2.10 Convected Coordinates

An introduction to curvilinear coordinate was given in section 1.16, which serves as an introduction to this section. As mentioned there, the formulation of almost all mechanics problems, and their numerical implementation and solution, can be achieved using a description of the problem in terms of Cartesian coordinates. However, use of curvilinear coordinates allows for a deeper insight into a number of important concepts and aspects of, in particular, large strain mechanics problems. These include the notions of the Push Forward operation, Lie derivatives and objective rates.

As will become clear, note that all the tensor relations expressed in symbolic notation already discussed, such as $\mathbf{U} = \sqrt{\mathbf{C}}$, $\mathbf{F}\hat{\mathbf{N}}_i = \lambda_i \mathbf{n}_i$, $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$, etc., are independent of coordinate system, and hold also for the convected coordinates discussed here.

2.10.1 Convected Coordinates

In the Cartesian system, orthogonal coordinates X^i, x^i were used. Here, introduce the curvilinear coordinates Θ^i. The material coordinates can then be written as

$$\mathbf{X} = \mathbf{X}(\Theta^1, \Theta^2, \Theta^3) \tag{2.10.1}$$

so $\mathbf{X} = X^i \mathbf{E}_i$ and

$$d\mathbf{X} = dX^i \mathbf{E}_i = d\Theta^i \mathbf{G}_i, \tag{2.10.2}$$

where $\mathbf{G}_i$ are the covariant base vectors in the reference configuration, with corresponding contravariant base vectors $\mathbf{G}^i$, Fig. 2.10.1, with

$$\mathbf{G}^i \cdot \mathbf{G}_j = \delta^i_j \tag{2.10.3}$$

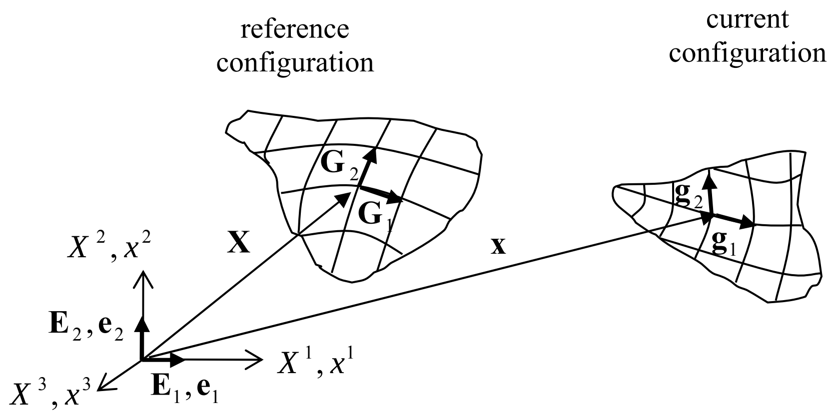

Figure 2.10.1: Curvilinear Coordinates

The coordinate curves form a net in the undeformed configuration (over the surfaces of constant Θ^i). One says that the curvilinear coordinates are **convected** or **embedded**, that is, the coordinate curves are attached to material particles and deform with the body, so that each material particle *has the same values* of the coordinates Θ^i in both the reference and current configurations. The covariant base vectors are tangent the coordinate curves.

In the current configuration, the spatial coordinates can be expressed in terms of a new, "current", set of curvilinear coordinates

$$\mathbf{x} = \mathbf{x}(\Theta^1, \Theta^2, \Theta^3, t), \tag{2.10.4}$$

with corresponding covariant base vectors $\mathbf{g}_i$ and contravariant base vectors $\mathbf{g}^i$, with

$$d\mathbf{x} = dx^i \mathbf{e}_i = d\Theta^i \mathbf{g}_i \quad , \tag{2.10.5}$$

As the material deforms, the covariant base vectors $\mathbf{g}_i$ deform with the body, being "attached" to the body. However, note that the contravariant base vectors $\mathbf{g}^i$ are not as such attached; they have to be re-evaluated at each step of the deformation anew, so as to ensure that the relevant relations, e.g. $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, are always satisfied.

Example 1

Consider a pure shear deformation, where a square deforms into a parallelogram, as illustrated in Fig. 2.10.2. In this scenario, a unit vector $\mathbf{E}_2$ in the "square" gets mapped to a vector $\mathbf{g}_2$ in the parallelogram[1]. The magnitude of $\mathbf{g}_2$ is $1/\sin\alpha$.

[1] This differs from the example worked through in section 1.16; there, the vector $\mathbf{g}_2$ maintained unit magnitude.

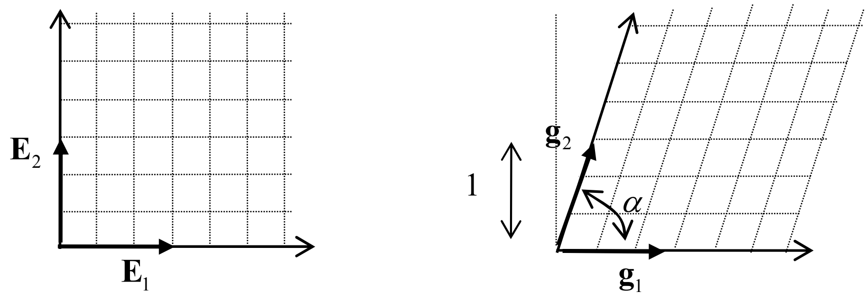

Figure 2.10.2: A pure shear deformation

Consider now a parallelogram (initial condition) deforming into a new parallelogram (the current configuration), as shown in Fig. 2.10.3.

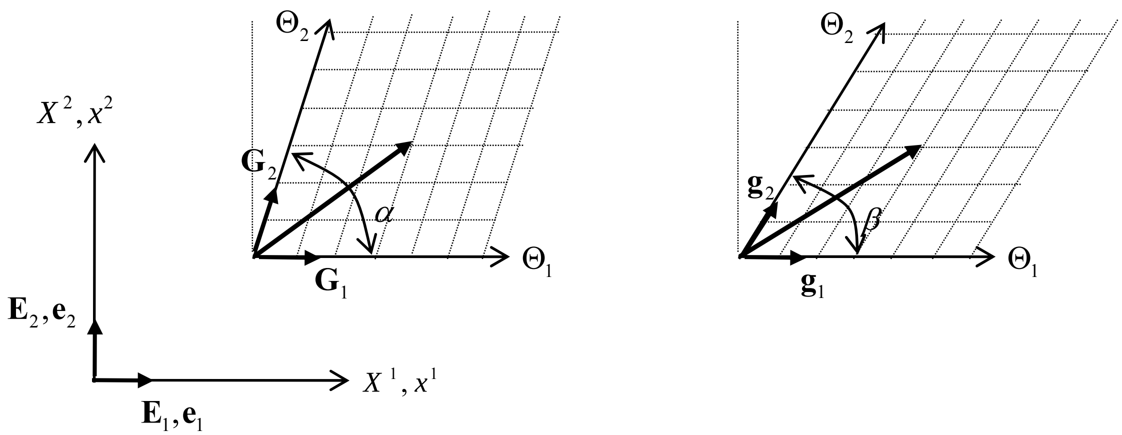

Figure 2.10.3: A pure shear deformation of one parallelogram into another

Keeping in mind that the vector $\mathbf{g}_2$ will be of magnitude $1/\sin\alpha$, the transformation equations 2.10.1 for the configurations shown in Fig. 2.10.3 are[2]

$$
\begin{gathered}
\Theta^1 = X^1 - \frac{1}{\tan\alpha} X^2, \quad \Theta^2 = X^2, \quad \Theta^3 = X^3 \\
X^1 = \Theta^1 + \frac{1}{\tan\alpha}\Theta^2, \quad X^2 = \Theta^2, \quad X^3 = \Theta^3 \\
\\
\Theta^1 = x^1 - \frac{1}{\tan\beta} x^2, \quad \Theta^2 = x^2, \quad \Theta^3 = x^3 \\
x^1 = \Theta^1 + \frac{1}{\tan\beta}\Theta^2, \quad x^2 = \Theta^2, \quad x^3 = \Theta^3
\end{gathered}
\tag{2.10.6}
$$

[2] Constants have been omitted from these expressions (which represent the translation of the "parallelogram origin" from the Cartesian origin).

Following on from §1.16, Eqns. 1.16.19, the covariant base vectors are:

$$\begin{aligned}
\mathbf{G}_i &= \frac{\partial X^m}{\partial \Theta^i}\mathbf{E}_m, \qquad \mathbf{G}_1 = \mathbf{E}_1, \quad \mathbf{G}_2 = \frac{1}{\tan\alpha}\mathbf{E}_1 + \mathbf{E}_2, \quad \mathbf{G}_3 = \mathbf{E}_3 \\
\mathbf{g}_i &= \frac{\partial X^m}{\partial \Theta^i}\mathbf{e}_m, \qquad \mathbf{g}_1 = \mathbf{e}_1, \quad \mathbf{g}_2 = \frac{1}{\tan\beta}\mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{g}_3 = \mathbf{e}_3
\end{aligned} \tag{2.10.7}$$

and the inverse expressions

$$\begin{aligned}
\mathbf{E}_1 &= \mathbf{G}_1, \qquad \mathbf{E}_2 = -\frac{1}{\tan\alpha}\mathbf{G}_1 + \mathbf{G}_2, \qquad \mathbf{E}_3 = \mathbf{G}_3 \\
\mathbf{e}_1 &= \mathbf{g}_1, \qquad \mathbf{e}_2 = -\frac{1}{\tan\beta}\mathbf{g}_1 + \mathbf{g}_2, \qquad \mathbf{e}_3 = \mathbf{g}_3
\end{aligned} \tag{2.10.8}$$

Line elements in the configurations can now be expressed as

$$\begin{aligned}
d\mathbf{X} &= dX^i\mathbf{E}_i = \frac{d\mathbf{X}}{\partial \Theta^i}d\Theta^i = d\Theta^i\mathbf{G}_i \\
d\mathbf{x} &= dx^i\mathbf{e}_i = \frac{d\mathbf{x}}{\partial \Theta^i}d\Theta^i = d\Theta^i\mathbf{g}_i
\end{aligned} \tag{2.10.9}$$

The scale factors, i.e. the magnitudes of the covariant base vectors, are (see Eqns. 1.16.36)

$$\begin{aligned}
H_1 &= \left|\mathbf{G}_1\right| = 1, \quad H_2 = \left|\mathbf{G}_2\right| = \frac{1}{\sin\alpha} \\
h_1 &= \left|\mathbf{g}_1\right| = 1, \quad h_2 = \left|\mathbf{g}_2\right| = \frac{1}{\sin\beta}
\end{aligned} \tag{2.10.10}$$

The contravariant base vectors are (see Eqn. 1.16.23)

$$\begin{aligned}
\mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^m}\mathbf{E}_m, \qquad \mathbf{G}^1 = \mathbf{E}_1 - \frac{1}{\tan\alpha}\mathbf{E}_2, \quad \mathbf{G}^2 = \mathbf{E}_2, \quad \mathbf{G}^3 = \mathbf{E}_3 \\
\mathbf{g}^i &= \frac{\partial \Theta^i}{\partial X^m}\mathbf{e}_m, \qquad \mathbf{g}^1 = \mathbf{e}_1 - \frac{1}{\tan\beta}\mathbf{e}_2, \quad \mathbf{g}^2 = \mathbf{e}_2, \quad \mathbf{g}^3 = \mathbf{e}_3
\end{aligned} \tag{2.10.11}$$

and the inverse expressions

$$\mathbf{E}_1 = \mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2, \qquad \mathbf{E}_2 = \mathbf{G}^2, \qquad \mathbf{E}_3 = \mathbf{G}^3$$
$$\mathbf{e}_1 = \mathbf{g}^1 + \frac{1}{\tan\beta}\mathbf{g}^2, \qquad \mathbf{e}_2 = \mathbf{g}^2, \qquad \mathbf{e}_3 = \mathbf{g}^3 \tag{2.10.12}$$

The magnitudes of the contravariant base vectors, are

$$H^1 = \left|\mathbf{G}^1\right| = \frac{1}{\sin\alpha}, \quad H^2 = \left|\mathbf{G}^2\right| = 1$$
$$h^1 = \left|\mathbf{g}^1\right| = \frac{1}{\sin\beta}, \quad h^2 = \left|\mathbf{g}^2\right| = 1 \tag{2.10.13}$$

The metric coefficients are (see Eqns. 1.16.27)

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j = \begin{bmatrix} 1 & \frac{1}{\tan\alpha} & 0 \\ \frac{1}{\tan\alpha} & \frac{1}{\sin^2\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G^{ij} = \mathbf{G}^i \cdot \mathbf{G}^j = \begin{bmatrix} \frac{1}{\sin^2\alpha} & -\frac{1}{\tan\alpha} & 0 \\ -\frac{1}{\tan\alpha} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \begin{bmatrix} 1 & \frac{1}{\tan\beta} & 0 \\ \frac{1}{\tan\beta} & \frac{1}{\sin^2\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \begin{bmatrix} \frac{1}{\sin^2\beta} & -\frac{1}{\tan\beta} & 0 \\ -\frac{1}{\tan\beta} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10.14}$$

The transformation determinants are (consistent with zero volume change), from Eqns. 1.16.32-34,

$$G = \det\left[G_{ij}\right] = \frac{1}{\det\left[G^{ij}\right]} = \left(\det\left[\frac{\partial X^i}{\partial \Theta^j}\right]\right)^2 = J_G^2 = 1$$
$$g = \det\left[g_{ij}\right] = \frac{1}{\det\left[g^{ij}\right]} = \left(\det\left[\frac{\partial x^i}{\partial \Theta^j}\right]\right)^2 = J_g^2 = 1 \tag{2.10.15}$$

■

Example 2

Consider a motion whereby a cube of material, with sides of length L_0, is transformed into a cylinder of radius R and height H, Fig. 2.10.4.

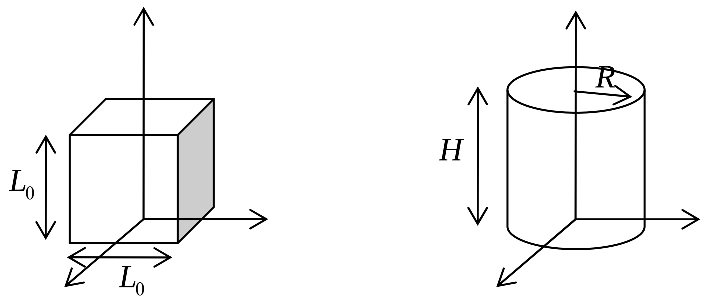

Figure 2.10.4: a cube deformed into a cylinder

A plane view of one quarter of the cube and cylinder are shown in Fig. 2.10.5.

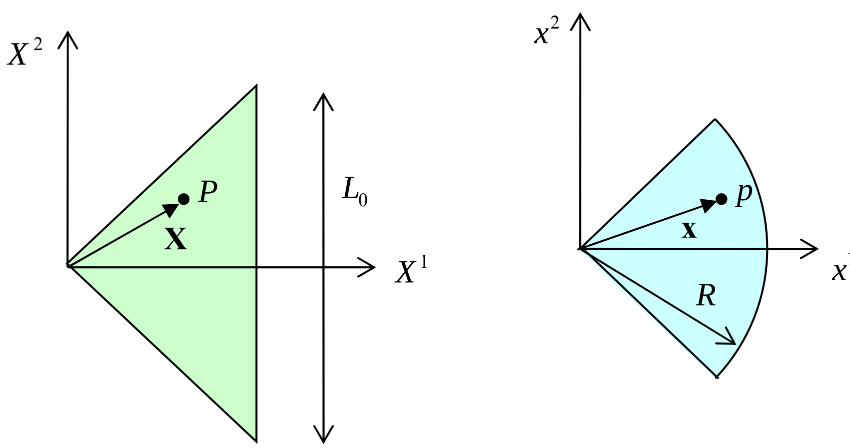

Figure 2.10.5: a cube deformed into a cylinder

The motion and inverse motion are given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \qquad \begin{aligned} x^1 &= \frac{2R}{L_0}\frac{\left(X^1\right)^2}{\sqrt{\left(X^1\right)^2 + \left(X^2\right)^2}} \\ x^2 &= \frac{2R}{L_0}\frac{X^1 X^2}{\sqrt{\left(X^1\right)^2 + \left(X^2\right)^2}} \\ x^3 &= \frac{H}{L_0}X^3 \end{aligned} \qquad (\text{basis: } \mathbf{e}_i) \tag{2.10.16}$$

and

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}), \qquad \begin{aligned} X^1 &= \frac{L_0}{2R}\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} \\ X^2 &= \frac{L_0}{2R}\frac{x^2}{x^1}\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} \\ X^3 &= \frac{L_0}{H}x^3 \end{aligned} \qquad (\text{basis: } \mathbf{E}_i) \tag{2.10.17}$$

Introducing a set of convected coordinates, Fig. 2.10.6, the material and spatial coordinates are

$$\mathbf{X} = \mathbf{X}(\Theta^1, \Theta^2, \Theta^3), \qquad \begin{aligned} X^1 &= \left(\frac{L_0}{2R}\right)\Theta^1 \\ X^2 &= \left(\frac{L_0}{2R}\right)\Theta^1 \tan\Theta^2 \\ X^3 &= \frac{L_0}{H}\Theta^3 \end{aligned} \tag{2.10.18}$$

and (these are simply cylindrical coordinates)

$$\mathbf{x} = \mathbf{x}(\Theta^1, \Theta^2, \Theta^3), \qquad \begin{aligned} x^1 &= \Theta^1 \cos\Theta^2 \\ x^2 &= \Theta^1 \sin\Theta^2 \\ x^3 &= \Theta^3 \end{aligned} \tag{2.10.19}$$

A typical material particle (denoted by p) is shown in Fig. 2.10.6. Note that the position vectors for p have the same Θ^i values, since they represent the same material particle.

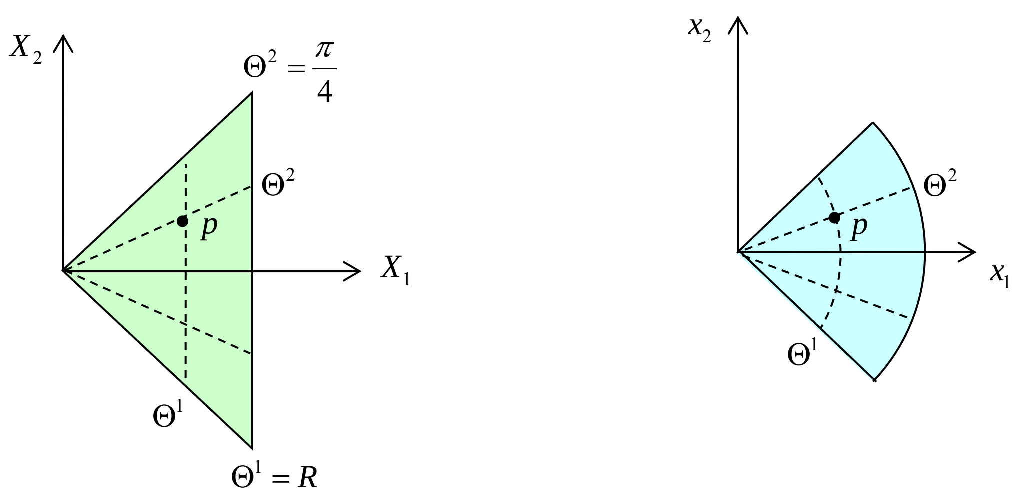

Figure 2.10.6: curvilinear coordinate curves

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2.10.2 The Deformation Gradient

With convected curvilinear coordinates, the deformation gradient is

$$\begin{aligned}\mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i \\ &= \mathbf{g}_1 \otimes \mathbf{G}^1 + \mathbf{g}_2 \otimes \mathbf{G}^2 + \mathbf{g}_3 \otimes \mathbf{G}^3 , \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\mathbf{g}_i \otimes \mathbf{G}^j \right)\end{aligned} \tag{2.10.20}$$

The deformation gradient operates on a material vector (with contravariant components) $\mathbf{V} = V^i \mathbf{G}_i$, resulting in a spatial tensor $\mathbf{v} = v^i \mathbf{g}_i$ (with the same components $V = v^i$), for example,

$$\mathbf{F} d\mathbf{X} = \left(\mathbf{g}_i \otimes \mathbf{G}^i \right) d\Theta^j \mathbf{G}_j = d\Theta^i \mathbf{g}_i = d\mathbf{x} \tag{2.10.21}$$

To emphasise the point, line elements mapped between the configurations have the same coordinates Θ^i: a line element $d\Theta^1 \mathbf{G}_1 + d\Theta^2 \mathbf{G}_2 + d\Theta^3 \mathbf{G}_3$ gets mapped to

$$\left(\mathbf{g}_1 \otimes \mathbf{G}^1 + \mathbf{g}_2 \otimes \mathbf{G}^2 + \mathbf{g}_3 \otimes \mathbf{G}^3 \right) \left(d\Theta^1 \mathbf{G}_1 + d\Theta^2 \mathbf{G}_2 + d\Theta^3 \mathbf{G}_3 \right) = d\Theta^1 \mathbf{g}_1 + d\Theta^2 \mathbf{g}_2 + d\Theta^3 \mathbf{g}_3 \tag{2.10.22}$$

This shows also that line elements tangent to the coordinate curves are mapped to new elements tangent to the new coordinate curves; the covariant base vectors $\mathbf{G}_i$ are a field of tangent vectors which get mapped to the new field of tangent vectors $\mathbf{g}_i$, as illustrated in Fig. 2.10.7.

Figure 2.10.7: Vectors tangent to coordinate curves

The deformation gradient **F**, the transpose $\mathbf{F}^{\mathrm{T}}$ and the inverses $\mathbf{F}^{-1}, \mathbf{F}^{-\mathrm{T}}$, map the base vectors in one configuration onto the base vectors in the other configuration (that the $\mathbf{F}^{-1}$ and $\mathbf{F}^{-\mathrm{T}}$ in this equation are indeed the inverses of $\mathbf{F}$ and $\mathbf{F}^{\mathrm{T}}$ follows from 1.16.63):

$$\boxed{\begin{aligned}\mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i \\ \mathbf{F}^{-1} &= \mathbf{G}_i \otimes \mathbf{g}^i \\ \mathbf{F}^{-\mathrm{T}} &= \mathbf{g}^i \otimes \mathbf{G}_i \\ \mathbf{F}^{\mathrm{T}} &= \mathbf{G}^i \otimes \mathbf{g}_i\end{aligned}} \quad \rightarrow \quad \boxed{\begin{aligned}\mathbf{F}\mathbf{G}_i &= \mathbf{g}_i \\ \mathbf{F}^{-1}\mathbf{g}_i &= \mathbf{G}_i \\ \mathbf{F}^{-\mathrm{T}}\mathbf{G}^i &= \mathbf{g}^i \\ \mathbf{F}^{\mathrm{T}}\mathbf{g}^i &= \mathbf{G}^i\end{aligned}} \qquad \textbf{Deformation Gradient} \qquad (2.10.23)$$

Thus the tensors $\mathbf{F}$ and $\mathbf{F}^{-1}$ map the covariant base vectors into each other, whereas the tensors $\mathbf{F}^{-\mathrm{T}}$ and $\mathbf{F}^{\mathrm{T}}$ map the contravariant base vectors into each other, as illustrated in Fig. 2.10.8.

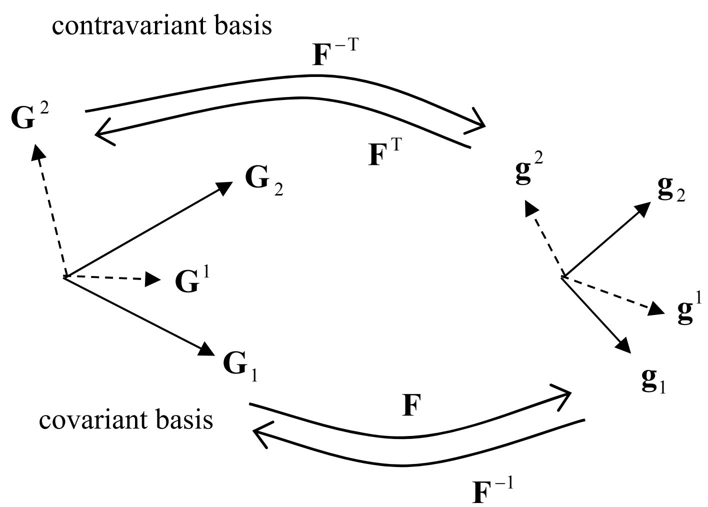

Figure 2.10.8: the deformation gradient, its transpose and the inverses

It was mentioned above how the deformation gradient maps base vectors tangential to the coordinate curves into new vectors tangential to the coordinate curves in the current configuration. In the same way, contravariant base vectors, which are normal to coordinate surfaces, get mapped to normal vectors in the current configuration. For example, the contravariant vector $\mathbf{G}^1$ is normal to the surface of constant Θ^1, and gets mapped through $\mathbf{F}^{-\mathrm{T}}$ to the new vector $\mathbf{g}^1$, which is normal to the surface of constant Θ^1 in the current configuration.

Example 1 continued

Carrying on Example 1 from above, in Cartesian coordinates, 4 corners of an initial parallelogram (see Fig. 2.10.3) get mapped as follows:

$$\begin{gathered}(0,0)\to(0,0)\\(1,0)\to(1,0)\\(1/\tan\alpha,1)\to(1/\tan\beta,1)\\(1+1/\tan\alpha,1)\to(1+1/\tan\beta,1)\end{gathered}\tag{2.10.24}$$

This corresponds to a deformation gradient with respect to the Cartesian bases:

$$\mathbf{F}=\begin{bmatrix}1&\Pi\\0&1\end{bmatrix}\quad(\mathbf{E}_i\otimes\mathbf{E}_j),(\mathbf{e}_i\otimes\mathbf{e}_j)\tag{2.10.25}$$

where

$$\Pi=\frac{1}{\tan\beta}-\frac{1}{\tan\alpha}\tag{2.10.26}$$

From the earlier work with example 1, the deformation gradient can be re-expressed in terms of different base vectors:

$$\begin{aligned}\mathbf{F}&=(\mathbf{E}_1\otimes\mathbf{E}_1)+\Pi(\mathbf{E}_1\otimes\mathbf{E}_2)+(\mathbf{E}_2\otimes\mathbf{E}_2)\\&=(\mathbf{e}_1\otimes\mathbf{E}_1)+\Pi(\mathbf{e}_1\otimes\mathbf{E}_2)+(\mathbf{e}_2\otimes\mathbf{E}_2)\\&=\mathbf{g}_1\otimes\left(\mathbf{G}^1+\frac{1}{\tan\alpha}\mathbf{G}^2\right)+\Pi(\mathbf{g}_1\otimes\mathbf{G}^2)+\left(-\frac{1}{\tan\beta}\mathbf{g}_1+\mathbf{g}_2\right)\otimes\mathbf{G}^2\\&=\mathbf{g}_i\otimes\mathbf{G}^i=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}(\mathbf{g}_i\otimes\mathbf{G}^j)\end{aligned}\tag{2.10.27}$$

which is Eqn. 2.10.20.

In fact, **F** can be expressed in a multitude of different ways, depending on which base vectors are used. For example, from the above, **F** can also be expressed as

$$\begin{aligned}
\mathbf{F} &= (\mathbf{E}_1 \otimes \mathbf{E}_1) + \Pi(\mathbf{E}_1 \otimes \mathbf{E}_2) + (\mathbf{E}_2 \otimes \mathbf{E}_2) \\
&= \left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) \otimes \left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) + \Pi\left[\left(\mathbf{G}^1 + \frac{1}{\tan\alpha}\mathbf{G}^2\right) \otimes \left(\mathbf{G}^2\right)\right] + \left[\mathbf{G}^2 \otimes \mathbf{G}^2\right] \\
&= \begin{bmatrix} 1 & \dfrac{1}{\tan\beta} & 0 \\ \dfrac{1}{\tan\alpha} & \dfrac{1}{\tan\alpha}\dfrac{1}{\tan\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\mathbf{G}^i \otimes \mathbf{G}^j\right)
\end{aligned} \tag{2.10.28}$$

(This can be verified using Eqn. 2.10.30a below.)

Components of F

The various components of **F** and its inverses and the transposes, with respect to the different bases, are:

$$\begin{aligned}
\mathbf{F} &= F_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = F^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = F_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = F_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= f_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = f^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = f_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = f_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\[1ex]
\mathbf{F}^{-1} &= \left(F^{-1}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{-1}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{-1}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{-1}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{-1}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{-1}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{-1}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{-1}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\[1ex]
\mathbf{F}^{\mathrm{T}} &= \left(F^{\mathrm{T}}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{\mathrm{T}}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{\mathrm{T}}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{\mathrm{T}}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{\mathrm{T}}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{\mathrm{T}}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j \\[1ex]
\mathbf{F}^{-\mathrm{T}} &= \left(F^{-\mathrm{T}}\right)_{ij}\mathbf{G}^i \otimes \mathbf{G}^j = \left(F^{-\mathrm{T}}\right)^{ij}\mathbf{G}_i \otimes \mathbf{G}_j = \left(F^{-\mathrm{T}}\right)_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j = \left(F^{-\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{G}_i \otimes \mathbf{G}^j \\
&= \left(f^{-\mathrm{T}}\right)_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \left(f^{-\mathrm{T}}\right)^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \left(f^{-\mathrm{T}}\right)_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \left(f^{-\mathrm{T}}\right)_{\cdot j}^{i}\mathbf{g}_i \otimes \mathbf{g}^j
\end{aligned} \tag{2.10.29}$$

The components of **F** with respect to the reference bases $\{\mathbf{G}_i\}, \{\mathbf{G}^i\}$ are

$$
\begin{aligned}
F_{ij} &= \mathbf{G}_i \mathbf{F} \mathbf{G}_j = \mathbf{G}_i \cdot \mathbf{g}_j = \frac{\partial X^m}{\partial \Theta^i} \frac{\partial x^m}{\partial \Theta^j} \\
F^{ij} &= \mathbf{G}^i \mathbf{F} \mathbf{G}^j = G^{jk} \mathbf{G}^i \cdot \mathbf{g}_k \\
F_i^{\cdot j} &= \mathbf{G}_i \mathbf{F} \mathbf{G}^j = G^{jk} \mathbf{G}_i \cdot \mathbf{g}_k \\
F^i_{\cdot j} &= \mathbf{G}^i \mathbf{F} \mathbf{G}_j = \mathbf{G}^i \cdot \mathbf{g}_j = \frac{\partial \Theta^i}{\partial X^m} \frac{\partial x^m}{\partial \Theta^j}
\end{aligned}
\qquad (2.10.30)
$$

and similarly for the components with respect to the current bases.

Components of the Base Vectors in different Bases

The base vectors themselves can be expressed alternately:

$$
\begin{aligned}
\mathbf{g}_i = \mathbf{F} \mathbf{G}_i &= F_{mj} \left(\mathbf{G}^m \otimes \mathbf{G}^j \right) \mathbf{G}_i &&= F^{\cdot m}_{\cdot j} \left(\mathbf{G}_m \otimes \mathbf{G}^j \right) \mathbf{G}_i \\
&= F_{mj} \mathbf{G}^m \delta_i^j &&= F^{\cdot m}_{\cdot j} \mathbf{G}_m \delta_i^j \\
&= F_{mi} \mathbf{G}^m &&= F^{\cdot m}_{\cdot i} \mathbf{G}_m
\end{aligned}
\qquad (2.10.31)
$$

showing that some of the components of the deformation gradient can be viewed also as components of the base vectors. Similarly,

$$
\mathbf{G}_i = \mathbf{F}^{-1} \mathbf{g}_i = \left(f^{-1} \right)_{mi} \mathbf{g}^m = \left(f^{-1} \right)^{m}_{\cdot i} \mathbf{g}_m \qquad (2.10.32)
$$

For the contravariant base vectors, one has

$$
\begin{aligned}
\mathbf{g}^i = \mathbf{F}^{-\mathrm{T}} \mathbf{G}^i &= \left(F^{-\mathrm{T}} \right)^{mj} \left(\mathbf{G}_m \otimes \mathbf{G}_j \right) \mathbf{G}^i &&= \left(F^{-\mathrm{T}} \right)_m^{\cdot j} \left(\mathbf{G}^m \otimes \mathbf{G}_j \right) \mathbf{G}^i \\
&= \left(F^{-\mathrm{T}} \right)^{mj} \mathbf{G}_m \delta_j^i &&= \left(F^{-\mathrm{T}} \right)_m^{\cdot j} \mathbf{G}^m \delta_j^i \\
&= \left(F^{-\mathrm{T}} \right)^{mi} \mathbf{G}_m &&= \left(F^{-\mathrm{T}} \right)_m^{\cdot i} \mathbf{G}^m
\end{aligned}
\qquad (2.10.33)
$$

and

$$
\mathbf{G}^i = \mathbf{F}^{\mathrm{T}} \mathbf{g}^i = \left(f^{\mathrm{T}} \right)^{mi} \mathbf{g}_m = \left(f^{\mathrm{T}} \right)_m^{\cdot i} \mathbf{g}^m \qquad (2.10.34)
$$

2.10.3 Reduction to Material and Spatial Coordinates

Material Coordinates

Suppose that the material coordinates X^i with Cartesian basis are used (rather than the convected coordinates with curvilinear basis $\mathbf{G}_i$), Fig. 2.10.9. Then

$$\Theta^i \to X^i, \qquad \begin{aligned} \mathbf{G}_i &= \frac{\partial X^j}{\partial \Theta^i}\mathbf{E}_j = \frac{\partial X^j}{\partial X^i}\mathbf{E}_j = \mathbf{E}_i \\ \mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^j}\mathbf{E}^j = \frac{\partial X^i}{\partial X^j}\mathbf{E}^j = \mathbf{E}^i \end{aligned}, \qquad \begin{aligned} \mathbf{g}_i &= \frac{\partial x^j}{\partial \Theta^i}\mathbf{e}_j = \frac{\partial x^j}{\partial X^i}\mathbf{e}_j \\ \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial x^j}\mathbf{e}^j = \frac{\partial X^i}{\partial x^j}\mathbf{e}^j \end{aligned} \qquad (2.10.35)$$

and

$$\begin{aligned} \mathbf{F} &= \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{g}_i \otimes \mathbf{E}^i = \frac{\partial x^j}{\partial X^i}\mathbf{e}_j \otimes \mathbf{E}^i = \mathrm{Grad}\mathbf{x} \\ \mathbf{F}^{-1} &= \mathbf{G}_i \otimes \mathbf{g}^i = \mathbf{E}_i \otimes \mathbf{g}^i = \frac{\partial X^i}{\partial x^j}\mathbf{E}_i \otimes \mathbf{e}^j = \mathrm{grad}\mathbf{X} \end{aligned} \qquad (2.10.36)$$

which are Eqns. 2.2.2, 2.2.4. Thus $\mathrm{Grad}\mathbf{x}$ is the notation for **F** and $\mathrm{grad}\mathbf{X}$ is the notation for $\mathbf{F}^{-1}$, to be used when the material coordinates X_i are used to describe the deformation.

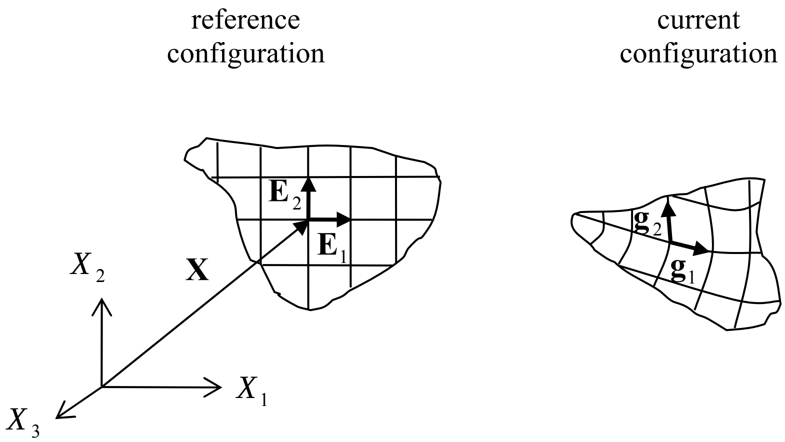

Figure 2.10.9: Material coordinates and deformed basis

Spatial Coordinates

Similarly, when the spatial coordinates x^i are to be used as independent variables, then

$$\Theta^i \to x^i, \qquad \begin{aligned} \mathbf{G}_i &= \frac{\partial X^j}{\partial \Theta^i}\mathbf{E}_j = \frac{\partial X^j}{\partial x^i}\mathbf{E}_j \\ \mathbf{G}^i &= \frac{\partial \Theta^i}{\partial X^j}\mathbf{E}^j = \frac{\partial x^i}{\partial X^j}\mathbf{E}^j \end{aligned}, \qquad \begin{aligned} \mathbf{g}_i &= \frac{\partial x^j}{\partial \Theta^i}\mathbf{e}_j = \frac{\partial x^j}{\partial x^i}\mathbf{e}_j = \mathbf{e}_i \\ \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial x^j}\mathbf{e}^j = \frac{\partial x^i}{\partial x^j}\mathbf{e}^j = \mathbf{e}^i \end{aligned} \qquad (2.10.37)$$

and

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{e}_i \otimes \mathbf{G}^i = \frac{\partial x^i}{\partial X^j}\mathbf{e}_i \otimes \mathbf{E}^j = \mathrm{Grad}\mathbf{x}$$
$$\mathbf{F}^{-1} = \mathbf{G}_i \otimes \mathbf{g}^i = \mathbf{G}_i \otimes \mathbf{e}^i = \frac{\partial X^j}{\partial x^i}\mathbf{E}_j \otimes \mathbf{e}^i = \mathrm{grad}\mathbf{X} \qquad (2.10.38)$$

The descriptions are illustrated in Fig. 2.10.10. Note that the base vectors $\mathbf{G}_i$, $\mathbf{g}_i$ are not the same in each of these cases (curvilinear, material and spatial).

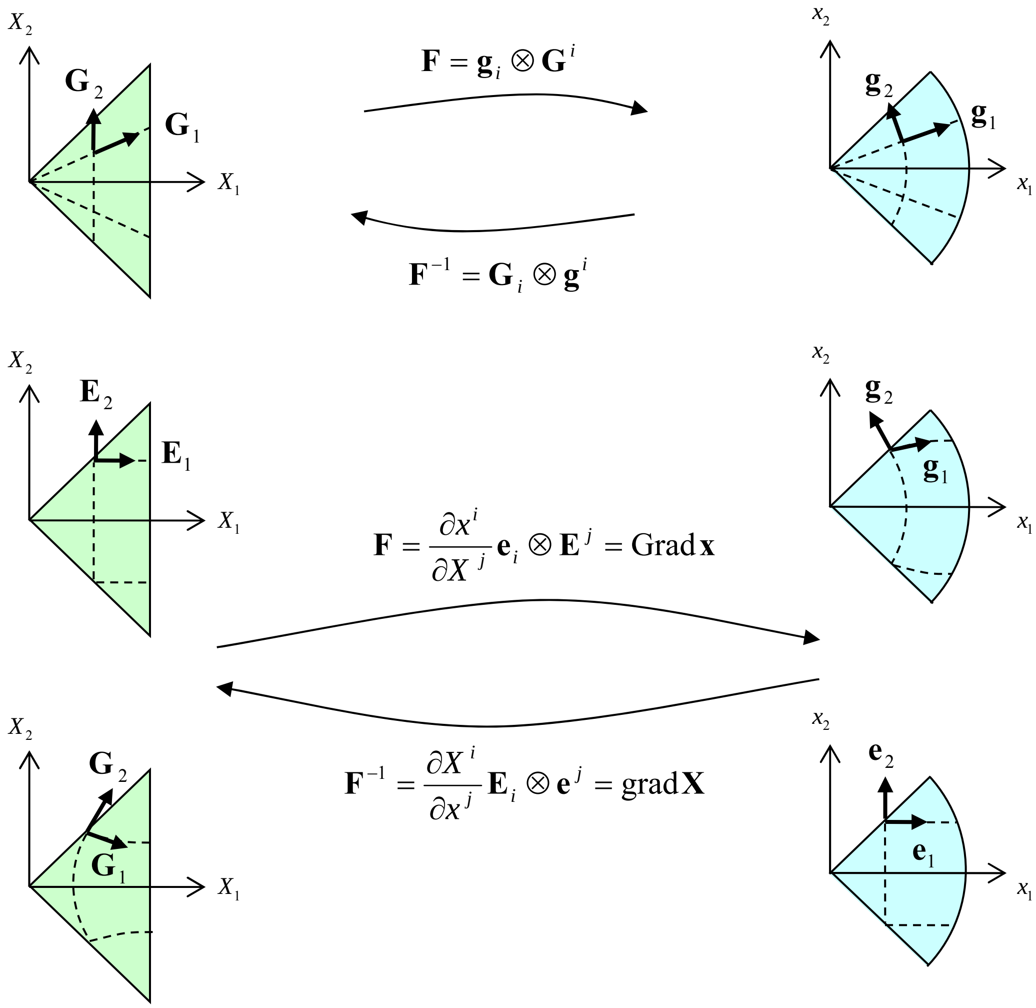

Figure 2.10.10: deformation described using different independent variables

2.10.4 Strain Tensors

The Cauchy-Green tensors

The right Cauchy-Green tensor **C** and the left Cauchy-Green tensor **b** are defined by Eqns. 2.2.10, 2.2.13,

$$\begin{aligned}
\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \quad &= (\mathbf{G}^i \otimes \mathbf{g}_i)(\mathbf{g}_j \otimes \mathbf{G}^j) = g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \equiv C_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \\
\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} &= (\mathbf{G}_i \otimes \mathbf{g}^i)(\mathbf{g}^j \otimes \mathbf{G}_j) = g^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \equiv (C^{-1})^{ij}\mathbf{G}_i \otimes \mathbf{G}_j \\
\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} \quad &= (\mathbf{g}_i \otimes \mathbf{G}^i)(\mathbf{G}^j \otimes \mathbf{g}_j) = G^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \equiv b^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \\
\mathbf{b}^{-1} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} &= (\mathbf{g}^i \otimes \mathbf{G}_i)(\mathbf{G}_j \otimes \mathbf{g}^j) = G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \equiv (b^{-1})_{ij}\mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned} \tag{2.10.39}$$

Thus the covariant components of the right Cauchy-Green tensor are the metric coefficients g_{ij}. This highlights the importance of **C**: the $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ give a clear measure of the deformation occurring. (It is possible to evaluate other components of **C**, e.g. C^{ij}, and also its components with respect to the current basis, but only the components C_{ij} with respect to the reference basis are (normally) used in the analysis.)

The Stretch

Now, analogous to 2.2.9, 2.2.12,

$$\begin{aligned}
ds^2 &= d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X}\mathbf{C}d\mathbf{X} \\
dS^2 &= d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x}\mathbf{b}^{-1}d\mathbf{x}
\end{aligned} \tag{2.10.40}$$

so that the stretches are, analogous to 2.2.17,

$$\begin{aligned}
\lambda^2 &= \frac{ds^2}{dS^2} = \frac{d\mathbf{X}}{|d\mathbf{X}|}\mathbf{C}\frac{d\mathbf{X}}{|d\mathbf{X}|} = d\hat{\mathbf{X}}\mathbf{C}d\hat{\mathbf{X}} \qquad \to d\hat{X}^i C_{ij} d\hat{X}^j \\
\frac{1}{\lambda^2} &= \frac{dS^2}{ds^2} = \frac{d\mathbf{x}}{|d\mathbf{x}|}\mathbf{b}^{-1}\frac{d\mathbf{x}}{|d\mathbf{x}|} = d\hat{\mathbf{x}}\mathbf{b}^{-1}d\hat{\mathbf{x}} \qquad \to d\hat{x}^i (b^{-1})_{ij} d\hat{x}^j
\end{aligned} \tag{2.10.41}$$

The Green-Lagrange and Euler-Almansi Tensors

The Green-Lagrange strain tensor **E** and the Euler-Almansi strain tensor **e** are defined through 2.2.22, 2.2.24,

$$\frac{ds^2 - dS^2}{2} = d\mathbf{X}\frac{1}{2}(\mathbf{C} - \mathbf{I})d\mathbf{X} \equiv d\mathbf{X}\mathbf{E}d\mathbf{X}$$
$$\frac{ds^2 - dS^2}{2} = d\mathbf{x}\frac{1}{2}\left(\mathbf{I} - \mathbf{b}^{-1}\right)d\mathbf{x} \equiv d\mathbf{x}\mathbf{e}d\mathbf{x} \tag{2.10.42}$$

The components of **E** and **e** can be evaluated through (writing $\mathbf{G} \equiv \mathbf{I}$, the identity tensor expressed in terms of the base vectors in the reference configuration, and $\mathbf{g} \equiv \mathbf{I}$, the identity tensor expressed in terms of the base vectors in the current configuration)

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G}) = \frac{1}{2}\left(g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j - G_{ij}\mathbf{G}^i \otimes \mathbf{G}^j\right) = \frac{1}{2}\left(g_{ij} - G_{ij}\right)\mathbf{G}^i \otimes \mathbf{G}^j \equiv E_{ij}\mathbf{G}^i \otimes \mathbf{G}^j$$
$$\mathbf{e} = \frac{1}{2}\left(\mathbf{g} - \mathbf{b}^{-1}\right) = \frac{1}{2}\left(g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j - G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right) = \frac{1}{2}\left(g_{ij} - G_{ij}\right)\mathbf{g}^i \otimes \mathbf{g}^j \equiv e_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \tag{2.10.43}$$

Note that the components of **E** and **e** with respect to their bases are equal, $E_{ij} = e_{ij}$ (although this is not true regarding their other components, e.g. $E^{ij} \neq e^{ij}$).

Example 1 continued

Carrying on Example 1 from above, consider now an example vector

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad (\mathbf{E}_i) \tag{2.10.44}$$

The contravariant and covariant components are

$$\mathbf{V} = \begin{bmatrix} V_x - \dfrac{1}{\tan\alpha}V_y \\ V_y \end{bmatrix} \quad (\mathbf{G}_i), \qquad \mathbf{V} = \begin{bmatrix} V_x \\ \dfrac{1}{\tan\alpha}V_x + V_y \end{bmatrix} \quad (\mathbf{G}^i) \tag{2.10.45}$$

The magnitude of the vector can be calculated through (see Eqn. 1.16.52 and 1.16.49)

$$\begin{aligned} |\mathbf{V}| &= \sqrt{\overset{\mathbf{E}_i}{\mathbf{V}}\cdot\overset{\mathbf{E}_i}{\mathbf{V}}} = \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{\overset{\mathbf{G}_i}{\mathbf{V}}\cdot\overset{\mathbf{G}_i}{\mathbf{V}}} = \sqrt{G_{ij}V^iV^j} = \sqrt{\left(V_x - \frac{V_y}{\tan\alpha}\right)^2 G_{11} + 2\left(V_x - \frac{V_y}{\tan\alpha}\right)V_y G_{12} + V_y^2 G_{22}} \\ &= \sqrt{\overset{\mathbf{G}^i}{\mathbf{V}}\cdot\overset{\mathbf{G}^i}{\mathbf{V}}} = \sqrt{G^{ij}V_iV_j} = \sqrt{V_x^2 G^{11} + 2V_x\left(\frac{V_x}{\tan\alpha} + V_y\right)G^{12} + \left(\frac{V_x}{\tan\alpha} + V_y\right)^2 G^{22}} \end{aligned} \tag{2.10.46}$$

The new vector is obtained from the deformation gradient:

$$
\begin{aligned}
\mathbf{v} = \mathbf{F}\overset{\mathbf{E}_i}{\mathbf{V}} &= \begin{bmatrix} 1 & \Pi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} &&= \begin{bmatrix} V_x + \Pi V_y \\ V_y \end{bmatrix} (\mathbf{e}_i) \\
= \mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_x - \dfrac{V_y}{\tan\alpha} \\ V_y \end{bmatrix} &&= \begin{bmatrix} V_x - \dfrac{1}{\tan\alpha} V_y \\ V_y \end{bmatrix} (\mathbf{g}_i)
\end{aligned}
\tag{2.10.47}
$$

In terms of the contravariant vectors:

$$
\mathbf{v} = v_j g^j = \begin{bmatrix} V_x + \Pi V_y \\ \dfrac{1}{\tan\beta} V_x + \left(1 + \dfrac{1}{\tan\beta}\Pi\right) V_y \end{bmatrix} (\mathbf{g}^i)
\tag{2.10.48}
$$

Note that the contravariant components do not change with the deformation, but the covariant components do in general change with the deformation.

The magnitudes of the vectors before and after deformation are given by the Cauchy-Green strain tensors, whose coefficients are those of the metric tensors (the first of these is the same as 2.10.46)

$$
\begin{aligned}
\mathbf{V}\cdot\mathbf{V} &= \mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}}\cdot\mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = \overset{\mathbf{g}_i}{\mathbf{v}}\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = \overset{\mathbf{g}_i}{\mathbf{v}}\mathbf{b}^{-1}\overset{\mathbf{g}_i}{\mathbf{v}} = v^k \mathbf{g}_k G_{ij}\mathbf{g}^i \otimes \mathbf{g}^j v^l \mathbf{g}_l = G_{ij} v^i v^j \\
\mathbf{v}\cdot\mathbf{v} &= \mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}}\cdot\mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} = \overset{\mathbf{G}_i}{\mathbf{V}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\overset{\mathbf{G}_i}{\mathbf{V}} = \overset{\mathbf{G}_i}{\mathbf{V}}\mathbf{C}\overset{\mathbf{G}_i}{\mathbf{V}} = V^k \mathbf{G}_k g_{ij}\mathbf{G}^i \otimes \mathbf{G}^j V^l \mathbf{G}_l = g_{ij} V^i V^j
\end{aligned}
\tag{2.10.49}
$$

From this, the magnitude of the vector after deformation is

$$
\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{g_{ij}V^iV^j} = \sqrt{\left(V_x^2 + V_y^2\right) + \Pi V_y \left(2V_x + \Pi V_y\right)}
\tag{2.10.50}
$$

2.10.5 Intermediate Configurations

Stretch and Rotation Tensors

The polar decompositions $\mathbf{F} = \mathbf{RU} = \mathbf{vR}$ have been described in §2.2.5. The decompositions are illustrated in Fig. 2.10.11. In the material decomposition, the material is first stretched by $\mathbf{U}$ and then rotated by $\mathbf{R}$. Let the base vectors in the associated intermediate configuration be $\{\hat{\mathbf{g}}_i\}$. Similarly, in the spatial decomposition, the material is first rotated by $\mathbf{R}$ and then stretched by $\mathbf{v}$. Let the base vectors in the associated intermediate configuration in this case be $\{\mathbf{G}_i\}$. Then, analogous to Eqn. 2.10.23, {▲Problem 1}

$$
\begin{aligned}
\mathbf{U} &= \hat{\mathbf{g}}_i \otimes \mathbf{G}^i \\
\mathbf{U}^{-1} &= \mathbf{G}_i \otimes \hat{\mathbf{g}}^i \\
\mathbf{U}^{-\mathrm{T}} &= \hat{\mathbf{g}}^i \otimes \mathbf{G}_i \\
\mathbf{U}^{\mathrm{T}} &= \mathbf{G}^i \otimes \hat{\mathbf{g}}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\mathbf{U}\mathbf{G}_i &= \hat{\mathbf{g}}_i \\
\mathbf{U}^{-1}\hat{\mathbf{g}}_i &= \mathbf{G}_i \\
\mathbf{U}^{-\mathrm{T}}\mathbf{G}^i &= \hat{\mathbf{g}}^i \\
\mathbf{U}^{\mathrm{T}}\hat{\mathbf{g}}^i &= \mathbf{G}^i
\end{aligned}
\tag{2.10.51}
$$

$$
\begin{aligned}
\mathbf{v} &= \mathbf{g}_i \otimes \hat{\mathbf{G}}^i \\
\mathbf{v}^{-1} &= \hat{\mathbf{G}}_i \otimes \mathbf{g}^i \\
\mathbf{v}^{-\mathrm{T}} &= \mathbf{g}^i \otimes \hat{\mathbf{G}}_i \\
\mathbf{v}^{\mathrm{T}} &= \hat{\mathbf{G}}^i \otimes \mathbf{g}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\mathbf{v}\hat{\mathbf{G}}_i &= \mathbf{g}_i \\
\mathbf{v}^{-1}\mathbf{g}_i &= \hat{\mathbf{G}}_i \\
\mathbf{v}^{-\mathrm{T}}\hat{\mathbf{G}}^i &= \mathbf{g}^i \\
\mathbf{v}^{\mathrm{T}}\mathbf{g}^i &= \hat{\mathbf{G}}^i
\end{aligned}
\tag{2.10.52}
$$

$$
\begin{tikzcd}
 & \{\hat{\mathbf{G}}_i\} \arrow[dr, bend left, "\mathbf{v}"] & \\
\{\mathbf{G}_i\} \arrow[ur, bend left, "\mathbf{R}"] \arrow[dr, bend right, "\mathbf{U}"'] & & \{\mathbf{g}_i\} \\
 & \{\hat{\mathbf{g}}_i\} \arrow[ur, bend right, "\mathbf{R}"'] &
\end{tikzcd}
$$

Figure 2.10.11: the material and spatial polar decompositions

Note that $\mathbf{U}$ and $\mathbf{v}$ symmetric, $\mathbf{U} = \mathbf{U}^{\mathrm{T}}$, $\mathbf{v} = \mathbf{v}^{\mathrm{T}}$, so

$$
\begin{aligned}
\mathbf{U} &= \hat{\mathbf{g}}_i \otimes \mathbf{G}^i = \mathbf{G}^i \otimes \hat{\mathbf{g}}_i \\
\mathbf{U}^{-1} &= \mathbf{G}_i \otimes \hat{\mathbf{g}}^i = \hat{\mathbf{g}}^i \otimes \mathbf{G}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
&\mathbf{U}\mathbf{G}_i = \hat{\mathbf{g}}_i, \quad \mathbf{U}\hat{\mathbf{g}}^i = \mathbf{G}^i \\
&\mathbf{U}^{-1}\hat{\mathbf{g}}_i = \mathbf{G}_i, \quad \mathbf{U}^{-1}\mathbf{G}^i = \hat{\mathbf{g}}^i
\end{aligned}
\tag{2.10.53}
$$

$$
\begin{aligned}
\mathbf{v} &= \mathbf{g}_i \otimes \hat{\mathbf{G}}^i = \hat{\mathbf{G}}^i \otimes \mathbf{g}_i \\
\mathbf{v}^{-1} &= \hat{\mathbf{G}}_i \otimes \mathbf{g}^i = \mathbf{g}^i \otimes \hat{\mathbf{G}}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
&\mathbf{v}\hat{\mathbf{G}}_i = \mathbf{g}_i, \quad \mathbf{v}\mathbf{g}^i = \hat{\mathbf{G}}^i \\
&\mathbf{v}^{-1}\mathbf{g}_i = \hat{\mathbf{G}}_i, \quad \mathbf{v}^{-1}\hat{\mathbf{G}}^i = \mathbf{g}^i
\end{aligned}
\tag{2.10.54}
$$

Similarly, for the rotation tensor, with $\mathbf{R}$ orthogonal, $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$,

$$
\begin{aligned}
\mathbf{R} &= \hat{\mathbf{G}}_i \otimes \mathbf{G}^i = \hat{\mathbf{G}}^i \otimes \mathbf{G}_i \\
\mathbf{R}^{\mathrm{T}} &= \mathbf{G}_i \otimes \hat{\mathbf{G}}^i = \mathbf{G}^i \otimes \hat{\mathbf{G}}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
&\mathbf{R}\mathbf{G}_i = \hat{\mathbf{G}}_i, \quad \mathbf{R}\mathbf{G}^i = \hat{\mathbf{G}}^i \\
&\mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}_i = \mathbf{G}_i, \quad \mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}^i = \mathbf{G}^i
\end{aligned}
\tag{2.10.55}
$$

$$
\begin{aligned}
\mathbf{R} &= \mathbf{g}_i \otimes \hat{\mathbf{g}}^i = \mathbf{g}^i \otimes \hat{\mathbf{g}}_i \\
\mathbf{R}^{\mathrm{T}} &= \hat{\mathbf{g}}_i \otimes \mathbf{g}^i = \hat{\mathbf{g}}^i \otimes \mathbf{g}_i
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
&\mathbf{R}\hat{\mathbf{g}}_i = \mathbf{g}_i, \quad \mathbf{R}\hat{\mathbf{g}}^i = \mathbf{g}^i \\
&\mathbf{R}^{\mathrm{T}}\mathbf{g}_i = \hat{\mathbf{g}}_i, \quad \mathbf{R}^{\mathrm{T}}\mathbf{g}^i = \hat{\mathbf{g}}^i
\end{aligned}
\tag{2.10.56}
$$

The above relations can be checked using Eqns. 2.10.23 and $\mathbf{F} = \mathbf{RU}$, $\mathbf{F} = \mathbf{vR}$, $\mathbf{v}^{-1} = \mathbf{RF}^{-1}$, etc.

Various relations between the base vectors can be derived, for example,

$$\begin{aligned}
\hat{\mathbf{G}}_i \cdot \mathbf{g}_j &= (\mathbf{R}\mathbf{G}_i)\cdot(\mathbf{R}\hat{\mathbf{g}}_j) = \mathbf{G}_i\mathbf{R}^{\mathrm{T}}\mathbf{R}\hat{\mathbf{g}}_j &&= \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\hat{\mathbf{G}}^i \cdot \mathbf{g}^j &= \qquad\qquad \cdots &&= \mathbf{G}^i \cdot \hat{\mathbf{g}}^j \\
\hat{\mathbf{G}}^i \cdot \mathbf{g}_j &= \qquad\qquad \cdots &&= \mathbf{G}^i \cdot \hat{\mathbf{g}}_j \\
\hat{\mathbf{G}}_i \cdot \mathbf{g}^j &= \qquad\qquad \cdots &&= \mathbf{G}_i \cdot \hat{\mathbf{g}}^j
\end{aligned} \tag{2.10.57}$$

Deformation Gradient Relationship between Bases

The various base vectors are related above through the stretch and rotation tensors. The intermediate bases are related directly through the deformation gradient. For example, from 2.10.53a, 2.10.55b,

$$\hat{\mathbf{g}}_i = \mathbf{U}\mathbf{G}_i = \mathbf{U}\mathbf{R}^{\mathrm{T}}\hat{\mathbf{G}}_i = \mathbf{F}^{\mathrm{T}}\hat{\mathbf{G}}_i \tag{2.10.58}$$

In the same way,

$$\begin{aligned}
\hat{\mathbf{g}}_i &= \mathbf{F}^{\mathrm{T}}\hat{\mathbf{G}}_i \\
\hat{\mathbf{g}}^i &= \mathbf{F}^{-1}\hat{\mathbf{G}}^i \\
\hat{\mathbf{G}}_i &= \mathbf{F}^{-\mathrm{T}}\hat{\mathbf{g}}_i \\
\hat{\mathbf{G}}^i &= \mathbf{F}\hat{\mathbf{g}}^i
\end{aligned} \tag{2.10.59}$$

Tensor Components

The stretch and rotation tensors can be decomposed along any of the bases. For $\mathbf{U}$ the most natural bases would be $\{\mathbf{G}_i\}$ and $\{\mathbf{G}^i\}$, for example,

$$\begin{aligned}
\mathbf{U} &= U_{ij}\mathbf{G}^i \otimes \mathbf{G}^j, \quad & U_{ij} &= \mathbf{G}_i\mathbf{U}\mathbf{G}_j = \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U} &= U^{ij}\mathbf{G}_i \otimes \mathbf{G}_j, \quad & U^{ij} &= \mathbf{G}^i\mathbf{U}\mathbf{G}^j = G^{im}\mathbf{G}^j \cdot \hat{\mathbf{g}}_m \\
\mathbf{U} &= U^i_{\cdot j}\mathbf{G}_i \otimes \mathbf{G}^j, \quad & U^i_{\cdot j} &= \mathbf{G}^i\mathbf{U}\mathbf{G}_j = \mathbf{G}^i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U} &= U_i^{\cdot j}\mathbf{G}^i \otimes \mathbf{G}_j, \quad & U_i^{\cdot j} &= \mathbf{G}_i\mathbf{U}\mathbf{G}^j = \hat{\mathbf{g}}_i \cdot \mathbf{G}^j
\end{aligned} \tag{2.10.60}$$

with $U_{ij} = U_{ji}, U^{ij} = U^{ji}, U^i_{\cdot j} = U_j^{\cdot i}, U_i^{\cdot j} = U^j_{\cdot i}$. One also has

$$\begin{aligned}
\mathbf{v} &= v_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j, \quad v_{ij} = \hat{\mathbf{G}}_i \mathbf{v} \hat{\mathbf{G}}_j = \hat{\mathbf{G}}_i \cdot \mathbf{g}_j \\
\mathbf{v} &= v^{ij}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}_j, \quad v^{ij} = \hat{\mathbf{G}}^i \mathbf{v} \hat{\mathbf{G}}^j = \hat{G}^{im}\hat{\mathbf{G}}^j \cdot \mathbf{g}_m \\
\mathbf{v} &= v^i_{\cdot j}\hat{\mathbf{G}}_i \otimes \hat{\mathbf{G}}^j, \quad v^i_{\cdot j} = \hat{\mathbf{G}}^i \mathbf{v} \hat{\mathbf{G}}_j = \hat{\mathbf{G}}^i \cdot \mathbf{g}_j \\
\mathbf{v} &= v_i^{\cdot j}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}_j, \quad v_i^{\cdot j} = \hat{\mathbf{G}}_i \mathbf{v} \hat{\mathbf{G}}^j = \mathbf{g}_i \cdot \hat{\mathbf{G}}^j
\end{aligned} \tag{2.10.61}$$

with similar symmetry. Also,

$$\begin{aligned}
\mathbf{U}^{-1} &= \left(U^{-1}\right)_{ij} \hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j, \quad \left(U^{-1}\right)_{ij} = \hat{\mathbf{g}}_i \mathbf{U}^{-1} \hat{\mathbf{g}}_j = \mathbf{G}_i \cdot \hat{\mathbf{g}}_j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)^{ij} \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j, \quad \left(U^{-1}\right)^{ij} = \hat{\mathbf{g}}^i \mathbf{U}^{-1} \hat{\mathbf{g}}^j = \hat{g}^{im}\mathbf{G}_m \cdot \hat{\mathbf{g}}^j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)^i_{\cdot j} \hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}^j, \quad \left(U^{-1}\right)^i_{\cdot j} = \hat{\mathbf{g}}^i \mathbf{U}^{-1} \hat{\mathbf{g}}_j = \hat{\mathbf{g}}^i \cdot \mathbf{G}_j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)_i^{\cdot j} \hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}_j, \quad \left(U^{-1}\right)_i^{\cdot j} = \hat{\mathbf{g}}_i \mathbf{U}^{-1} \hat{\mathbf{g}}^j = \mathbf{G}_i \cdot \hat{\mathbf{g}}^j
\end{aligned} \tag{2.10.62}$$

and

$$\begin{aligned}
\mathbf{v}^{-1} &= \left(v^{-1}\right)_{ij} \mathbf{g}^i \otimes \mathbf{g}^j, \quad \left(v^{-1}\right)_{ij} = \mathbf{g}_i \mathbf{v}^{-1} \mathbf{g}_j = \hat{\mathbf{G}}_i \cdot \mathbf{g}_j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)^{ij} \mathbf{g}_i \otimes \mathbf{g}_j, \quad \left(v^{-1}\right)^{ij} = \mathbf{g}^i \mathbf{v}^{-1} \mathbf{g}^j = g^{mj}\hat{\mathbf{G}}_m \cdot \mathbf{g}^i \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)^i_{\cdot j} \mathbf{g}_i \otimes \mathbf{g}^j, \quad \left(v^{-1}\right)^i_{\cdot j} = \mathbf{g}^i \mathbf{v}^{-1} \mathbf{g}_j = \mathbf{g}^i \cdot \hat{\mathbf{G}}_j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)_i^{\cdot j} \mathbf{g}^i \otimes \mathbf{g}_j, \quad \left(v^{-1}\right)_i^{\cdot j} = \mathbf{g}_i \mathbf{v}^{-1} \mathbf{g}^j = \hat{\mathbf{G}}_i \cdot \mathbf{g}^j
\end{aligned} \tag{2.10.63}$$

with similar symmetry. Note that, comparing 2.10.60a, 2.10.61a, 2.10.62a, 2.10.63a and using 2.10.57,

$$\begin{aligned}
\mathbf{U} &= U_{ij}\mathbf{G}^i \otimes \mathbf{G}^j \\
\mathbf{v} &= v_{ij}\hat{\mathbf{G}}^i \otimes \hat{\mathbf{G}}^j \\
\mathbf{U}^{-1} &= \left(U^{-1}\right)_{ij} \hat{\mathbf{g}}^i \otimes \hat{\mathbf{g}}^j \\
\mathbf{v}^{-1} &= \left(v^{-1}\right)_{ij} \mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned} \qquad U_{ij} = \left(U^{-1}\right)_{ij} = v_{ij} = \left(v^{-1}\right)_{ij} \tag{2.10.64}$$

Now note that rotations preserve vectors lengths and, in particular, preserve the metric, i.e.,

$$\begin{aligned}
G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \quad &= \quad \hat{G}_{ij} = \hat{\mathbf{G}}_i \cdot \hat{\mathbf{G}}_j \\
g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \quad &= \quad \hat{g}_{ij} = \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j
\end{aligned} \tag{2.10.65}$$

Thus, again using 2.10.57, and 2.10.60-2.10.63, the contravariant components of the above tensors are also equal, $U^{ij} = \left(U^{-1}\right)^{ij} = v^{ij} = \left(v^{-1}\right)^{ij}$.

As mentioned, the tensors can be decomposed along other bases, for example,

$$\mathbf{v} = v^{ij}\mathbf{g}_i \otimes \mathbf{g}_j, \quad v^{ij} = \mathbf{g}^i \mathbf{v} \mathbf{g}^j = \hat{\mathbf{G}}^i \cdot \mathbf{g}^j \tag{2.10.66}$$

2.10.6 Eigenvectors and Eigenvalues

Analogous to §2.2.5, the eigenvalues of **C** are determined from the eigenvalue problem

$$\det\left(\mathbf{C} - \lambda_{\mathbf{C}}\mathbf{I}\right) = 0 \tag{2.10.67}$$

leading to the characteristic equation 1.11.5

$$\lambda_{\mathbf{C}}^3 - \mathrm{I}_{\mathbf{C}}\lambda_{\mathbf{C}}^2 + \mathrm{II}_{\mathbf{C}}\lambda_{\mathbf{C}} - \mathrm{III}_{\mathbf{C}} = 0 \tag{2.10.68}$$

with principal scalar invariants 1.11.6-7

$$\begin{aligned} \mathrm{I}_{\mathbf{C}} &= \mathrm{tr}\mathbf{C} = A_i^i = \lambda_{\mathbf{C}1} + \lambda_{\mathbf{C}2} + \lambda_{\mathbf{C}3} \\ \mathrm{II}_{\mathbf{C}} &= \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right] = \tfrac{1}{2}\left(C_i^i C_j^j - C_j^i C_i^j\right) = \lambda_{\mathbf{C}1}\lambda_{\mathbf{C}2} + \lambda_{\mathbf{C}2}\lambda_{\mathbf{C}3} + \lambda_{\mathbf{C}3}\lambda_{\mathbf{C}1} \\ \mathrm{III}_{\mathbf{C}} &= \det\mathbf{C} = \varepsilon_{ijk} C_1^i C_2^j C_3^k = \lambda_{\mathbf{C}1}\lambda_{\mathbf{C}2}\lambda_{\mathbf{C}3} \end{aligned} \tag{2.10.69}$$

The eigenvectors are the principal material directions $\hat{\mathbf{N}}_i$, with

$$\left(\mathbf{C} - \lambda_i \mathbf{I}\right)\hat{\mathbf{N}}_i = \mathbf{0} \tag{2.10.70}$$

The spectral decomposition is then

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \hat{\mathbf{N}}_i \otimes \hat{\mathbf{N}}_i \tag{2.10.71}$$

where $\lambda_{\mathbf{C}i} = \lambda_i^2$ and the λ_i are the stretches. The remaining spectral decompositions in 2.2.37 hold also. Note also that the rotation tensor in terms of principal directions is (see 2.2.35)

$$\mathbf{R} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{N}}^i = \hat{\mathbf{n}}^i \otimes \hat{\mathbf{N}}_i \tag{2.10.72}$$

where $\hat{\mathbf{n}}_i$ are the spatial principal directions.

2.10.7 Displacement and Displacement Gradients

Consider the displacement **u** of a material particle. This can be written in terms of covariant components U_i and u_i:

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \equiv U_i \mathbf{G}^i = u_i \mathbf{g}^i \,. \tag{2.10.73}$$

The covariant derivative of **u** can be expressed as

$$\frac{\partial \mathbf{u}}{\partial \Theta^i} = U_m\big|_i \mathbf{G}^m = u_m\big\|_i \mathbf{g}^m \tag{2.10.74}$$

The single line refers to covariant differentiation with respect to the undeformed basis, i.e. the Christoffel symbols to use are functions of the G_{ij}. The double line refers to covariant differentiation with respect to the deformed basis, i.e. the Christoffel symbols to use are functions of the g_{ij}.

Alternatively, the covariant derivative can be expressed as

$$\frac{\partial \mathbf{u}}{\partial \Theta^i} = \frac{\partial \mathbf{x}}{\partial \Theta^i} - \frac{\partial \mathbf{X}}{\partial \Theta^i} = \mathbf{g}_i - \mathbf{G}_i \tag{2.10.75}$$

and so

$$\begin{aligned}
\mathbf{g}_i &= \mathbf{G}_i + U_m\big|_i \mathbf{G}^m = \left(\delta_i^m + U^m\big|_i\right)\mathbf{G}_m = F_{\cdot i}^{m}\mathbf{G}_m \\
\mathbf{G}_i &= \mathbf{g}_i - u_m\big\|_i \mathbf{g}^m = \left(\delta_i^m - u^m\big\|_i\right)\mathbf{g}_m = \left(f^{-1}\right)_{\cdot i}^{m}\mathbf{g}_m
\end{aligned} \tag{2.10.76}$$

The last equalities following from 2.10.31-32.

The components of the Green-Lagrange and Euler-Almansi strain tensors 2.10.43 can be written in terms of displacements using relations 2.10.76 {▲Problem 2}:

$$\begin{aligned}
E_{ij} &= \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(U_i\big|_j + U_j\big|_i + U_n\big|_i U^n\big|_j\right) \\
e_{ij} &= \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(u_i\big\|_j + u_j\big\|_i - u_n\big\|_i u^n\big\|_j\right)
\end{aligned} \tag{2.10.77}$$

In terms of spatial coordinates, $\Theta^i = X^i$, $\mathbf{G}_i = \mathbf{E}_i$, $\mathbf{g}_i = \left(\partial x^j / \partial X^i\right)\mathbf{e}_j$, $U_i\big|_j = \partial U_i / \partial X^j$, the components of the Euler-Lagrange strain tensor are

$$E_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(\frac{\partial x^m}{\partial X^i}\frac{\partial x^n}{\partial X^j}\delta_{mn} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X^j} + \frac{\partial U_j}{\partial X^i} + \frac{\partial U_k}{\partial X^i}\frac{\partial U_k}{\partial X^j}\right) \qquad (2.10.78)$$

which is 2.2.46.

2.10.8 The Deformation of Area and Volume Elements

Differential Volume Element

Consider a differential volume element formed by the elements $d\Theta^i \mathbf{G}_i$ in the undeformed configuration, Eqn. 1.16.43:

$$dV = \sqrt{G}\, d\Theta^1 d\Theta^2 d\Theta^3 \qquad (2.10.79)$$

where, Eqn. 1.16.31,1.16.34,

$$\sqrt{G} = \sqrt{\det\left[G_{ij}\right]}, \quad G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \qquad (2.10.80)$$

The *same* volume element in the deformed configuration is determined by the elements $d\Theta^i \mathbf{g}_i$:

$$dv = \sqrt{g}\, d\Theta^1 d\Theta^2 d\Theta^3 \qquad (2.10.81)$$

where

$$\sqrt{g} = \sqrt{\det\left[g_{ij}\right]}, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \qquad (2.8.82)$$

From 1.16.53 *et seq.*, 2.10.11,

$$\begin{aligned} \sqrt{g} &= \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \\ &= F^{i}_{\cdot 1} F^{j}_{\cdot 2} F^{k}_{\cdot 3} \mathbf{G}_i \cdot \mathbf{G}_j \times \mathbf{G}_k \\ &= F^{i}_{\cdot 1} F^{j}_{\cdot 2} F^{k}_{\cdot 3} \varepsilon_{ijk} \sqrt{G} \\ &= \sqrt{G}\det \mathbf{F} \end{aligned} \qquad (2.10.83)$$

where ε_{ijk} is the Cartesian permutation symbol, and so the Jacobian determinant is (see 2.2.53)

$$J = \frac{dv}{dV} = \frac{\sqrt{g}}{\sqrt{G}} = \det \mathbf{F} \qquad (2.10.84)$$

and $\det \mathbf{F}$ is the determinant of the matrix with components $F^{i}_{\cdot j}$.

Differential Area Element

Consider a differential surface (parallelogram) element in the undeformed configuration, bounded by two vector elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, and with unit normal $\hat{\mathbf{N}}$. Then the vector normal to the surface element and with magnitude equal to the area of the surface is, using 1.16.54, given by

$$\hat{\mathbf{N}}dS = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = d\Theta^{(1)i}\mathbf{G}_i \times d\Theta^{(2)j}\mathbf{G}_j = e^{(\mathbf{G})}_{ijk} d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{G}^k \qquad (2.10.85)$$

where $e^{(\mathbf{G})}_{ijk}$ is the permutation symbol associated with the basis $\mathbf{G}_i$, i.e.

$$e^{(\mathbf{G})}_{ijk} = \varepsilon_{ijk}\mathbf{G}_i \cdot \mathbf{G}_j \times \mathbf{G}_k = \varepsilon_{ijk}\sqrt{G}\,. \qquad (2.10.86)$$

Using $\mathbf{G}^k = \mathbf{F}^{\mathrm{T}}\mathbf{g}^k$, one has

$$\hat{\mathbf{N}}dS = \varepsilon_{ijk}\sqrt{G}\, d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{F}^{\mathrm{T}}\mathbf{g}^k \qquad (2.10.87)$$

Similarly, the surface vector in the deformed configuration with unit normal $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}}ds = d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = d\Theta^{(1)i}\mathbf{g}_i \times d\Theta^{(2)j}\mathbf{g}_j = e^{(\mathbf{g})}_{ijk} d\Theta^{(1)i} d\Theta^{(2)j}\mathbf{g}^k \qquad (2.10.88)$$

where $e^{(\mathbf{g})}_{ijk}$ is the permutation symbol associated with the basis $\mathbf{g}_i$, i.e.

$$e^{(\mathbf{g})}_{ijk} = \varepsilon_{ijk}\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \varepsilon_{ijk}\sqrt{g}\,. \qquad (2.10.89)$$

Comparing the two expressions for the areas in the undeformed and deformed configurations, 2.10.87-88, one finds that

$$\hat{\mathbf{n}}ds = \sqrt{\frac{g}{G}}\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS = (\det \mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{N}dS \qquad (2.10.90)$$

which is Nanson's relation, Eqn. 2.2.59. This is consistent with was said earlier in relation to Fig. 2.10.8 and the contravariant bases: $\mathbf{F}^{-\mathrm{T}}$ maps vectors normal to the coordinate curves in the initial configuration into corresponding vectors normal to the coordinate curves in the current configuration.

2.10.9 Problems

1. Derive the relations 2.10.51.
2. Use relations 2.10.76, with $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$, to derive 2.10.77

$$E_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(U_i\big|_j + U_j\big|_i + U_n\big|_i U^n\big|_j\right)$$

$$e_{ij} = \frac{1}{2}\left(g_{ij} - G_{ij}\right) = \frac{1}{2}\left(u_i\big\|_j + u_j\big\|_i - u_n\big\|_i u^n\big\|_j\right)$$