

## 3.1 Conservation of Mass

### 3.1.1 Mass and Density

**Mass** is a non-negative scalar measure of a body's tendency to resist a change in motion.

Consider a small volume element  $\Delta v$  whose mass is  $\Delta m$ . Define the average **density** of this volume element by the ratio

$$\rho_{\text{AVE}} = \frac{\Delta m}{\Delta v} \quad (3.1.1)$$

If  $p$  is some point within the volume element, then define the **spatial mass density** at  $p$  to be the limiting value of this ratio as the volume shrinks down to the point,

$$\boxed{\rho(\mathbf{x}, t) = \lim_{\Delta v \rightarrow 0} \frac{\Delta m}{\Delta v}} \quad \text{Spatial Density} \quad (3.1.2)$$

In a real material, the incremental volume element  $\Delta v$  must not actually get too small since then the limit  $\rho$  would depend on the atomistic structure of the material; the volume is only allowed to decrease to some minimum value which contains a large number of molecules. The spatial mass density is a representative average obtained by having  $\Delta v$  large compared to the atomic scale, but small compared to a typical length scale of the problem under consideration.

The density, as with displacement, velocity, and other quantities, is defined for *specific particles* of a continuum, and is a continuous function of coordinates and time,  $\rho = \rho(\mathbf{x}, t)$ . However, the mass is not defined this way – one writes for the mass of an infinitesimal volume of material – a **mass element**,

$$dm = \rho(\mathbf{x}, t)dv \quad (3.1.3)$$

or, for the mass of a volume  $v$  of material at time  $t$ ,

$$m = \int_v \rho(\mathbf{x}, t)dv \quad (3.1.4)$$

### 3.1.2 Conservation of Mass

The law of conservation of mass states that mass can neither be created nor destroyed.

Consider a collection of matter located somewhere in space. This quantity of matter with well-defined boundaries is termed a **system**. The law of conservation of mass then implies that the mass of this given system remains constant,

$$\boxed{\frac{Dm}{Dt} = 0} \quad \text{Conservation of Mass} \quad (3.1.5)$$

The volume occupied by the matter may be changing and the density of the matter within the system may be changing, but the mass remains constant.

Considering a differential mass element at position  $\mathbf{X}$  in the reference configuration and at  $\mathbf{x}$  in the current configuration, Eqn. 3.1.5 can be rewritten as

$$dm(\mathbf{X}) = dm(\mathbf{x}, t) \quad (3.1.6)$$

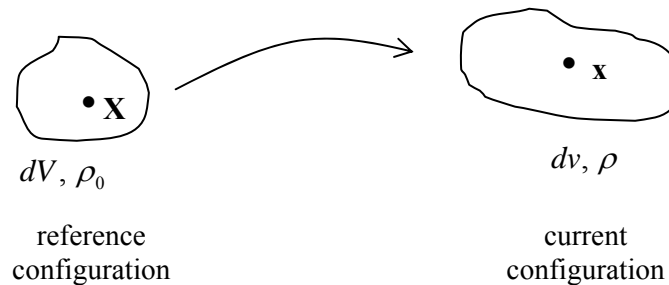
The conservation of mass equation can be expressed in terms of densities. First, introduce  $\rho_0$ , the **reference mass density** (or simply the **density**), defined through

$$\boxed{\rho_0(\mathbf{X}) = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V}} \quad \text{Density} \quad (3.1.7)$$

Note that the density  $\rho_0$  and the spatial mass density  $\rho$  are *not* the same quantities<sup>1</sup>.

Thus the **local** (or **differential**) **form** of the conservation of mass can be expressed as (see Fig. 3.1.1)

$$dm = \rho_0(\mathbf{X})dV = \rho(\mathbf{x}, t)dv = \text{const} \quad (3.1.8)$$



**Figure 3.1.1: Conservation of Mass for a deforming mass element**

Integration over a finite region of material gives the **global** (or **integral**) **form**,

$$m = \int_V \rho_0(\mathbf{X})dV = \int_v \rho(\mathbf{x}, t)dv = \text{const} \quad (3.1.9)$$

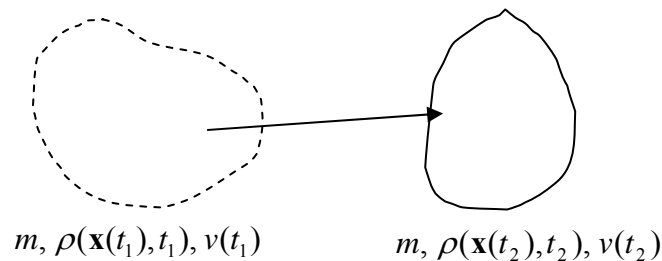
or

$$\dot{m} = \frac{dm}{dt} = \frac{d}{dt} \int_v \rho(\mathbf{x}, t)dv = 0 \quad (3.1.10)$$

<sup>1</sup> they not only are functions of different variables, but also have different values; they are not different representations of the same thing, as were, for example, the velocities  $\mathbf{v}$  and  $\mathbf{V}$ . One could introduce a material mass density,  $P(X, t) = \rho(x(X, t), t)$ , but such a quantity is not useful in analysis

### 3.1.3 Control Mass and Control Volume

A **control mass** is a *fixed mass* of material whose volume and density may change, and which may move through space, Fig. 3.1.2. There is no mass transport through the moving surface of the control mass. For such a system, Eqn. 3.1.10 holds.



**Figure 3.1.2: Control Mass**

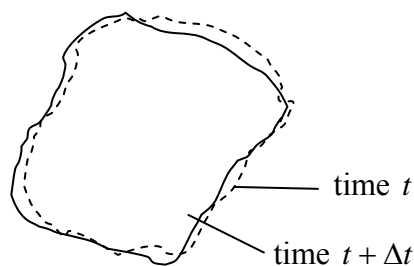
By definition, the derivative in 3.1.10 is the time derivative of a property (in this case mass) of a collection of material particles as they move through space, and when they instantaneously occupy the volume  $v$ , Fig. 3.1.3, or

$$\frac{d}{dt} \int_{dv} \rho dv = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_{v(t+\Delta t)} \rho(\mathbf{x}, t + \Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t) dv \right\} = 0 \quad (3.1.11)$$

Alternatively, one can take the material derivative inside the integral sign:

$$\frac{dm}{dt} = \int_v \frac{d}{dt} [\rho(\mathbf{x}, t) dv] = 0 \quad (3.1.12)$$

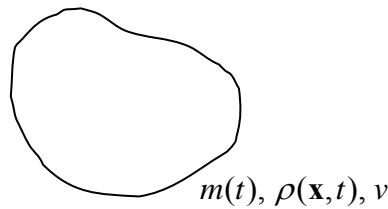
This is now equivalent to the sum of the rates of change of mass of the mass elements occupying the volume  $v$ .



**Figure 3.1.3: Control Mass occupying different volumes at different times**

A **control volume**, on the other hand, is a *fixed volume* (region) of space through which material may flow, Fig. 3.1.4, and for which *the mass may change*. For such a system, one has

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int_v \rho(\mathbf{x}, t) dv = \int_v \frac{\partial}{\partial t} [\rho(\mathbf{x}, t)] dv \neq 0 \quad (3.1.13)$$



**Figure 3.1.4: Control Volume**

### 3.1.4 The Continuity Equation (Spatial Form)

A consequence of the law of conservation of mass is the **continuity equation**, which (in the spatial form) relates the density and velocity of any material particle during motion. This equation can be derived in a number of ways:

#### Derivation of the Continuity Equation using a Control Volume (Global Form)

The continuity equation can be derived directly by considering a control volume - this is the derivation appropriate to fluid mechanics. Mass inside this fixed volume cannot be created or destroyed, so that the rate of increase of mass in the volume must equal the rate at which mass is flowing into the volume through its bounding surface.

The rate of increase of mass inside the fixed volume  $v$  is

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int_v \rho(\mathbf{x}, t) dv = \int_v \frac{\partial \rho}{\partial t} dv \quad (3.1.14)$$

The **mass flux** (rate of flow of mass) out through the surface is given by Eqn. 1.7.9,

$$\int_s \rho \mathbf{v} \cdot \mathbf{n} ds, \quad \int_s \rho v_i n_i ds$$

where  $\mathbf{n}$  is the unit outward normal to the surface and  $\mathbf{v}$  is the velocity. It follows that

$$\int_v \frac{\partial \rho}{\partial t} dv + \int_s \rho \mathbf{v} \cdot \mathbf{n} ds = 0, \quad \int_v \frac{\partial \rho}{\partial t} dv + \int_s \rho v_i n_i ds = 0 \quad (3.1.15)$$

Use of the divergence theorem 1.7.12 leads to

$$\int_v \left[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) \right] dv = 0, \quad \int_v \left[ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} \right] dv = 0 \quad (3.1.16)$$

leading to the continuity equation,

$$\begin{array}{l}
 \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \\
 \frac{d\rho}{dt} + \rho \text{div} \mathbf{v} = 0 \\
 \frac{\partial \rho}{\partial t} + \text{grad} \rho \cdot \mathbf{v} + \rho \text{div} \mathbf{v} = 0
 \end{array}
 \qquad
 \begin{array}{l}
 \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \\
 \frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} = 0 \\
 \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i + \rho \frac{\partial v_i}{\partial x_i} = 0
 \end{array}
 \quad \text{Continuity Equation}$$

(3.1.17)

This is (these are) the continuity equation in spatial form. The second and third forms of the equation are obtained by re-writing the local derivative in terms of the material derivative 2.4.7 (see also 1.6.23b).

If the material is incompressible, so the density remains constant in the neighbourhood of a particle as it moves, then the continuity equation reduces to

$$\boxed{\text{div} \mathbf{v} = 0, \quad \frac{\partial v_i}{\partial x_i} = 0} \quad \text{Continuity Eqn. for Incompressible Material} \quad (3.1.18)$$

### Derivation of the Continuity Equation using a Control Mass

Here follow two ways to derive the continuity equation using a control mass.

#### 1. Derivation using the Formal Definition

From 3.1.11, adding and subtracting a term:

$$\begin{aligned}
 \frac{d}{dt} \int_{dv} \rho dv = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \left[ \int_{v(t+\Delta t)} \rho(\mathbf{x}, t + \Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t + \Delta t) dv \right] \right. \\
 \left. + \left[ \int_{v(t)} \rho(\mathbf{x}, t + \Delta t) dv - \int_{v(t)} \rho(\mathbf{x}, t) dv \right] \right\} \quad (3.1.19)
 \end{aligned}$$

The terms in the second square bracket correspond to holding the volume  $v$  fixed and evidently equals the local rate of change:

$$\frac{d}{dt} \int_{dv} \rho dv = \int_v \frac{\partial \rho}{\partial t} dv + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{v(t+\Delta t)-v(t)} \rho(\mathbf{x}, t + \Delta t) dv \quad (3.1.20)$$

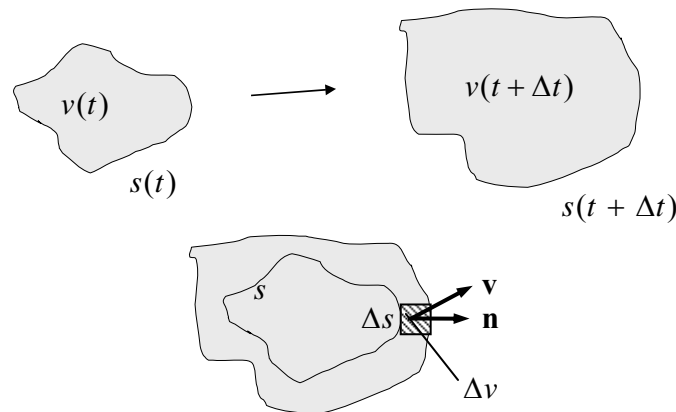
The region  $v(t + \Delta t) - v(t)$  is swept out in time  $\Delta t$ . Superimposing the volumes  $v(t)$  and  $v(t + \Delta t)$ , Fig. 3.1.5, it can be seen that a small element  $\Delta v$  of  $v(t + \Delta t) - v(t)$  is given by (see the example associated with Fig. 1.7.7)

$$\Delta v = \Delta t \mathbf{v} \cdot \mathbf{n} \Delta s \quad (3.1.21)$$

where  $s$  is the surface. Thus

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{v(t+\Delta t)-v(t)} \rho(\mathbf{x}, t + \Delta t) dv = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_s \Delta t \rho(\mathbf{x}, t + \Delta t) \mathbf{v} \cdot \mathbf{n} ds = \int_s \rho(\mathbf{x}, t) \mathbf{v} \cdot \mathbf{n} ds \quad (3.1.22)$$

and 3.1.15 is again obtained, from which the continuity equation results from use of the divergence theorem.



**Figure 3.1.5: Evaluation of Eqn. 3.1.22**

## 2. Derivation by Converting to Mass Elements

This derivation requires the kinematic relation for the material time derivative of a volume element, 2.5.23:  $d(dv)/dt = \text{div} \mathbf{v} dv$ . One has

$$\frac{dm}{dt} = \frac{d}{dt} \int_v \rho(\mathbf{x}, t) dv = \int_v \frac{d}{dt} (\rho dv) = \int_v \left( \dot{\rho} dv + \rho \dot{dv} \right) = \int_v (\dot{\rho} + \text{div} \mathbf{v} \rho) dv \equiv 0 \quad (3.1.23)$$

The continuity equation then follows, since this must hold for any arbitrary region of the volume  $v$ .

### Derivation of the Continuity Equation using a Control Volume (Local Form)

The continuity equation can also be derived using a differential control volume element. This calculation is similar to that given in §1.6.6, with the velocity  $\mathbf{v}$  replaced by  $\rho \mathbf{v}$ .

## 3.1.5 The Continuity Equation (Material Form)

From 3.1.9, and using 2.2.53,  $dv = JdV$ ,

$$\int_v [\rho_0(\mathbf{X}) - \rho(\chi(\mathbf{X}, t), t) J(\mathbf{X}, t)] dV = 0 \quad (3.1.24)$$

Since  $V$  is an arbitrary region, the integrand must vanish everywhere, so that

$$\boxed{\rho_0(\mathbf{X}) = \rho(\chi(\mathbf{X}, t), t) J(\mathbf{X}, t)} \quad \text{Continuity Equation (Material Form)} \quad (3.1.25)$$

This is known as the continuity (mass) equation in the material description. Since  $\dot{\rho}_0 = 0$ , the rate form of this equation is simply

$$\frac{d}{dt}(\rho J) = 0 \quad (3.1.26)$$

The material form of the continuity equation,  $\rho_0 = \rho J$ , is an algebraic equation, unlike the partial differential equation in the spatial form. However, the two must be equivalent, and indeed the spatial form can be derived directly from this material form: using 2.5.20,  $dJ/dt = J \operatorname{div} \mathbf{v}$ ,

$$\begin{aligned} \frac{d}{dt}(\rho J) &= \dot{\rho} J + \rho \dot{J} \\ &= J(\dot{\rho} + \rho \operatorname{div} \mathbf{v}) \end{aligned} \quad (3.1.27)$$

This is zero, and  $J > 0$ , and the spatial continuity equation follows.

### Example (of Conservation of Mass)

Consider a bar of material of length  $l_0$ , with density in the undeformed configuration  $\rho_0$  and spatial mass density  $\rho(x, t)$ , undergoing the 1-D motion  $\mathbf{X} = \mathbf{x}/(1 + At)$ ,  $\mathbf{x} = \mathbf{X} + At\mathbf{X}$ . The volume ratio (taking unit cross-sectional area) is  $J = 1 + At$ . The continuity equation in the material form 3.1.25 specifies that

$$\rho_0 = \rho(1 + At)$$

Suppose now that

$$\rho_0(\mathbf{X}) = \frac{2m}{l_0^2} \mathbf{X}$$

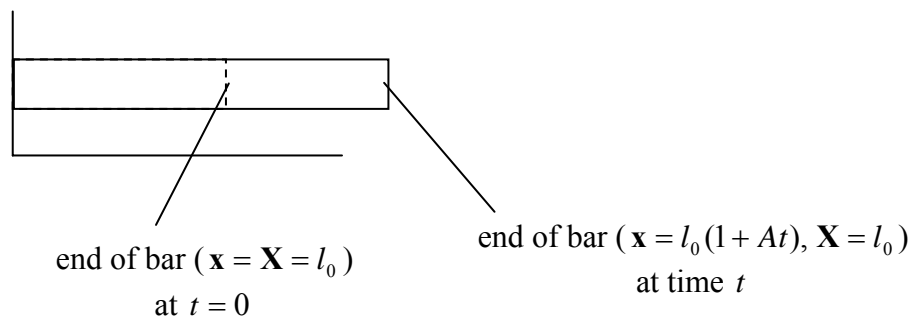
so that the total mass of the bar is  $\int_0^{l_0} \rho_0(\mathbf{X}) d\mathbf{X} = m$ . It follows that the spatial mass density is

$$\rho = \frac{\rho_0}{(1 + At)} = \frac{2m}{l_0^2} \frac{\mathbf{X}}{1 + At} = \frac{2m}{l_0^2} \frac{\mathbf{x}}{(1 + At)^2}$$

Evaluating the total mass of the bar at time  $t$  leads to

$$\int_0^{l_0(1+At)} \rho(\mathbf{x}, t) d\mathbf{x} = \frac{2m}{l_0^2} \frac{1}{(1 + At)^2} \int_0^{l_0(1+At)} \mathbf{x} d\mathbf{x}$$

which is again  $m$ , as required.



**Figure 3.1.6: a stretching bar**

The density could have been derived from the equation of continuity in the spatial form: since the velocity is

$$\mathbf{V}(\mathbf{X}, t) = \frac{d\mathbf{x}(\mathbf{X}, t)}{dt} = A\mathbf{X}, \quad \mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t) = \frac{A\mathbf{x}}{1 + At}$$

one has

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial \mathbf{x}} + \rho \frac{\partial v}{\partial \mathbf{x}} = \frac{\partial \rho}{\partial t} + \frac{A\mathbf{x}}{1 + At} \frac{\partial \rho}{\partial \mathbf{x}} + \rho \frac{A}{1 + At} = 0$$

Without attempting to solve this first order partial differential equation, it can be seen by substitution that the value for  $\rho$  obtained previously satisfies the equation. ■

### 3.1.6 Material Derivatives of Integrals

#### Reynold's Transport Theorem

In the above, the material derivative of the total mass carried by a control mass,

$$\frac{d}{dt} \int_v \rho(\mathbf{x}, t) dv,$$

was considered. It is quite often that one needs to evaluate material time derivatives of similar volume (and line and surface) integrals, involving other properties, for example momentum or energy. Thus, suppose that  $\mathbf{A}(\mathbf{x}, t)$  is the distribution of some property (per unit volume) throughout a volume  $v$  ( $\mathbf{A}$  is taken to be a second order tensor, but what follows applies also to vectors and scalars). Then the rate of change of the total amount of the property carried by the mass system is



$$\frac{d}{dt} \int_v \mathbf{A}(\mathbf{x}, t) dv$$

Again, this integral can be evaluated in a number of ways. For example, one could evaluate it using the formal definition of the material derivative, as done above for  $\mathbf{A} = \rho$ . Alternatively, one can evaluate it using the relation 2.5.23,  $d(dv)/dt = \text{div} \mathbf{v} dv$ , through

$$\frac{d}{dt} \int_v \mathbf{A}(\mathbf{x}, t) dv = \int_v \frac{d}{dt} [\mathbf{A}(\mathbf{x}, t) dv] = \int_v \left[ \dot{\mathbf{A}} dv + \mathbf{A} \frac{d}{dt} dv \right] = \int_v \left[ \dot{\mathbf{A}} + \text{div} \mathbf{v} \mathbf{A} \right] dv \quad (3.1.28)$$

Thus one arrives at **Reynold's transport theorem**

$$\frac{d}{dt} \int_v \mathbf{A}(\mathbf{x}, t) dv = \begin{cases} \int_v \left[ \frac{d\mathbf{A}}{dt} + \text{div} \mathbf{v} \mathbf{A} \right] dv & \int_v \left[ \frac{dA_{ij}}{dt} + \frac{\partial v_k}{\partial x_k} A_{ij} \right] dv \\ \int_v \left[ \frac{\partial \mathbf{A}}{\partial t} + \text{grad} \mathbf{A} \cdot \mathbf{v} + \text{div} \mathbf{v} \mathbf{A} \right] dv & \int_v \left[ \frac{\partial A_{ij}}{\partial t} + \frac{\partial A_{ij}}{\partial x_k} v_k + \frac{\partial v_k}{\partial x_k} A_{ij} \right] dv \\ \int_v \left[ \frac{\partial \mathbf{A}}{\partial t} + \text{div}(\mathbf{A} \otimes \mathbf{v}) \right] dv & \int_v \left[ \frac{\partial A_{ij}}{\partial t} + \frac{\partial (A_{ij} v_k)}{\partial x_k} \right] dv \\ \int_v \frac{\partial \mathbf{A}}{\partial t} dv + \int_s \mathbf{A}(\mathbf{v} \cdot \mathbf{n}) ds & \int_v \frac{\partial A_{ij}}{\partial t} dv + \int_s A_{ij} v_k n_k ds \end{cases}$$

$$\text{Reynold's Transport Theorem} \quad (3.1.29)$$

The index notation is shown for the case when  $\mathbf{A}$  is a second order tensor. In the last of these forms<sup>2</sup> (obtained by application of the divergence theorem), the first term represents the amount (of  $\mathbf{A}$ ) created within the volume  $v$  whereas the second term (the flux term) represents the (volume) rate of flow of the property through the surface. In the last three versions, Reynold's transport theorem gives the material derivative of the moving control mass in terms of the derivative of the instantaneous fixed volume in space (the first term).

Of course when  $\mathbf{A} = \rho$ , the continuity equation is recovered.

Another way to derive this result is to first convert to the reference configuration, so that integration and differentiation commute (since  $dV$  is independent of time):

$$\begin{aligned} \frac{d}{dt} \int_v \mathbf{A}(\mathbf{x}, t) dv &= \frac{d}{dt} \int_v \mathbf{A}(\mathbf{X}, t) J dV = \int_v \frac{d}{dt} (\mathbf{A}(\mathbf{X}, t) J) dV \\ &= \int_v (\dot{\mathbf{A}} J + \mathbf{A} \dot{J}) dV = \int_v (\dot{\mathbf{A}} + \text{div} \mathbf{v} \mathbf{A}) J dV \\ &= \int_v (\dot{\mathbf{A}}(\mathbf{x}, t) + \text{div} \mathbf{v} \mathbf{A}(\mathbf{x}, t)) dv \end{aligned} \quad (3.1.30)$$

<sup>2</sup> also known as the **Leibniz formula**

### Reynold's Transport Theorem for Specific Properties

A property that is given per unit mass is called a **specific property**. For example, specific heat is the heat per unit mass. Consider then a property  $\mathbf{B}$ , a scalar, vector or tensor, which is defined per unit mass through a volume. Then the rate of change of the total amount of the property carried by the mass system is simply

$$\frac{d}{dt} \int_v \rho \mathbf{B}(\mathbf{x}, t) dv = \int_v \frac{d}{dt} [\mathbf{B} \rho dv] = \int_v \frac{d}{dt} [\mathbf{B} dm] = \int_v \frac{d\mathbf{B}}{dt} dm = \int_v \rho \frac{d\mathbf{B}}{dt} dv \quad (3.1.31)$$

### Material Derivatives of Line and Surface Integrals

Material derivatives of line and surface integrals can also be evaluated. From 2.5.8,  $d(d\mathbf{x})/dt = \mathbf{l}d\mathbf{x}$ ,

$$\frac{d}{dt} \int \mathbf{A}(\mathbf{x}, t) d\mathbf{x} = \int [\dot{\mathbf{A}} + \mathbf{A}\mathbf{l}] d\mathbf{x} \quad (3.1.32)$$

and, using 2.5.22,  $d(\hat{\mathbf{n}}ds)/dt = (\text{div}\mathbf{v} - \mathbf{l}^T)\hat{\mathbf{n}}ds$ ,

$$\frac{d}{dt} \int_s \mathbf{A}(\mathbf{x}, t) \hat{\mathbf{n}} ds = \int_s [\dot{\mathbf{A}} + \mathbf{A}(\text{div}\mathbf{v} - \mathbf{l}^T)] \hat{\mathbf{n}} ds \quad (3.1.33)$$

### 3.1.7 Problems

1. A motion is given by the equations

$$x_1 = X_1 + 3X_2t, \quad x_2 = -X_1t^2 + X_2(t+1), \quad x_3 = X_3$$

- Calculate the spatial mass density  $\rho$  in terms of the density  $\rho_0$
- Derive a first order ordinary differential equation for the density  $\rho$  (in terms of  $\mathbf{x}$  and  $t$  only) assuming that it is independent of position  $\mathbf{x}$