2.1 Motion

2.1.1 The Material Body and Motion

Physical materials in the real world are modeled using an abstract mathematical entity called a **body**. This body consists of an infinite number of **material particles**[1]. Shown in Fig. 2.1.1a is a body *B* with material particle *P*. One distinguishes between this body and the space in which it resides and through which it travels. Shown in Fig. 2.1.1b is a certain **point** $\mathbf{x}$ in Euclidean point space *E*.

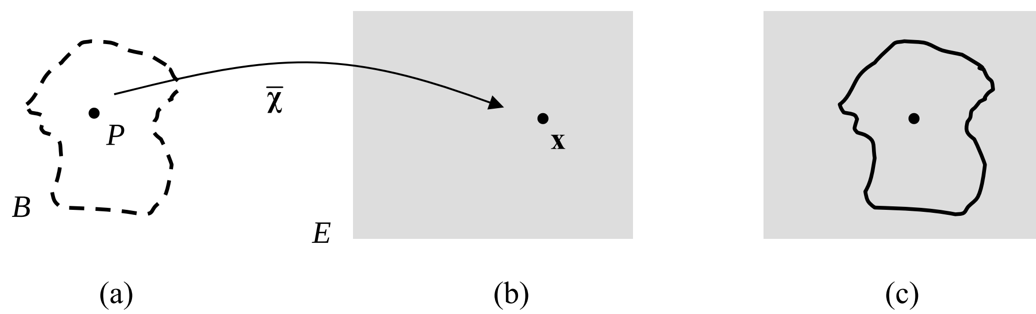

Figure 2.1.1: (a) a material particle in a body, (b) a place in space, (c) a configuration of the body

By fixing the material particles of the body to points in space, one has a **configuration** of the body $\bar{\chi}$, Fig. 2.1.1c. A configuration can be expressed as a mapping of the particles P to the point $\mathbf{x}$,

$$\mathbf{x} = \bar{\chi}(P) \tag{2.1.1}$$

A **motion** of the body is a *family* of configurations parameterised by time *t*,

$$\mathbf{x} = \bar{\chi}(P, t) \tag{2.1.2}$$

At any time *t*, Eqn. 2.1.2 gives the location in space $\mathbf{x}$ of the material particle P, Fig. 2.1.2.

[1] these particles are not the discrete mass particles of Newtonian mechanics, rather they are very small portions of continuous matter; the meaning of particle is made precise in the definitions which follow

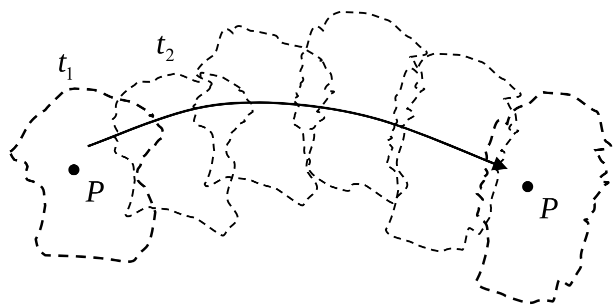

Figure 2.1.2: a motion of material

The Reference and Current Configurations

Choose now some **reference configuration**, Fig. 2.1.3. The motion can then be measured relative to this configuration. The reference configuration might be the configuration occupied by the material at time $t = 0$, in which case it is often called the **initial configuration**. For a solid, it might be natural to choose a configuration for which the material is stress-free, in which case it is often called the **undeformed configuration**. However, the choice of reference configuration is completely arbitrary.

Introduce a Cartesian coordinate system with base vectors $\mathbf{E}_i$ for the reference configuration. A material particle P in the reference configuration can then be assigned a unique position vector $\mathbf{X} = X_i \mathbf{E}_i$ relative to the origin of the axes. The coordinates (X_1, X_2, X_3) of the particle are called **material coordinates** (or **Lagrangian coordinates** or **referential coordinates**).

Some time later, say at time *t*, the material occupies a different configuration, which will be called the **current configuration** (or **deformed configuration**). Introduce a second Cartesian coordinate system with base vectors $\mathbf{e}_i$ for the current configuration, Fig. 2.1.3. In the current configuration, the same particle P now occupies the location $\mathbf{x}$, which can now also be assigned a position vector $\mathbf{x} = x_i \mathbf{e}_i$. The coordinates (x_1, x_2, x_3) are called **spatial coordinates** (or **Eulerian coordinates**).

Each particle thus has two sets of coordinates associated with it. The particle's material coordinates stay with it throughout its motion. The particle's spatial coordinates change as it moves.

The coordinate systems do not have to be Cartesian. For example, suppose one has a rectangular block which deforms into a curved beam (part of a circle). In that case it would be sensible to employ a rectangular Cartesian coordinate system with coordinates (X_1, X_2, X_3) to describe the reference configuration, and a polar coordinate system (r, θ, z) to describe the current configuration.

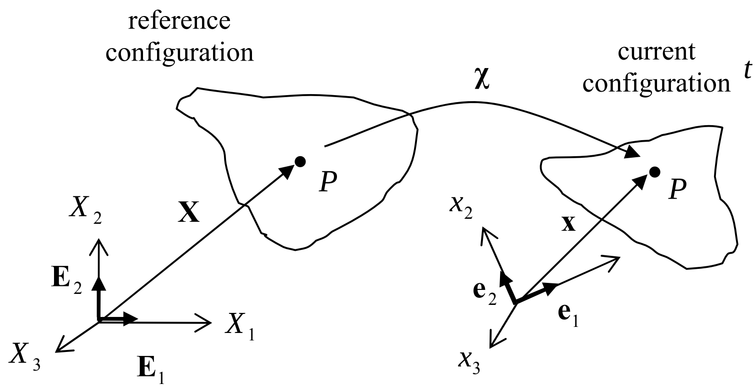

Figure 2.1.3: reference and current configurations

In practice, the material and spatial axes are usually taken to be coincident so that the base vectors $\mathbf{E}_i$ and $\mathbf{e}_i$ are the same, as in Fig. 2.1.4. Nevertheless, the use of different base vectors $\mathbf{E}$ and $\mathbf{e}$ for the reference and current configurations is useful even when the material and spatial axes are coincident, since it helps distinguish between quantities associated with the reference configuration and those associated with the spatial configuration (see later).

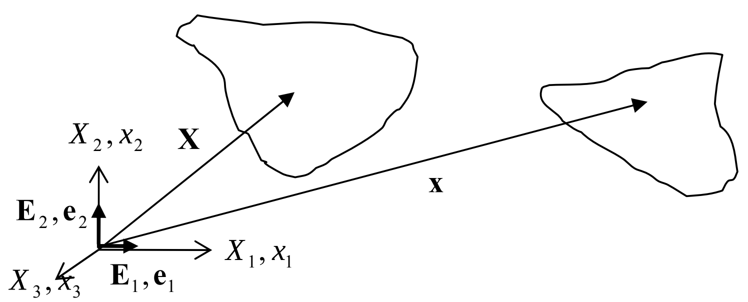

Figure 2.1.4: reference and current configurations with coincident axes

In terms of the position vectors, the motion 2.1.2 can be expressed as a relationship between the material and spatial coordinates,

$$\boxed{\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad x_i = \chi_i\left(X_1, X_2, X_3, t\right)} \qquad \textbf{Material description} \qquad (2.1.3)$$

or the inverse relation

$$\boxed{\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \qquad X_i = \chi_i^{-1}\left(x_1, x_2, x_3, t\right)} \qquad \textbf{Spatial description} \qquad (2.1.4)$$

If one knows the material coordinates of a particle then its position in the current configuration can be determined from 2.1.3. Alternatively, if one focuses on some location in space, in the current configuration, then the material particle occupying that position can be determined from 2.1.4. This is illustrated in the following example.

Example (Extension of a Bar)

Consider the motion

$$x_1 = 3X_1 t + X_1 + t, \quad x_2 = X_2, \quad x_3 = X_3 \tag{2.1.5}$$

These equations are of the form 2.1.3 and say that "the particle that was originally at position **X** is now, at time *t*, at position **x**". They represent a simple translation and uniaxial extension of material as shown in Fig. 2.1.5. Note that $\mathbf{X} = \mathbf{x}$ at $t = 0$.

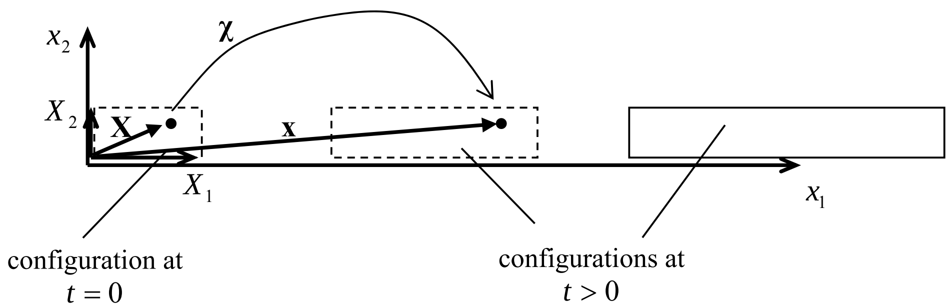

Figure 2.1.5: translation and extension of material

Relations of the form 2.1.4 can be obtained by inverting 2.1.5:

$$X_1 = \frac{x_1 - t}{1 + 3t}, \quad X_2 = x_2, \quad X_3 = x_3$$

These equations say that "the particle that is now, at time *t*, at position **x** was originally at position **X**".

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Convected Coordinates

The material and spatial coordinate systems used here are fixed Cartesian systems. An alternative method of describing a motion is to *attach* the material coordinate system to the material and let it deform with the material. The motion is then described by defining how this coordinate system changes. This is the **convected coordinate system**. In general, the axes of a convected system will not remain mutually orthogonal and a curvilinear system is required. Convected coordinates will be examined in §2.10.

2.1.2 The Material and Spatial Descriptions

Any physical property (such as density, temperature, etc.) or kinematic property (such as displacement or velocity) of a body can be described in terms of either the material coordinates **X** or the spatial coordinates **x**, since they can be transformed into each other using 2.1.3-4. A **material** (or **Lagrangian**) **description** of events is one where the material coordinates are the independent variables. A **spatial** (or **Eulerian**) description of events is one where the spatial coordinates are used.

Example (Temperature of a Body)

Suppose the temperature θ of a body is, in material coordinates,

$$\theta(\mathbf{X},t) = 3X_1 - X_3 \tag{2.1.6}$$

but, in the spatial description,

$$\theta(\mathbf{x},t) = \frac{x_1}{t} - 1 - x_3 \,. \tag{2.1.7}$$

According to the material description 2.1.6, the temperature is different for different particles, but the temperature of each particle remains constant over time. The spatial description 2.1.7 describes the time-dependent temperature at a specific location in space, **x**, Fig. 2.1.6. Different material particles are flowing through this location over time.

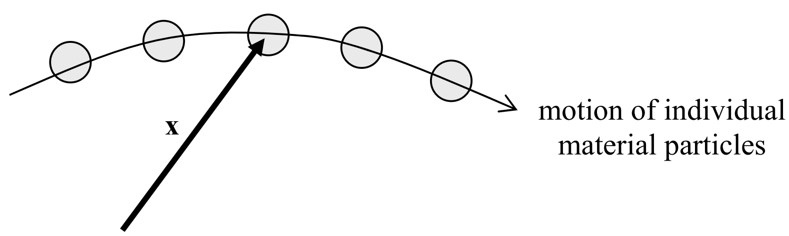

Figure 2.1.6: particles flowing through space

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In the material description, then, attention is focused on specific *material*. The piece of matter under consideration may change shape, density, velocity, and so on, but it is always the same piece of material. On the other hand, in the spatial description, attention is focused on a fixed location in *space*. Material may pass through this location during the motion, so different material is under consideration at different times.

The spatial description is the one most often used in Fluid Mechanics since there is no natural reference configuration of the material as it is continuously moving. However, both the material and spatial descriptions are used in Solid Mechanics, where the reference configuration is usually the stress-free configuration.

2.1.3 Small Perturbations

A large number of important problems involve materials which deform only by a relatively small amount. An example would be the steel structural columns in a building under modest loading. In this type of problem there is virtually no distinction to be made between the two viewpoints taken above and the analysis is simplified greatly (see later, on Small Strain Theory, §2.7).

2.1.4 Problems

1. The density of a material is given by $\rho = 3X_1 + X_2$ and the motion is given by the equations $X_1 = x_1,\ X_2 = x_2 - t,\ X_3 = x_3 - t$.
   (a) what kind of description is this for the density, and what kind of description is this for the motion?
   (b) re-write the density in terms of **x** – what is the name given to this description of the density?
   (c) is the density of any given material particle changing with time?
   (d) invert the motion equations so that **X** is the independent variable – what is the name given to this description of the motion?
   (e) draw the line element joining the origin to $(1,1,0)$ and sketch the position of this element of material at times $t = 1$ and $t = 2$.