1.17 Curvilinear Coordinates: Transformation Laws

1.17.1 Coordinate Transformation Rules

Suppose that one has a second set of curvilinear coordinates $(\overline{\Theta}^1, \overline{\Theta}^2, \overline{\Theta}^3)$, with

$$\Theta^i = \Theta^i(\overline{\Theta}^1, \overline{\Theta}^2, \overline{\Theta}^3), \qquad \overline{\Theta}^i = \overline{\Theta}^i(\Theta^1, \Theta^2, \Theta^3) \tag{1.17.1}$$

By the chain rule, the covariant base vectors in the second coordinate system are given by

$$\overline{\mathbf{g}}_i = \frac{\partial \mathbf{x}}{\partial \overline{\Theta}^i} = \frac{\partial \Theta^j}{\partial \overline{\Theta}^i} \frac{\partial \mathbf{x}}{\partial \Theta^j} = \frac{\partial \Theta^j}{\partial \overline{\Theta}^i} \mathbf{g}_j$$

A similar calculation can be carried out for the inverse relation and for the contravariant base vectors, giving

$$\begin{aligned} \overline{\mathbf{g}}_i &= \frac{\partial \Theta^j}{\partial \overline{\Theta}^i} \mathbf{g}_j, \qquad & \mathbf{g}_i &= \frac{\partial \overline{\Theta}^j}{\partial \Theta^i} \overline{\mathbf{g}}_j \\ \overline{\mathbf{g}}^i &= \frac{\partial \overline{\Theta}^i}{\partial \Theta^j} \mathbf{g}^j, \qquad & \mathbf{g}^i &= \frac{\partial \Theta^i}{\partial \overline{\Theta}^j} \overline{\mathbf{g}}^j \end{aligned} \tag{1.17.2}$$

The coordinate transformation formulae for vectors **u** can be obtained from $\mathbf{u} = u^i \mathbf{g}_i = \overline{u}^i \overline{\mathbf{g}}_i$ and $\mathbf{u} = u_i \mathbf{g}^i = \overline{u}_i \overline{\mathbf{g}}^i$:

$$\boxed{\begin{aligned} \overline{u}^i &= \frac{\partial \overline{\Theta}^i}{\partial \Theta^j} u^j, \qquad & u^i &= \frac{\partial \Theta^i}{\partial \overline{\Theta}^j} \overline{u}^j \\ \overline{u}_i &= \frac{\partial \Theta^j}{\partial \overline{\Theta}^i} u_j, \qquad & u_i &= \frac{\partial \overline{\Theta}^j}{\partial \Theta^i} \overline{u}_j \end{aligned}} \quad \textbf{Vector Transformation Rule} \tag{1.17.3}$$

These transformation laws have a simple structure and pattern – the subscripts/superscripts on the transformed coordinates $\overline{\Theta}$ quantities match those on the transformed quantities, $\overline{u}, \overline{\mathbf{g}}$, and similarly for the first coordinate system.

<u>Note:</u>

- Covariant and contravariant vectors (and other quantities) are often *defined* in terms of the transformation rules which they obey. For example, a covariant vector can be defined as one whose components transform according to the rules in the second line of the box Eqn. 1.17.3

The transformation laws can be extended to higher-order tensors,

$$\begin{aligned}
\overline{A}_{ij} &= \frac{\partial\Theta^m}{\partial\overline{\Theta}^i}\frac{\partial\Theta^n}{\partial\overline{\Theta}^j}A_{mn}, & A_{ij} &= \frac{\partial\overline{\Theta}^m}{\partial\Theta^i}\frac{\partial\overline{\Theta}^n}{\partial\Theta^j}\overline{A}_{mn} \\
\overline{A}^{ij} &= \frac{\partial\overline{\Theta}^i}{\partial\Theta^m}\frac{\partial\overline{\Theta}^j}{\partial\Theta^n}A^{mn}, & A^{ij} &= \frac{\partial\Theta^i}{\partial\overline{\Theta}^m}\frac{\partial\Theta^j}{\partial\overline{\Theta}^n}\overline{A}^{mn} \\
\overline{A}^i_{\cdot j} &= \frac{\partial\overline{\Theta}^i}{\partial\Theta^m}\frac{\partial\Theta^n}{\partial\overline{\Theta}^j}A^m_{\cdot n}, & A^i_{\cdot j} &= \frac{\partial\Theta^i}{\partial\overline{\Theta}^m}\frac{\partial\overline{\Theta}^n}{\partial\Theta^j}\overline{A}^m_{\cdot n} \\
\overline{A}_i^{\cdot j} &= \frac{\partial\overline{\Theta}^j}{\partial\Theta^n}\frac{\partial\Theta^m}{\partial\overline{\Theta}^i}A_m^{\cdot n}, & A_i^{\cdot j} &= \frac{\partial\Theta^j}{\partial\overline{\Theta}^n}\frac{\partial\overline{\Theta}^m}{\partial\Theta^i}\overline{A}_m^{\cdot n}
\end{aligned}$$

Tensor Transformation Rule (1.17.4)

From these transformation expressions, the following important theorem can be deduced:

If the tensor components are zero in any one coordinate system, they also vanish in any other coordinate system

Reduction to Cartesian Coordinates

For the Cartesian system, let $\mathbf{e}_i = \mathbf{g}_i = \mathbf{g}^i$, $\mathbf{e}'_i = \overline{\mathbf{g}}_i = \overline{\mathbf{g}}^i$ and

$$Q_{ij} = \frac{\partial\Theta^i}{\partial\overline{\Theta}^j} = \frac{\partial x_i}{\partial x'_j} \tag{1.17.5}$$

It follows from 1.17.2 that

$$\frac{\partial\Theta^j}{\partial\overline{\Theta}^i} = \frac{\partial\overline{\Theta}^i}{\partial\Theta^j} \quad \rightarrow \quad Q_{ji} = Q_{ij}^{-1} \tag{1.17.6}$$

so the transformation is orthogonal, as expected. Also, as in Eqns. 1.5.11 and 1.5.13.

$$\begin{aligned}
u^i &= \frac{\partial\Theta^i}{\partial\overline{\Theta}^j}\overline{u}^j \quad \rightarrow \quad u_i = Q_{ij}u'_j \\
\overline{u}^i &= \frac{\partial\overline{\Theta}^i}{\partial\Theta^j}u^j \quad \rightarrow \quad u'_i = Q_{ij}^{-1}u_j = Q_{ji}u_j
\end{aligned} \tag{1.17.7}$$

Transformation Matrix

Transforming coordinates from $\mathbf{g}_i \rightarrow \overline{\mathbf{g}}_i$, one can write

$$\mathbf{g}_i = M_i^{\cdot j}\overline{\mathbf{g}}_j = \left(\mathbf{g}_i \cdot \overline{\mathbf{g}}^j\right)\overline{\mathbf{g}}_j \tag{1.17.8}$$

The transformation for a vector can then be expressed, in index notation and matrix notation, as

$$v_i = M_i^{\cdot j}\overline{v}_j, \qquad [v_i] = \left[M_i^{\cdot j}\right]\left[\overline{v}_j\right] \tag{1.17.9}$$

and the transformation matrix is

$$\boxed{\left[M_i^{\cdot j}\right] = \left[\frac{\partial\overline{\Theta}^j}{\partial\Theta^i}\right] = \left[\mathbf{g}_i \cdot \overline{\mathbf{g}}^j\right]} \qquad \textbf{Transformation Matrix} \qquad (1.17.10)$$

The rule for contravariant components is then, from 1.17.4,

$$\overline{A}^{ij} = M_m^{\cdot i} M_n^{\cdot j} A^{mn}, \qquad \left[\overline{A}^{ij}\right] = \left[M_m^{\cdot i}\right]^{\mathrm{T}} \left[A^{mn}\right]\left[M_n^{\cdot j}\right] \qquad (1.17.11)$$

The Identity Tensor

The identity tensor transforms as

$$\mathbf{I} = \delta_j^i \mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{g}_i \otimes \mathbf{g}^i = \frac{\partial\overline{\Theta}^j}{\partial\Theta^i}\frac{\partial\Theta^i}{\partial\overline{\Theta}^k}\overline{\mathbf{g}}_j \otimes \overline{\mathbf{g}}^k = \delta_k^j \overline{\mathbf{g}}_j \otimes \overline{\mathbf{g}}^k = \overline{\mathbf{g}}_i \otimes \overline{\mathbf{g}}^i \qquad (1.17.12)$$

Note that

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial\overline{\Theta}^m}{\partial\Theta^i}\frac{\partial\overline{\Theta}^n}{\partial\Theta^j}\overline{\mathbf{g}}_m \cdot \overline{\mathbf{g}}_n = \frac{\partial\overline{\Theta}^m}{\partial\Theta^i}\frac{\partial\overline{\Theta}^n}{\partial\Theta^j}\overline{g}_{mn} \\ \overline{g}_{ij} &= \overline{\mathbf{g}}_i \cdot \overline{\mathbf{g}}_j = \frac{\partial\Theta^m}{\partial\overline{\Theta}^i}\frac{\partial\Theta^n}{\partial\overline{\Theta}^j}\mathbf{g}_m \cdot \mathbf{g}_n = \frac{\partial\Theta^m}{\partial\overline{\Theta}^i}\frac{\partial\Theta^n}{\partial\overline{\Theta}^j}g_{mn} \end{aligned} \qquad (1.17.13)$$

so that, for example,

$$\mathbf{I} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \frac{\partial\overline{\Theta}^m}{\partial\Theta^i}\frac{\partial\overline{\Theta}^n}{\partial\Theta^j}\overline{g}_{mn}\frac{\partial\Theta^i}{\partial\overline{\Theta}^m}\overline{\mathbf{g}}^m \frac{\partial\Theta^j}{\partial\overline{\Theta}^n} \otimes \overline{\mathbf{g}}^n = \overline{g}_{mn}\overline{\mathbf{g}}^m \otimes \overline{\mathbf{g}}^n \qquad (1.17.14)$$

1.17.2 The Metric of the Space

In a second coordinate system, the metric 1.16.38 transforms to

$$\begin{aligned} \overline{(\Delta s)^2} &= \overline{g}_{ij}\overline{\Delta\Theta}^i\overline{\Delta\Theta}^j \\ &= \overline{\mathbf{g}_i \cdot \mathbf{g}_j}\,\overline{\Delta\Theta}^i\overline{\Delta\Theta}^j \\ &= \left(\frac{\partial\Theta^k}{\partial\overline{\Theta}^i}\mathbf{g}_k \cdot \frac{\partial\Theta^m}{\partial\overline{\Theta}^j}\mathbf{g}_m\right)\frac{\partial\overline{\Theta}^i}{\partial\Theta^p}\Delta\Theta^p \frac{\partial\overline{\Theta}^j}{\partial\Theta^q}\Delta\Theta^q \\ &= \delta_p^k \delta_q^m \left(\mathbf{g}_k \cdot \mathbf{g}_m\right)\Delta\Theta^p \Delta\Theta^q \\ &= g_{pq}\Delta\Theta^p\Delta\Theta^q \\ &= (\Delta s)^2 \end{aligned} \qquad (1.17.15)$$

confirming that the metric is a scalar invariant.

1.17.3 Problems

1 Show that $g_{mn}u^m v^n$ is an invariant.

2 How does $\sqrt{g}$ transform between different coordinate systems (in terms of the Jacobian of the transformation, $J = \det\left[\partial\Theta^m / \partial\overline{\Theta}^p\right]$)? [Note that g, although a scalar, is not invariant; it is thus called a **pseudoscalar**.]

3 The components A_{ij} of a tensor **A** are

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

in cylindrical coordinates, at the point $r = 1, \theta = \pi / 4, z = \sqrt{3}$. Find the contravariant components of **A** at this point in spherical coordinates. [Hint: use matrix multiplication.]