1.B Appendix to Chapter 1

1.B.1 The Ordinary Calculus

Here are listed some important concepts from the ordinary calculus.

The Derivative

Consider *u*, a function *f* of one independent variable *x*. The *derivative* of *u* at *x* is defined by

$$\frac{du}{dx} \equiv f'(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (1.B.1)$$

where Δu is the **increment** in *u* due to an increment Δx in *x*.

The Differential

The **differential** of *u* is defined by

$$du = f'(x)\Delta x \qquad (1.B.2)$$

By considering the special case of $u = f(x) = x$, one has $du = dx = \Delta x$, so the differential of the independent variable is equivalent to the increment. $dx = \Delta x$. Thus, in general, the differential can be written as $du = f'(x)dx$. The differential of *u* and increment in *u* are only approximately equal, $du \approx \Delta u$, and approach one another as $\Delta x \to 0$. This is illustrated in Fig. 1.B.1.

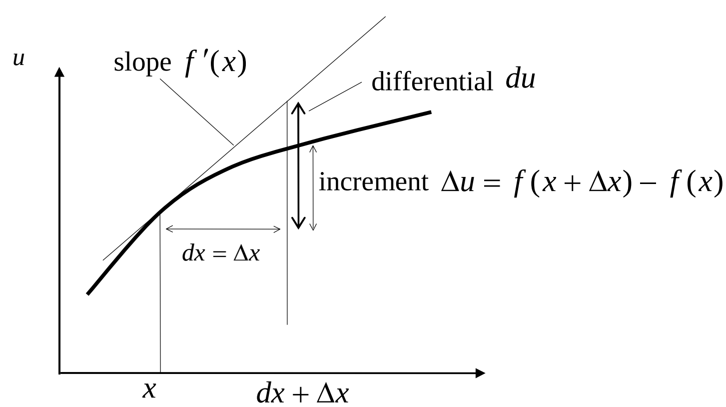

Figure 1.B.1: the differential

If *x* is itself a function of another variable, *t* say, $u(x(t))$, then the **chain rule** of differentiation gives

$$\frac{du}{dt} = f'(x)\frac{dx}{dt} \tag{1.B.3}$$

Arc Length

The length of an arc, measured from a fixed point *a* on the arc, to *x*, is, from the definition of the integral,

$$s = \int_a^x ds = \int_a^x \sec\psi dx = \int_a^x \sqrt{1+(dy/dx)^2}\,dx \tag{1.B.4}$$

where ψ is the angle the tangent to the arc makes with the *x* axis, Fig 1.B.2, with $(dy/dx) = \tan\psi$ and $(ds)^2 = (dx)^2 + (dy)^2$ (ds is the length of the dotted line in Fig. 1.B.2b). Also, it can be seen that

$$\begin{aligned}\lim_{\Delta s\to 0}\frac{|pq|_{\text{chord}}}{|pq|_{\text{arc}}} &= \lim_{\Delta s\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta s} \\ &= \lim_{\Delta s\to 0}\sqrt{\left(\frac{\Delta x}{\Delta s}\right)^2+\left(\frac{\Delta y}{\Delta s}\right)^2} \\ &= \sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2} = 1\end{aligned} \tag{1.B.5}$$

so that, if the increment Δs is small, $(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$.

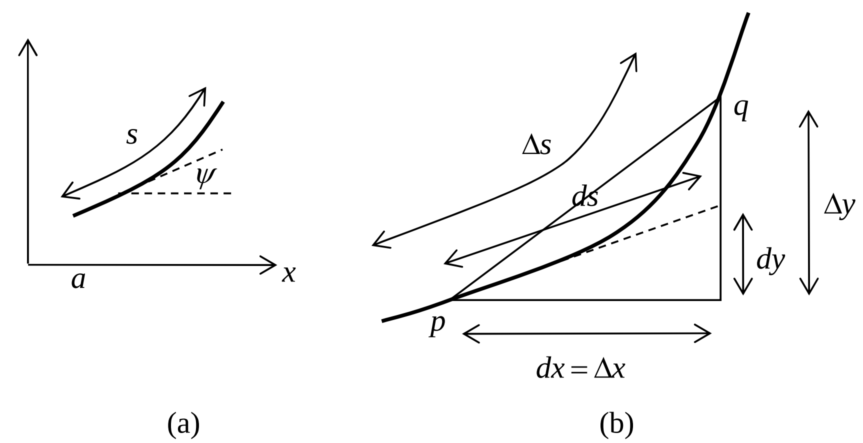

Figure 1.B.2: arc length

The Calculus of Two or More Variables

Consider now two independent variables, $u = f(x, y)$. We can define **partial derivatives** so that, for example,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \left. \frac{\Delta u}{\Delta x} \right|_{y \text{ constant}} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{1.B.6}$$

The **total differential** du due to increments in both *x* and *y* can in this case be shown to be

$$du = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \tag{1.B.7}$$

which is written as $du = (\partial u / \partial x)dx + (\partial u / \partial y)dy$, by setting $dx = \Delta x, dy = \Delta y$. Again, the differential du is only an approximation to the actual increment Δu (the increment and differential are shown in Fig. 1.B.3 for the case $dy = \Delta y = 0$).

It can be shown that this expression for the differential du holds whether *x* and *y* are independent, or whether they are functions themselves of an independent variable *t*, $u \equiv u(x(t), y(t))$, in which case one has the total derivative of *u* with respect to *t*,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \tag{1.B.8}$$

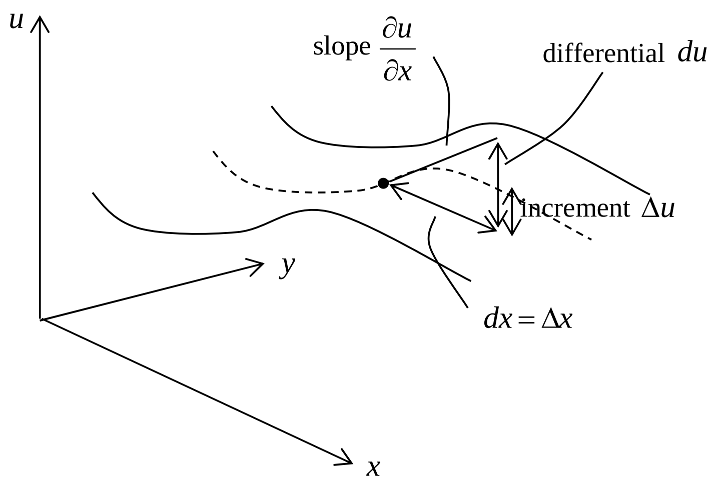

Figure 1.B.3: the partial derivative

The Chain rule for Two or More Variables

Consider the case where *u* is a function of the two variables x, y, $u = f(x, y)$, but also that *x* and *y* are functions of the two independent variables *s* and *t*, $u = f(x(s,t), y(s,t))$. Then

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt\right) \\ &= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)ds + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)dt \end{aligned} \tag{1.B.9}$$

But, also,

$$du = \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt \tag{1.B.10}$$

Comparing the two, and since ds, dt are independent and arbitrary, one obtains the chain rule

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \end{aligned} \tag{1.B.11}$$

In the special case when x and y are functions of only *one* variable, t say, so that $u = f[x(t), y(t)]$, the above reduces to the total derivative given earlier.

One can further specialise: In the case when u is a function of x and t, with $x = x(t)$, $u = f[x(t), t]$, one has

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} \tag{1.B.12}$$

When u is a function of one variable only, x say, so that $u = f[x(t)]$, the above reduces to the chain rule for ordinary differentiation.

Taylor's Theorem

Suppose the value of a function $f(x, y)$ is known at (x_0, y_0). Its value at a neighbouring point $(x_0 + \Delta x, y_0 + \Delta y)$ is then given by

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) &+ \left(\Delta x \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} + \Delta y \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}\right) \\ &+ \frac{1}{2}\left((\Delta x)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0, y_0)} + \Delta x \Delta y \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0, y_0)} + (\Delta x)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0, y_0)}\right) + \cdots \end{aligned} \tag{1.B.13}$$

The Mean Value Theorem

If $f(x)$ is continuous over an interval $a < x < b$, then

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \tag{1.B.14}$$

Geometrically, this is equivalent to saying that there exists at least one point in the interval for which the tangent line is parallel to the line joining $f(a)$ and $f(b)$. This result is known as the **mean value theorem**.

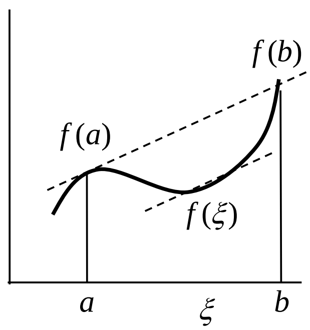

Figure 1.B.4: the mean value theorem

The law of the mean can also be written in terms of an integral: there is at least one point ξ in the interval $[a,b]$ such that

$$f(\xi) = \frac{1}{l}\int_a^b f(x)dx \tag{1.B.15}$$

where *l* is the length of the interval, $l = b - a$. The right hand side here can be interpreted as the average value of *f* over the interval. The theorem therefore states that the average value of the function lies somewhere in the interval. The equivalent expression for a double integral is that there is at least one point (ξ_1, ξ_2) in a region *R* such that

$$f(\xi_1, \xi_2) = \frac{1}{A}\iint_R f(x_1, x_2)dx_1 dx_2 \tag{1.B.16}$$

where *A* is the area of the region of integration *R*, and similarly for a triple/volume integral.

1.B.2 Transformation of Coordinate System

Let the coordinates of a point in space be (x_1, x_2, x_3). Introduce a second set of coordinates $(\Theta_1, \Theta_2, \Theta_3)$, related to the first set through the transformation equations

$$\Theta_i = f_i(x_1, x_2, x_3) \tag{1.B.17}$$

with the inverse equations

$$x_i = g_i(\Theta_1, \Theta_2, \Theta_3) \tag{1.B.18}$$

A transformation is termed an **admissible transformation** if the inverse transformation exists and is in one-to-one correspondence in a certain region of the variables (x_1, x_2, x_3), that is, each set of numbers $(\Theta_1, \Theta_2, \Theta_3)$ defines a unique set (x_1, x_2, x_3) in the region, and *vice versa*.

Now suppose that one has a point with coordinates x_i^0, Θ_i^0 which satisfy 1.B.17. Eqn. 1.B.17 will be in general non-linear, but differentiating leads to

$$d\Theta_i = \frac{\partial f_i}{\partial x_j} dx_j , \tag{1.B.19}$$

which is a system of three linear equations. From basic linear algebra, this system can be solved for the dx_j if and only if the determinant of the coefficients does not vanish, i.e

$$J = \det\left[\frac{\partial f_i}{\partial x_j}\right] \neq 0 , \tag{1.B.20}$$

with the partial derivatives evaluated at x_i^0 (the one dimensional situation is shown in Fig. 1.B.5). If $J \neq 0$, one can solve for the dx_i:

$$dx_i = A_{ij} d\Theta_j , \tag{1.B.21}$$

say. This is a linear approximation of the inverse equations 1.B.18 and so the inverse exists in a small region near (x_1^0, x_2^0, x_3^0). This argument can be extended to other neighbouring points and the region in which $J \neq 0$ will be the region for which the transformation will be admissible.

If the Jacobian is positive everywhere, then a right handed set will be transformed into another right handed set, and the transformation is said to be **proper**.

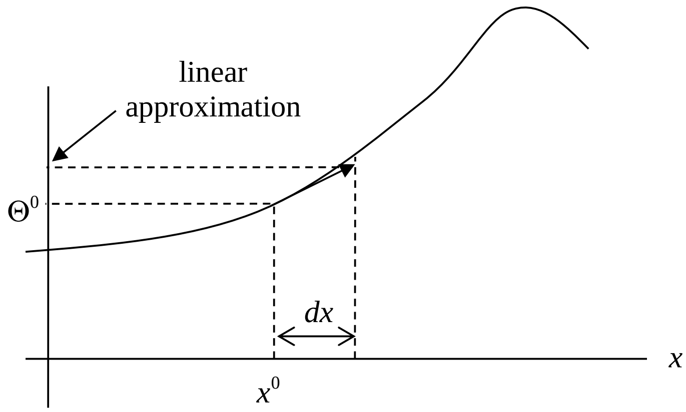

Figure 1.B.5: linear approximation