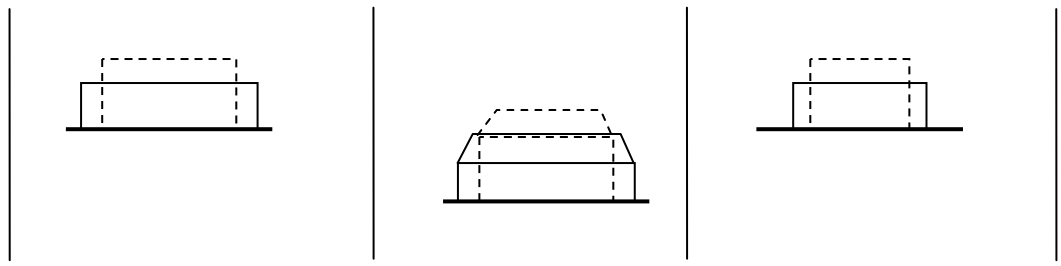
2.8 Objectivity and Objective Tensors

2.8.1 Dependence on Observer

Consider a rectangular block of material resting on a circular table. A person stands and observes the material deform, Fig. 2.8.1a. The dashed lines indicate the undeformed material whereas the solid line indicates the current state. A second observer is standing just behind the first, but on a step ladder – this observer sees the material as in 2.8.1b. A third observer is standing around the table, 45° from the first, and sees the material as in Fig. 2.8.1c.

The deformation can be described by each observer using concepts like displacement, velocity, strain and so on.. However, it is clear that the three observers will in general record different values for these measures, since their perspectives differ.

The goal in what follows is to determine which of the kinematical tensors are in fact *independent* of observer. Since the laws of physics describing the response of a deforming material must be independent of any observer, it is these particular tensors which will be more readily used in expressions to describe material response.



(a) (b) (c)

Figure 2.8.1: a deforming material as seen by different observers

Note that Fig. 2.8.1 can be interpreted in another, equivalent, way. One can imagine *one* static observer, but this time with the material moved into three different positions. This viewpoint will be returned to in the next section.

2.8.2 Change of Reference Frame

Consider two **frames of reference**, the first consisting of the origin **o** and the basis $\{\mathbf{e}_i\}$, the second consisting of the origin $\mathbf{o}^*$ and the basis $\{\mathbf{e}_i^*\}$, Fig. 2.8.2. A point **x** in space is then identified as having position vector $\mathbf{x} = x_i \mathbf{e}_i$ in the first frame and position vector $\mathbf{x}^* = x_i^* \mathbf{e}_i^*$ in the second frame.

When the origins **o** and $\mathbf{o}^*$ coincide, $\mathbf{x} = \mathbf{x}^*$ and the vector components x_i and x_i^* are related through Eqn. 1.5.3, $x_i = Q_{ij} x_j^*$, or $\mathbf{x} = x_i \mathbf{e}_i = Q_{ij} x_j^* \mathbf{e}_i$, where $[\mathbf{Q}]$ is the transformation matrix 1.5.4, $Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j^*$. Alternatively, one has Eqn. 1.5.5, $x_i^* = Q_{ji} x_j$, or $\mathbf{x}^* = x_i^* \mathbf{e}_i^* = Q_{ji} x_j \mathbf{e}_i^*$.

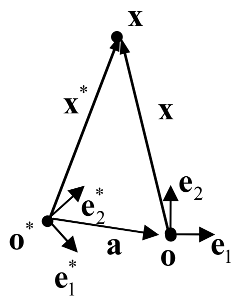


Figure 2.8.2: two frames of reference

With the shift in origin $\mathbf{a} = \mathbf{o} - \mathbf{o}^*$, one has

$$\mathbf{x}^* = x_i^* \mathbf{e}_i^* = Q_{ji} x_j \mathbf{e}_i^* + a_i^* \mathbf{e}_i^* \tag{2.8.1}$$

where $\mathbf{a} = a_i^* \mathbf{e}_i^*$. Alternatively,

$$\mathbf{x} = x_i \mathbf{e}_i = Q_{ij} x_j^* \mathbf{e}_i - a_i \mathbf{e}_i \tag{2.8.2}$$

where $\mathbf{a} = a_i \mathbf{e}_i$, with $a_i^* = Q_{ji} a_j$.

Formulae 2.8.1-2 relate the coordinates of the position vector to a point in space as observed from one frame of reference to the coordinates of the position vector to the *same* point as observed from a different frame of reference.

Finally, consider the position vector **x**, which is defined relative to the frame $(\mathbf{o}, \mathbf{e}_i)$. To an observer in the frame $(\mathbf{o}^*, \mathbf{e}_i^*)$, the *same* position vector would appear as $(\mathbf{x})^*$, Fig. 2.8.3. Rotating this vector $(\mathbf{x})^*$ through $\mathbf{Q}^{\mathrm{T}}$ (the tensor which rotates the basis $\{\mathbf{e}_i^*\}$ into the basis $\{\mathbf{e}_i\}$) and adding the vector **a** then produces $\mathbf{x}^*$:

$$\mathbf{x}^* = \mathbf{Q}^{\mathrm{T}} (\mathbf{x})^* + \mathbf{a} \tag{2.8.3}$$

This relation will be discussed further below.

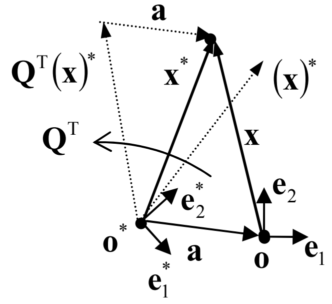


Figure 2.8.3: Relation between vectors in Eqn. 2.8.3

2.8.3 Change of Observer

The change of frame encompassed by Eqns. 2.8.1-2 is more precisely called a **passive change of frame**, and merely involves a transformation between vector components. One would say that there is one observer but that this observer is using two frames of reference. Here follows a different concept, an **active change of frame**, also called a **change in observer**, in which there are two observers, each with their own frame of reference.

An **observer** is someone who can measure relative positions in space (with a ruler) and instants of time (with a clock). An **event** in the physical world (for example a material particle) is perceived by an observer as occurring at a particular point in space and at a particular time. One can regard an observer O to be a map of an event E in the physical world to a point $\mathbf{x}$ in point space (*cf.* §1.2.5) and a real number t. A *single* event E is recorded as the pair $(\mathbf{x}, t)$ by an observer O and, in general, by a *different* pair $(\mathbf{x}^*, t^*)$ by a second observer O^*, Fig. 2.8.4.

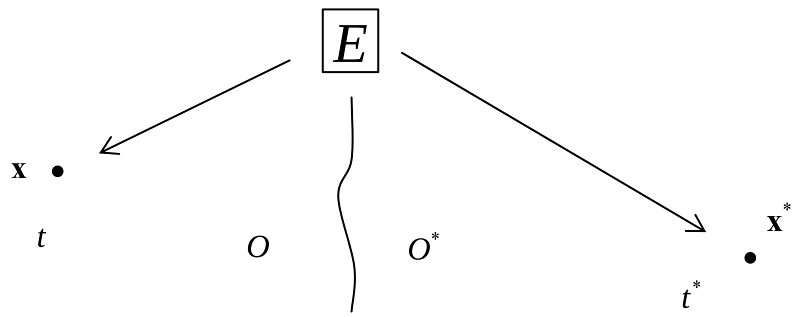


Figure 2.8.4: recordings by two observers of the same event

Let the two observers record three points corresponding to three events, Fig. 2.8.5. These points define vectors in space, as the difference between the points (*cf.* §1.2.5). It is assumed that both observers "see" the same Euclidean geometry, that is, if one observer sees an ellipse, then the other observer will see the same ellipse, but perhaps positioned differently in space. To ensure that this is so, observed vectors must be related through some orthogonal tensor $\mathbf{Q}$, for example,

$$\mathbf{x}^* - \mathbf{x}_0^* = \mathbf{Q}(\mathbf{x} - \mathbf{x}_0) \tag{2.8.4}$$

since this transformation will automatically preserve distances between points, and angles between vectors (see §1.10.7), for example,

$$\left(\mathbf{x}_1^* - \mathbf{x}_0^*\right) \cdot \left(\mathbf{x}^* - \mathbf{x}_0^*\right) = \mathbf{Q}\left(\mathbf{x}_1 - \mathbf{x}_0\right) \cdot \mathbf{Q}\left(\mathbf{x} - \mathbf{x}_0\right) = \left(\mathbf{x}_1 - \mathbf{x}_0\right) \cdot \left(\mathbf{x} - \mathbf{x}_0\right) \qquad (2.8.5)$$

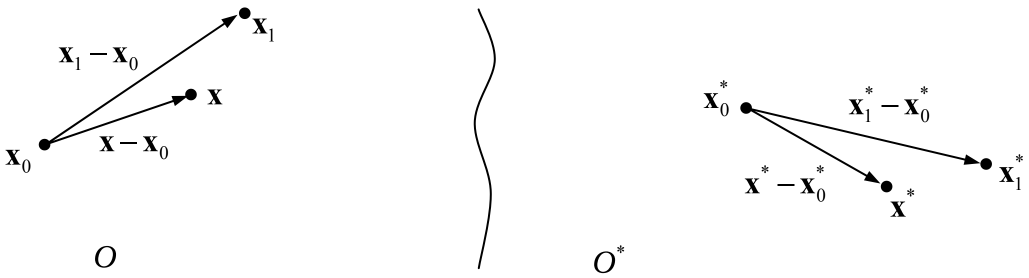


Figure 2.8.5: recordings of two observers of three separate events

Although all orthogonal tensors $\mathbf{Q}$ do indeed preserve length and angles, it is taken that the $\mathbf{Q}$ in 2.8.4-5 is proper orthogonal, i.e. a rotation tensor (*cf.* §1.10.8), so that orientation is also preserved. Further, it is assumed that $\mathbf{Q} = \mathbf{Q}(t)$, which expresses the fact that the observers can move relative to each other over time.

Observers must also agree on time intervals between events. Let an observer *O* record a certain event at time *t* and a second observer O^* record the same event as occurring at time t^*. Then the times must be related through

$$\boxed{t^* = t + \alpha} \quad \textbf{Observer Time Transformation} \qquad (2.8.6)$$

where α is a *constant*. If now the observers record a second event as occurring at t_1 and t_1^* say, one has $t_1^* - t^* = t_1 - t$ as required.

The observer transformation 2.8.4 involves the vectors $\mathbf{x} - \mathbf{x}_0$ and $\mathbf{x}^* - \mathbf{x}_0^*$ and as such does not require the notion of origin or coordinate system; it is an abstract symbolic notation for an observer transformation. However, an origin $\mathbf{o}$ for O and $\mathbf{o}^*$ for O^* can be introduced and then the points $\mathbf{x}_0, \mathbf{x}, \mathbf{x}_0^*, \mathbf{x}^*$ can be regarded as *position vectors* in space, Fig. 2.8.6.

The transformation 2.8.4 can now be expressed in the oft-used format

$$\boxed{\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}} \quad \textbf{Observer (Spatial) Transformation} \quad (2.8.7)$$

where

$$\mathbf{c}(t) = \mathbf{x}_0^* - \mathbf{Q}(t)\mathbf{x}_0 \qquad (2.8.8)$$

The transformation 2.8.7 is called a **Euclidean transformation**, since it preserves the Euclidean geometry.

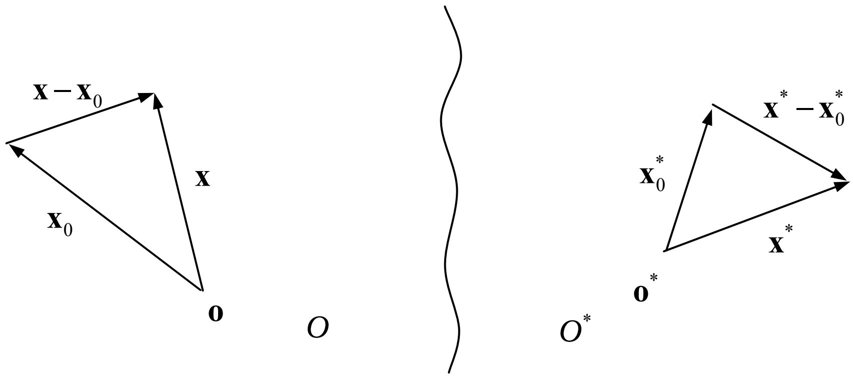


Figure 2.8.6: position vectors for two observers of the same events

Coordinate Systems

Each observer can introduce any Cartesian coordinate system, with basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i^*\}$ say. They can then resolve the position vectors into vector components. These basis vectors can be oriented with respect to each other in any way, that is, they will be related through $\mathbf{e}_i^* = \mathbf{R}\mathbf{e}_i$, where $\mathbf{R}$ is any rotation tensor. Indeed, each observer can change their basis, effecting a coordinate transformation. No attempt to introduce specific coordinate systems will be made here since they are completely unnecessary to the notion of observer transformation and would only greatly confuse the issue.

Relationship to Passive Change of Frame

Recall the passive change of frame encompassed in Eqns. 2.8.1-2. If one substitutes the *actual* $\mathbf{x}$ for $(\mathbf{x})^*$ in Eqn. 2.8.3, one has:

$$\mathbf{x}^* = \mathbf{Q}^{\mathrm{T}}\mathbf{x} + \mathbf{a} \tag{2.8.9}$$

This is clearly an observer transformation, relating the position vector as seen by one observer to the position vector as seen by a second observer, through an orthogonal tensor and a vector, as in Eqn. 2.8.7. In the passive change of frame, Q_{ij} are the components of the orthogonal tensor $\mathbf{Q} = \mathbf{e}_i^* \otimes \mathbf{e}_i$, Eqn. 1.10.25, which maps the bases onto each other: $\mathbf{e}_i^* = \mathbf{Q}\mathbf{e}_i$. Thus the transformation 2.8.1-2 can be defined uniquely by the pair $\mathbf{Q}$ and $\mathbf{a}$. In that sense, the passive change of frame does indeed define an active change of frame, i.e. a change of observer, through Eqn. 2.8.9. However, the concept of observer discussed above is the preferred way of defining an observer transformation.

2.8.4 Objective Vectors and Tensors

The observer transformation 2.8.7 encapsulates the different viewpoints observers have of the physical world. They will see the same objects, but in general they will see these objects oriented differently and located at different positions. The goal now is to see which of the kinematical tensors are independent of these different viewpoints. As a first step, next is introduced the concept of an **objective tensor**.

Suppose that different observers are examining a deforming material. In order to describe the material, the observers take measurements. This will involve measurements of *spatial* objects associated with the current configuration, for example the velocity or spin. It will also involve *material* objects associated with the reference configuration, for example line elements in that configuration. It will also involve *two-point* tensors such as the rotation or deformation gradient, which are associated with both the current and reference configurations.

It is assumed that all observers observe the reference configuration to be the same, that is, they record the same set of points for the material particles in the reference configuration[1]. The observers then move relative to each other and their measurements of objects associated with the current configuration will in general differ. One would expect (want) different observers to make the same measurement of material objects despite this relative movement; thus one says that material vectors and tensors are **objective (material) vectors** and **objective (material) tensors** if they remain unchanged under the observer transformation 2.8.6-7.

A spatial vector **u** on the other hand is said to be an **objective (spatial) vector** if it satisfies the observer transformation (see 2.8.4):[2]

$$\boxed{\mathbf{u}^* = \mathbf{Q}\mathbf{u}} \qquad \textbf{Objectivity Requirement for a Spatial Vector} \qquad (2.8.10)$$

for all rotation tensors **Q**. An **objective (spatial) tensor** is defined to be one which transforms an objective vector into an objective vector. Consider a tensor observed as $\mathbf{T}$ and $\mathbf{T}^*$ by two different observers. Take an objective vector which is observed as **v** and $\mathbf{v}^*$, and let $\mathbf{u} = \mathbf{T}\mathbf{v}$ and $\mathbf{u}^* = \mathbf{T}^*\mathbf{v}^*$. Then, for **u** to be objective,

$$\mathbf{u}^* = \mathbf{Q}\mathbf{u} = \mathbf{Q}\mathbf{T}\mathbf{v} = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{v}^* \qquad (2.8.11)$$

and so the tensor is objective provided

$$\boxed{\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}} \qquad \textbf{Objectivity Requirement for a Spatial Tensor} \qquad (2.8.12)$$

Various identities can be derived; for example, for objective vectors **a** and **b**, and objective tensors **A** and **B**, {▲Problem 1}

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[1] this does not affect the generality of what follows; the notion of objective tensor is independent of the chosen reference configuration

[2] the time transformation 2.8.6 is trivial and does not affect the relations to be derived

$$
\begin{aligned}
(\mathbf{a}+\mathbf{b})^* &= \mathbf{a}^* + \mathbf{b}^* \\
(\mathbf{a}\otimes\mathbf{b})^* &= \mathbf{a}^* \otimes \mathbf{b}^* \\
(\mathbf{a}\cdot\mathbf{b})^* &= \mathbf{a}^* \cdot \mathbf{b}^* \\
(\mathbf{A}\mathbf{b})^* &= \mathbf{A}^*\mathbf{b}^* \\
(\mathbf{A}\mathbf{B})^* &= \mathbf{A}^*\mathbf{B}^* \\
(\mathbf{A}^{-1})^* &= (\mathbf{A}^*)^{-1} \\
(\mathbf{A}\mathbf{B})^* &= \mathbf{A}^*\mathbf{B}^* \\
(\mathbf{A}:\mathbf{B})^* &= \mathbf{A}^* : \mathbf{B}^*
\end{aligned} \tag{2.8.13}
$$

For a scalar,

$$\boxed{\phi^* = \phi} \qquad \textbf{Objectivity Requirement for a Scalar} \tag{2.8.14}$$

In other words, an **objective scalar** is one which has the same value to all observers.

Finally, consider a two-point tensor. Such a tensor is said to be objective if it maps an objective material vector into an objective spatial vector. Consider then a two-point tensor observed as $\mathbf{T}$ and $\mathbf{T}^*$. Take an objective material vector which is observed as $\mathbf{v}$ and $\mathbf{v}^*$, and let $\mathbf{u} = \mathbf{T}\mathbf{v}$ and $\mathbf{u}^* = \mathbf{T}^*\mathbf{v}^*$. A material vector is objective if it is unaffected by an observer transformation, so

$$\mathbf{u}^* = \mathbf{Q}\mathbf{u} = \mathbf{Q}\mathbf{T}\mathbf{v} = \mathbf{Q}\mathbf{T}\mathbf{v}^* \tag{2.8.15}$$

and so the tensor is objective provided

$$\boxed{\mathbf{T}^* = \mathbf{Q}\mathbf{T}} \qquad \textbf{Objectivity Requirement for a Two-point Tensor} \tag{2.8.16}$$

Thus the objectivity requirement for a two-point tensor is the same as that for a spatial vector.

2.8.5 Objective Kinematics

Next are examined the various kinematic vectors and tensors introduced in the earlier sections, and their objectivity status is determined.

The motion is observed by one observer as $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ and by a second observer as $\mathbf{x}^* = \boldsymbol{\chi}(\mathbf{X}, t^*)$. The observer transformation gives

$$\boldsymbol{\chi}^*(\mathbf{X}, t^*) = \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{c}(t), \qquad t^* = t + \alpha \tag{2.8.17}$$

and so the motion is not an objective vector, i.e. $\boldsymbol{\chi}^* \neq \mathbf{Q}\boldsymbol{\chi}$.

The Velocity and Acceleration

Differentiating 2.8.17 (and using the notation $\dot{\mathbf{x}}$ instead of $\dot{\boldsymbol{\chi}}(\mathbf{X}, t)$ for brevity), the velocity under the observer transformation is

$$\dot{\mathbf{x}}^* = \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\dot{\mathbf{x}} + \dot{\mathbf{c}} \tag{2.8.18}$$

which does not comply with the objectivity requirement for spatial vectors, 2.8.10. In other words, different observers will measure different magnitudes for the velocity. The velocity expression can be put in a form similar to that of elementary mechanics (the "non-objective" terms are on the right),

$$\dot{\mathbf{x}}^* - \mathbf{Q}\dot{\mathbf{x}} = \boldsymbol{\Omega}_{\mathbf{Q}}\left(\mathbf{x}^* - \mathbf{c}\right) + \dot{\mathbf{c}} \tag{2.8.19}$$

where

$$\boldsymbol{\Omega}_{\mathbf{Q}} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} \tag{2.8.20}$$

is skew-symmetric (see Eqn. 1.14.2); this tensor represents the rigid body angular velocity between the observers (see Eqn. 2.6.1). Note that the velocity *is* objective provided $\dot{\mathbf{Q}} = \mathbf{0}$, $\dot{\mathbf{c}} = \mathbf{o}$, for which $\mathbf{x}^* = \mathbf{Q}_0\mathbf{x} + \mathbf{c}_0$, which is called a **time-independent rigid transformation**.

Similarly, for the acceleration, it can be shown that

$$\ddot{\mathbf{x}}^* - \mathbf{Q}\ddot{\mathbf{x}} = \dot{\boldsymbol{\Omega}}_{\mathbf{Q}}\left(\mathbf{x}^* - \mathbf{c}\right) - \boldsymbol{\Omega}_{\mathbf{Q}}^2\left(\mathbf{x}^* - \mathbf{c}\right) + 2\boldsymbol{\Omega}_{\mathbf{Q}}\left(\dot{\mathbf{x}} - \dot{\mathbf{c}}\right) + \ddot{\mathbf{c}} \tag{2.8.21}$$

The first three terms on the right-hand side are called the **Euler acceleration**, the **centrifugal acceleration** and the **Coriolis acceleration** respectively. The acceleration is objective provided $\dot{\mathbf{c}}$ and $\mathbf{Q}$ are constant, for which $\mathbf{x}^* = \mathbf{Q}_0\mathbf{x} + \mathbf{c}(t)$ with $\ddot{\mathbf{c}} = \mathbf{o}$, which is called a **Galilean transformation** – where the two configurations are related by a rigid rotation and a translational motion with constant velocity.

The Deformation Gradient

Consider the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. As mentioned, observers observe the reference configuration to be the same: $\mathbf{X}^* = \mathbf{X}$. The deformation is then observed as $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ and $d\mathbf{x}^* = \mathbf{F}^* d\mathbf{X}$, so that

$$d\mathbf{x}^* = \mathbf{Q}d\mathbf{x} = \mathbf{Q}\mathbf{F}d\mathbf{X} = \mathbf{Q}\mathbf{F}d\mathbf{X} \tag{2.8.22}$$

and

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F} \tag{2.8.23}$$

and so, according to 2.8.16, the deformation gradient is objective.

The Cauchy-Green Strain Tensors

For the right and left Cauchy-Green tensors,

$$\begin{aligned} \mathbf{C}^* &= \mathbf{F}^{*\mathrm{T}}\mathbf{F}^* = \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{F} = \mathbf{C} \\ \mathbf{b}^* &= \mathbf{F}^*\mathbf{F}^{*\mathrm{T}} = \mathbf{Q}\mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{b}\mathbf{Q}^{\mathrm{T}} \end{aligned} \tag{2.8.24}$$

Thus the material tensor **C** and the spatial tensor **b** are objective[3].

The Jacobian Determinant

For the Jacobian determinant, using 1.10.16a,

$$J^* = \det \mathbf{F}^* = \det(\mathbf{Q}\mathbf{F}) = \det \mathbf{Q} \det \mathbf{F} = \det \mathbf{F} = J \tag{2.8.25}$$

and[4] so is objective according to 2.8.14.

The Rotation and Stretch Tensors

The polar decomposition is $\mathbf{F} = \mathbf{R}\mathbf{U}$, where **R** is the orthogonal rotation tensor and **U** is the right stretch tensor. Then $\mathbf{F}^* = \mathbf{Q}\mathbf{F} = \mathbf{Q}\mathbf{R}\mathbf{U} \equiv \mathbf{R}^*\mathbf{U}^*$. Since $\mathbf{Q}\mathbf{R}$ is orthogonal, the expression $\mathbf{Q}\mathbf{R}\mathbf{U} = \mathbf{R}^*\mathbf{U}^*$ is valid provided

$$\mathbf{R}^* = \mathbf{Q}\mathbf{R}, \qquad \mathbf{U}^* = \mathbf{U} \tag{2.8.26}$$

Thus the two-point tensor **R** and the material tensor **U** are objective.

The Velocity Gradient

Allowing **Q** to be a function of time, for the velocity gradient, using 2.5.4, 1.9.18c,

$$\mathbf{l}^* = \dot{\overline{\mathbf{F}^*}}(\mathbf{F}^*)^{-1} = (\mathbf{Q}\dot{\mathbf{F}} + \dot{\mathbf{Q}}\mathbf{F})\mathbf{F}^{-1}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{l}\mathbf{Q}^{\mathrm{T}} + \mathbf{\Omega}_{\mathbf{Q}} \tag{2.8.27}$$

where $\mathbf{\Omega}_{\mathbf{Q}}$ is the angular velocity tensor 2.8.20. On the other hand, with $\mathbf{l} = \mathbf{d} + \mathbf{w}$, and separating out the symmetric and skew-symmetric parts,

$$\mathbf{d}^* = \mathbf{Q}\mathbf{d}\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{w}^* = \mathbf{Q}\mathbf{w}\mathbf{Q}^{\mathrm{T}} + \mathbf{\Omega}_{\mathbf{Q}} \tag{2.8.28}$$

Thus the velocity gradient is not objective. This is not surprising given that the velocity is not objective. However, significantly, the rate of deformation, a measure of the rate of stretching of material, *is* objective.

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[3] Some authors define a second order tensor to be objective only if 2.8.12 is satisfied, regardless of whether it is spatial, two-point or material; with this definition, **F** and **C** would be defined as non-objective
[4] Note that **Q** must be a rotation tensor, not just an orthogonal tensor, here

The Spatial Gradient

Consider the spatial gradient of an *objective vector* **t**:

$$\mathrm{grad}\,\mathbf{t} = \frac{\partial \mathbf{t}}{\partial \mathbf{x}}, \qquad (\mathrm{grad}\,\mathbf{t})^* = \frac{\partial \mathbf{t}^*}{\partial \mathbf{x}^*} \tag{2.8.29}$$

Since $\mathbf{t}^* = \mathbf{Qt}$, the chain rule gives

$$\frac{\partial \mathbf{t}^*}{\partial \mathbf{x}} = \frac{\partial \mathbf{t}^*}{\partial \mathbf{x}^*}\frac{\partial \mathbf{x}^*}{\partial \mathbf{x}} \equiv \frac{\partial (\mathbf{Qt})}{\partial \mathbf{x}} = \mathbf{Q}\frac{\partial \mathbf{t}}{\partial \mathbf{x}} \tag{2.8.30}$$

It follows that

$$(\mathrm{grad}\,\mathbf{t})^* = \mathbf{Q}\frac{\partial \mathbf{t}}{\partial \mathbf{x}}\mathbf{Q}^{\mathrm{T}} \tag{2.8.31}$$

Thus the spatial gradient is objective. In general, it can be shown that the spatial gradient of a tensor field of order *n* is objective, for example the gradient of a scalar ϕ, {▲Problem 2} $\mathrm{grad}\phi$. Further, for a vector **v**, {▲Problem 3} $\mathrm{div}\mathbf{v}$ is objective.

Objective Rates

Consider an objective vector field **u**. The material derivative $\dot{\mathbf{u}}$ is not objective. However, the co-rotational derivative, Eqn. 2.6.12, $\overset{\circ}{\mathbf{u}} = \dot{\mathbf{u}} - \mathbf{wu}$ *is* objective. To show this, contract 2.8.28b, $\mathbf{w}^* = \mathbf{QwQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$, to the right with $\mathbf{Q}$ to get an expression for $\dot{\mathbf{Q}}$:

$$\dot{\mathbf{Q}} = \mathbf{w}^*\mathbf{Q} - \mathbf{Qw} \tag{2.8.32}$$

and then

$$\mathbf{u}^* = \mathbf{Qu} \quad \to \quad \dot{\overline{\mathbf{u}^*}} = \dot{\mathbf{Q}}\mathbf{u} + \mathbf{Q}\dot{\mathbf{u}} = \mathbf{w}^*\mathbf{Qu} + \mathbf{Q}(\dot{\mathbf{u}} - \mathbf{wu}) = \mathbf{w}^*\mathbf{Qu} + \mathbf{Q}\overset{\circ}{\mathbf{u}} \tag{2.8.33}$$

Then $\dot{\overline{\mathbf{u}^*}} - \mathbf{w}^*\mathbf{u}^* = \mathbf{Q}\overset{\circ}{\mathbf{u}}$, or $(\overset{\circ}{\mathbf{u}})^* = \mathbf{Q}\overset{\circ}{\mathbf{u}}$, so that the co-rotational derivative of a vector is an objective vector.

Rates of spatial tensors can also be modified in order to construct objective rates. For example, consider an objective spatial tensor **T**, so $\mathbf{T}^* = \mathbf{QTQ}^{\mathrm{T}}$. Then

$$\dot{\overline{\mathbf{T}^*}} = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{TQ}^{\mathrm{T}} + \mathbf{QT}\dot{\mathbf{Q}}^{\mathrm{T}} \tag{2.8.34}$$

which is clearly not objective. However, this can be re-arranged using 2.8.32 into

$$\overline{\mathbf{T}^*}^{\cdot} - \mathbf{w}^*\mathbf{T}^* + \mathbf{T}^*\mathbf{w}^* = \mathbf{Q}\left(\dot{\mathbf{T}} - \mathbf{w}\mathbf{T} + \mathbf{T}\mathbf{w}\right)\mathbf{Q}^{\mathrm{T}} \tag{2.8.35}$$

and so the quantity

$$\dot{\mathbf{T}} - \mathbf{w}\mathbf{T} + \mathbf{T}\mathbf{w} \tag{2.8.36}$$

is an objective rate, called the **Jaumann rate**. Other objective rates of tensors can be constructed in a similar fashion, for example the **Cotter-Rivlin rate**, defined by {▲Problem 4}

$$\dot{\mathbf{T}} + \mathbf{l}^{\mathrm{T}}\mathbf{T} + \mathbf{T}\mathbf{l} \tag{2.8.37}$$

Summary of Objective Kinematic Objects

Table 2.8.1 summarises the objectivity of some important kinematic objects:

	objective	definition	Type	Transformation
Jacobian determinant	✓		Scalar	$J^* = J$
Deformation gradient	✓		2-point	$\mathbf{F}^* = \mathbf{QF}$
Rotation	✓	$\mathbf{R} = \mathbf{FU}^{-1} = \mathbf{v}^{-1}\mathbf{F}$	2-point	$\mathbf{R}^* = \mathbf{QR}$
Right Cauchy-Green strain	✓	$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	Material	$\mathbf{C}^* = \mathbf{C}$
Green-Lagrange strain	✓	$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	Material	$\mathbf{E}^* = \mathbf{E}$
Rate of Green-Lagrange strain	✓		Material	$\overline{\mathbf{E}^*}^{\cdot} = \dot{\mathbf{E}}$
Right Stretch	✓	$\mathbf{U} = \sqrt{\mathbf{C}}$	Material	$\mathbf{U}^* = \mathbf{U}$
Left Cauchy-Green strain	✓	$\mathbf{b} = \mathbf{FF}^{\mathrm{T}}$	Spatial	$\mathbf{b}^* = \mathbf{QbQ}^{\mathrm{T}}$
Euler-Almansi strain	✓	$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$	Spatial	$\mathbf{e}^* = \mathbf{QeQ}^{\mathrm{T}}$
Left Stretch	✓	$\mathbf{v} = \sqrt{\mathbf{b}}$	Spatial	$\mathbf{v}^* = \mathbf{QvQ}^{\mathrm{T}}$
Spatial Velocity Gradient	×	$\mathbf{l} = \mathrm{grad}\mathbf{v}$	Spatial	$\mathbf{l}^* = \mathbf{QlQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$
Rate of Deformation	✓	$\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^{\mathrm{T}})$	Spatial	$\mathbf{d}^* = \mathbf{QdQ}^{\mathrm{T}}$
Spin	×	$\mathbf{w} = \frac{1}{2}(\mathbf{l} - \mathbf{l}^{\mathrm{T}})$	Spatial	$\mathbf{w}^* = \mathbf{QwQ}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$

Table 2.8.1: Objective kinematic objects

2.8.6 Objective Functions

In a similar way, functions are defined to be objective as follows:

- A scalar-valued function ϕ of, for example, a tensor **A**, is objective if it transforms in the same way as an objective scalar,

$$\phi^*(\mathbf{A}) = \phi(\mathbf{A}) \tag{2.8.38}$$

- A (spatial) vector-valued function **a** of a tensor **A** is objective if it transforms in the same way as an objective vector

$$\mathbf{v}^*(\mathbf{A}) = \mathbf{Q}\mathbf{v}(\mathbf{A}) \tag{2.8.39}$$

- A (spatial) tensor-valued function **f** of a tensor **A** is objective if it transforms according to

$$\mathbf{f}^*(\mathbf{A}) = \mathbf{Q}\mathbf{f}(\mathbf{A})\mathbf{Q}^{\mathrm{T}} \tag{2.8.40}$$

Objective functions of the Deformation Gradient

Consider an objective scalar-valued function ϕ of the deformation gradient **F**, $\phi(\mathbf{F})$. The function is objective if $\phi^* = \phi(\mathbf{F})$. But also,

$$\phi^* = \phi(\mathbf{F}^*) = \phi(\mathbf{QF}) \tag{2.8.41}$$

Using the polar decomposition theorem, $\phi(\mathbf{RU}) = \phi(\mathbf{QRU})$. Choosing the particular rigid-body rotation $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ then leads to

$$\phi(\mathbf{RU}) = \phi(\mathbf{U}) \tag{2.8.42}$$

which leads to the **reduced form**

$$\phi(\mathbf{F}) = \phi(\mathbf{U}) \tag{2.8.43}$$

Thus for the scalar function ϕ to be objective, it must be independent of the rotational part of **F**, and depends only on the stretching part; it cannot be a function of the nine independent components of the deformation gradient, but only of the six independent components of the right stretch tensor.

Consider next an objective (spatial) tensor-valued function **f** of the deformation gradient **F**, $\mathbf{f}(\mathbf{F})$. According to the definition of objectivity of a second order tensor, 2.8.12:

$$\mathbf{f}^* = \mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^{\mathrm{T}} \tag{2.8.44}$$

But also,

$$\mathbf{f}^* = \mathbf{f}(\mathbf{F}^*) = \mathbf{f}(\mathbf{QF}) \tag{2.8.45}$$

Again, using the polar decomposition theorem and choosing the particular rigid-body rotation $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ leads to

$$\mathbf{f}(\mathbf{U}) = \mathbf{R}^{\mathrm{T}}\mathbf{f}(\mathbf{RU})\mathbf{R} \tag{2.8.46}$$

which leads to the reduced form

$$\mathbf{f}(\mathbf{F}) = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \tag{2.8.47}$$

Thus for **f** to be objective, its dependence on **F** must be through an arbitrary function of **U** together with a more explicit dependence on **R**, the rotation tensor

Example

Consider the tensor function $\mathbf{f}(\mathbf{F}) = \alpha\left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^2$. Then

$$\mathbf{f}(\mathbf{QF}) = \alpha\left[(\mathbf{QF})(\mathbf{QF})^{\mathrm{T}}\right]^2 = \mathbf{Q}\alpha\left[\mathbf{F}\mathbf{F}^{\mathrm{T}}\right]^2\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}$$

and so the objectivity requirement is satisfied. According to the above, then, one can evaluate $\mathbf{f}(\mathbf{U}) = \mathbf{R}^{\mathrm{T}}\mathbf{f}(\mathbf{RU})\mathbf{R} = \alpha\left(\mathbf{U}\mathbf{U}^{\mathrm{T}}\right)^2$, and the reduced form is

$$\mathbf{f} = \mathbf{R}\alpha\left(\mathbf{U}\mathbf{U}^{\mathrm{T}}\right)^2\mathbf{R}^{\mathrm{T}} = \alpha\mathbf{R}\mathbf{U}^4\mathbf{R}^{\mathrm{T}}$$

Also, since $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, alternative reduced forms are

$$\mathbf{f} = \mathbf{R}\mathbf{f}_2(\mathbf{C})\mathbf{R}^{\mathrm{T}}, \quad \mathbf{f} = \mathbf{R}\mathbf{f}_3(\mathbf{E})\mathbf{R}^{\mathrm{T}}$$

■

Finally, consider a *spatial* tensor function **f** of a *material* tensor **T**. Then

$$\mathbf{f}^*(\mathbf{T}) = \mathbf{Q}\mathbf{f}(\mathbf{T})\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{f}^*(\mathbf{T}) = \mathbf{f}(\mathbf{T}^*) = \mathbf{f}(\mathbf{T}) \tag{2.8.48}$$

It follows that

$$\mathbf{f} = \mathbf{Q}\mathbf{f}\mathbf{Q}^{\mathrm{T}} \tag{2.8.49}$$

This is true only in the special case $\mathbf{Q} = \mathbf{I}$ and so is not true in general. It follows that the function **f** is not objective.

2.8.7 Problems

1. Derive the relations 2.8.13
2. Show that the spatial gradient of a scalar ϕ is objective.
3. Show that the divergence of a spatial vector **v** is objective. [Hint: use the definition 1.11.9 and identity 1.9.10e]
4. Verify that the Rivlin-Cotter rate of a tensor **T**, $\dot{\mathbf{T}} + \mathbf{l}^{\mathrm{T}}\mathbf{T} + \mathbf{T}\mathbf{l}$, is objective.