l}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{l} & = \mathrm{L}_{\mathrm{v}}^{\mathrm{b}}\boldsymbol{\sigma} \\ \textbf{Jaumann stress rate} & \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} & = \tfrac{1}{2}\left(\mathrm{L}_{\mathrm{v}}^{\mathrm{b}}\boldsymbol{\sigma} + \mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma}\right) \\ \textbf{Oldroyd stress rate}^{2} & \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} & = \mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma} \end{array} \tag{3.5.24}$$

Stress rates of other spatial stress tensors can be defined in the same way, for example the Oldroyd rate of the Kirchhoff stress tensor is $\dot{\boldsymbol{\tau}} - \mathbf{l}\boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}$.

The material derivative of the material PK2 stress tensor, $\dot{\mathbf{S}}$, is objective. The push forward of $\dot{\mathbf{S}}$ is, from 2.12.9b,

$$\chi_*\left(\dot{\mathbf{S}}\right)^{\#} = \mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} \tag{3.5.25}$$

[2] this is sometimes called the contravariant Oldroyd stress rate, to distinguish it from the Cotter-Rivlin rate, which is also sometimes called the covariant Oldroyd stress rate

This push forward, scaled by the inverse of the Jacobian, $J^{-1}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}}$ is called the **Truesdell stress rate**. This can be expressed in terms of the Cauchy stress by using 3.5.9, and then 2.5.20, 2.5.5:

$$J^{-1}\mathbf{F}\frac{d}{dt}\left(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)\mathbf{F}^{\mathrm{T}} = J^{-1}\mathbf{F}\left(\dot{J}\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + J\,\dot{\overline{\mathbf{F}^{-1}}}\,\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\dot{\boldsymbol{\sigma}}\mathbf{F}^{-\mathrm{T}} + J\mathbf{F}^{-1}\boldsymbol{\sigma}\,\dot{\overline{\mathbf{F}^{-\mathrm{T}}}}\right)\mathbf{F}^{\mathrm{T}}$$

$$= \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{\mathrm{T}} + \mathrm{tr}(\mathbf{d})\boldsymbol{\sigma} \tag{3.5.26}$$

Thus far, objective rates have been constructed by pulling back, taking derivatives and pushing forward. One can construct objective rates also by pulling back and pushing forward with the rotation tensor **R** only, since it is the rotation which causes the stress rates to be non-objective. For example, $\mathrm{L}_{\mathrm{v}}^{\#}\boldsymbol{\sigma}$, setting $\mathbf{F} = \mathbf{R}$, is, from 3.5.17 and 2.12.27b,

$$\begin{aligned}
\chi_*\left(\frac{d}{dt}\left[\chi_*^{-1}(\boldsymbol{\sigma})^{\#}_{\mathbf{R}(\mathbf{g})}\right]\right)_{\mathbf{R}(\hat{\mathbf{g}})} &= \chi_*\left(\frac{d}{dt}\left[\hat{\boldsymbol{\sigma}}\right]\right)^{\#}_{\mathbf{R}(\hat{\mathbf{g}})} \\
&= \mathbf{R}\left(\dot{\mathbf{R}}^{\mathrm{T}}\boldsymbol{\sigma}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{\sigma}}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\boldsymbol{\sigma}\dot{\mathbf{R}}\right)\mathbf{R}^{\mathrm{T}} \\
&= \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\Omega}_R - \boldsymbol{\Omega}_R\boldsymbol{\sigma}
\end{aligned} \tag{3.5.27}$$

where $\boldsymbol{\Omega}_R = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$ is the skew-symmetric angular velocity tensor 2.6.3. The stress rate 3.5.27 is called the **Green-Naghdi stress rate**. From the above, the Green-Naghdi rate is the push forward of the time derivative of the corotational stress.

Example

Consider again the example discussed at the end of §3.5.3, only let the plate rotate at constant angular velocity ω, so

$$\mathbf{F} = \mathbf{R} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}, \quad \dot{\mathbf{F}} = \dot{\mathbf{R}} = \omega\begin{bmatrix} -\sin(\omega t) & -\cos(\omega t) \\ \cos(\omega t) & -\sin(\omega t) \end{bmatrix}$$

Again, using the stress transformation rule $[\boldsymbol{\sigma}] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}^*][\mathbf{Q}] = [\mathbf{R}][\boldsymbol{\sigma}^*][\mathbf{R}^{\mathrm{T}}]$,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1^0\cos^2(\omega t) + \sigma_2^0\sin^2(\omega t) & \cos(\omega t)\sin(\omega t)\left(\sigma_1^0 - \sigma_2^0\right) \\ \cos(\omega t)\sin(\omega t)\left(\sigma_1^0 - \sigma_2^0\right) & \sigma_1^0\sin^2(\omega t) + \sigma_2^0\cos^2(\omega t) \end{bmatrix}$$

and, with $J = 1$,

$$\boldsymbol{\tau} = \boldsymbol{\sigma}, \quad \mathbf{P} = \begin{bmatrix} \cos(\omega t)\sigma_1^0 & -\sin(\omega t)\sigma_2^0 \\ \sin(\omega t)\sigma_1^0 & \cos(\omega t)\sigma_2^0 \end{bmatrix}, \quad \mathbf{S} = \hat{\boldsymbol{\sigma}} = \mathbf{T}_{\mathrm{B}} = \begin{bmatrix} \sigma_1^0 & 0 \\ 0 & \sigma_2^0 \end{bmatrix}$$

Also,

$$\mathbf{l} = \mathbf{w} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \mathbf{\Omega}_{\mathbf{R}} = \omega\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then

$$\dot{\mathbf{P}} = \omega\begin{bmatrix} -\sin(\omega t)\sigma_1^0 & -\cos(\omega t)\sigma_2^0 \\ \cos(\omega t)\sigma_1^0 & -\sin(\omega t)\sigma_2^0 \end{bmatrix}, \quad \dot{\mathbf{S}} = \dot{\hat{\boldsymbol{\sigma}}} = \dot{\mathbf{T}}_{\mathrm{B}} = \mathbf{0}$$

and

$$\dot{\boldsymbol{\sigma}} = \omega\begin{bmatrix} -2\sin(\omega t)\cos(\omega t)\left(\sigma_1^0 - \sigma_2^0\right) & \left(\cos^2(\omega t) - \sin^2(\omega t)\right)\left(\sigma_1^0 - \sigma_2^0\right) \\ \left(\cos^2(\omega t) - \sin^2(\omega t)\right)\left(\sigma_1^0 - \sigma_2^0\right) & +2\sin(\omega t)\cos(\omega t)\left(\sigma_1^0 - \sigma_2^0\right) \end{bmatrix}$$

For a rigid body rotation, it can be seen that the definitions of the Cotter-Rivlin, Jaumann, Oldroyd, Truesdell and Green-Naghdi rates are equivalent, and they are all zero:

$$\dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} = \mathbf{0}$$

This is as expected since objective stress rates for two configurations which differ by a rigid body rotation will, by definition, be equal (the stress components will not change); they are zero in the reference configuration and so will be zero in the rotated configuration.

■

3.5.7 Problems

1. Consider the case of uniaxial stress, where a material with initial dimensions length l_0, breadth w_0 and height h_0 deforms into a component with dimensions length l, breadth w and height h. The only non-zero Cauchy stress component is σ_{11}, acting in the direction of the length of the component.
   (a) write down the motion equations in the material description, $\mathbf{x} = \chi(\mathbf{X})$
   (b) calculate the deformation gradient $\mathbf{F}$ and confirm that $J = \det\mathbf{F}$ is the ratio of the volume in the current configuration to that in the initial configuration
   (c) Calculate the PK1 stress. How is it related to the Cauchy stress for this uniaxial stress-state?
   (d) calculate the PK2 stress
2. A material undergoes the deformation
   $$x_1 = 3X_1 t, \quad x_2 = X_1 t + X_2, \quad x_3 = X_3$$
   The Cauchy stress at a point in the material is
   $$[\boldsymbol{\sigma}] = \begin{bmatrix} t & -2t & 0 \\ -2t & t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
   (a) Calculate the PK1 and PK2 stresses at the point (check that PK2 is symmetric)

(b) Calculate the expressions $\mathbf{P}:\dot{\mathbf{F}}$, $J\boldsymbol{\sigma}:\mathbf{d}$, $\mathbf{S}:\dot{\mathbf{E}}$ (for $\dot{\mathbf{E}}$, use the expression 2.5.18b, $\dot{\mathbf{E}}=\mathbf{F}^{\mathrm{T}}\mathbf{dF}$). In these expressions, **d** is the rate of deformation tensor. (You should get the same result for all three cases, since they all give the rate of internal work done by the stresses during the deformation, per unit reference volume – see later)

3. Show that the Oldroyd rate of the Kirchhoff stress, $\dot{\boldsymbol{\tau}}-\mathbf{l}\boldsymbol{\tau}-\boldsymbol{\tau}\mathbf{l}^{\mathrm{T}}$, is equal to the Jacobian times the Truesdell stress rate of the Cauchy stress, 3.5.26.