1.13 Coordinate Transformation of Tensor Components

This section generalises the results of §1.5, which dealt with vector coordinate transformations. It has been seen in §1.5.2 that the transformation equations for the components of a vector are $u_i = Q_{ij} u'_j$, where $[\mathbf{Q}]$ is the transformation matrix. Note that these Q_{ij}'s are *not the components of a tensor* – these Q_{ij}'s are mapping the components of a vector onto the components of the *same vector* in a second coordinate system – a (second-order) tensor, in general, maps one vector onto a different vector. The equation $u_i = Q_{ij} u'_j$ is in matrix element form, and is not to be confused with the index notation for vectors and tensors.

1.13.1 Relationship between Base Vectors

Consider two coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$. It has been seen in the context of vectors that, Eqn. 1.5.9,

$$\mathbf{e}_i \cdot \mathbf{e}'_j = Q_{ij} \equiv \cos(x_i, x'_j). \tag{1.13.1}$$

Recal that the *i*'s and *j*'s here are not referring to the three different components of a vector, but to *different* vectors (nine different vectors in all).

Note that the relationship 1.13.1 can also be derived as follows:

$$\begin{aligned} \mathbf{e}_i = \mathbf{I}\mathbf{e}_i &= (\mathbf{e}'_k \otimes \mathbf{e}'_k)\mathbf{e}_i \\ &= (\mathbf{e}'_k \cdot \mathbf{e}_i)\mathbf{e}'_k \\ &= Q_{ik}\mathbf{e}'_k \end{aligned} \tag{1.13.2}$$

Dotting each side here with $\mathbf{e}'_j$ then gives 1.13.1. Eqn. 1.13.2, together with the corresponding inverse relations, read

$$\mathbf{e}_i = Q_{ij}\mathbf{e}'_j, \qquad \mathbf{e}'_i = Q_{ji}\mathbf{e}_j \tag{1.13.3}$$

Note that the components of the transformation matrix $[\mathbf{Q}]$ are the same as the components of the change of basis tensor 1.10.24-25.

1.13.2 Tensor Transformation Rule

As with vectors, the components of a (second-order) tensor will change under a change of coordinate system. In this case, using 1.13.3,

$$\begin{aligned} T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j &\equiv T'_{pq}\mathbf{e}'_p \otimes \mathbf{e}'_q \\ &= T'_{pq} Q_{mp}\mathbf{e}_m \otimes Q_{nq}\mathbf{e}_n \\ &= Q_{mp} Q_{nq} T'_{pq}\mathbf{e}_m \otimes \mathbf{e}_n \end{aligned} \tag{1.13.4}$$

so that (and the inverse relationship)

$$\boxed{T_{ij} = Q_{ip} Q_{jq} T'_{pq}, \quad T'_{ij} = Q_{pi} Q_{qj} T_{pq}}$$ **Tensor Transformation Formulae** (1.13.5)

or, in matrix form,

$$[\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}], \qquad [\mathbf{T}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{T}][\mathbf{Q}] \tag{1.13.6}$$

Note:
- as with vectors, second-order tensors are often *defined* as mathematical entities whose components transform according to the rule 1.13.5.
- the transformation rule for higher order tensors can be established in the same way, for example, $T'_{ijk} = Q_{pi} Q_{qj} Q_{rk} T_{pqr}$, and so on.

Example (Mohr Transformation)

Consider a two-dimensional space with base vectors $\mathbf{e}_1, \mathbf{e}_2$. The second order tensor $\mathbf{S}$ can be written in component form as

$$\mathbf{S} = S_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + S_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + S_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + S_{22}\mathbf{e}_2 \otimes \mathbf{e}_2$$

Consider now a second coordinate system, with base vectors $\mathbf{e}'_1, \mathbf{e}'_2$, obtained from the first by a rotation θ. The components of the transformation matrix are

$$Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}'_1 & \mathbf{e}_1 \cdot \mathbf{e}'_2 \\ \mathbf{e}_2 \cdot \mathbf{e}'_1 & \mathbf{e}_2 \cdot \mathbf{e}'_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90+\theta) \\ \cos(90-\theta) & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and the components of $\mathbf{S}$ in the second coordinate system are $[\mathbf{S}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{S}][\mathbf{Q}]$, so

$$\begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For $\mathbf{S}$ symmetric, $S_{12} = S_{21}$, and this simplifies to

$$\boxed{\begin{aligned} S'_{11} &= S_{11}\cos^2\theta + S_{22}\sin^2\theta + S_{12}\sin 2\theta \\ S'_{22} &= S_{11}\sin^2\theta + S_{22}\cos^2\theta - S_{12}\sin 2\theta \\ S'_{12} &= (S_{22} - S_{11})\sin\theta\cos\theta + S_{12}\cos 2\theta \end{aligned}}$$ **The Mohr Transformation** (1.13.7)

■

1.13.3 Isotropic Tensors

An **isotropic tensor** is one whose components are the same under arbitrary rotation of the basis vectors, i.e. in any coordinate system.

All scalars are isotropic.

There is no isotropic vector (first-order tensor), i.e. there is no vector **u** such that $u_i = Q_{ij} u_j$ for all orthogonal $[\mathbf{Q}]$ (except for the zero vector **o**). To see this, consider the particular orthogonal transformation matrix

$$[\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (1.13.8)$$

which corresponds to a rotation of $\pi/2$ about $\mathbf{e}_3$. This implies that

$$[u_1 \quad u_2 \quad u_3]^{\mathrm{T}} = [u_2 \quad -u_1 \quad u_3]^{\mathrm{T}}$$

or $u_1 = u_2 = 0$. The matrix corresponding to a rotation of $\pi/2$ about $\mathbf{e}_1$ is

$$[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad (1.13.9)$$

which implies that $u_3 = 0$.

The only isotropic second-order tensor is $\alpha\mathbf{I} \equiv \alpha\delta_{ij}$, where α is a constant, that is, the spherical tensor, §1.10.12. To see this, first note that, by substituting $\alpha\mathbf{I}$ into 1.13.6, it can be seen that it is indeed isotropic. To see that it is the only isotropic second order tensor, first use 1.13.8 in 1.13.6 to get

$$[\mathbf{T}'] = \begin{bmatrix} T_{22} & -T_{21} & -T_{23} \\ -T_{12} & T_{11} & T_{13} \\ -T_{32} & T_{31} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \qquad (1.13.10)$$

which implies that $T_{11} = T_{22}, T_{12} = -T_{21}, T_{13} = T_{23} = T_{31} = T_{32} = 0$. Repeating this for 1.13.9 implies that $T_{11} = T_{33}, T_{12} = 0$, so

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{11} & 0 \\ 0 & 0 & T_{11} \end{bmatrix}$$

or $\mathbf{T} = T_{11}\mathbf{I}$. Multiplying by a scalar does not affect 1.13.6, so one has $\alpha\mathbf{I}$.

The only third-order isotropic tensors are scalar multiples of the permutation tensor, $\mathbf{E} = \varepsilon_{ijk}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$. Using the third order transformation rule, $T'_{ijk} = Q_{pi} Q_{qj} Q_{rk} T_{pqr}$, one has $\varepsilon'_{ijk} = Q_{pi}Q_{qj}Q_{rk}\varepsilon_{pqr}$. From 1.10.16e this reads $\varepsilon'_{ijk} = (\det\mathbf{Q})\varepsilon_{ijk}$, where **Q** is the change of basis tensor, with components Q_{ij}. When **Q** is proper orthogonal, i.e. a rotation tensor, one has indeed, $\varepsilon'_{ijk} = \varepsilon_{ijk}$. That it is the only isotropic tensor can be established by carrying out a few specific rotations as done above for the first and second order tensors.

Note that orthogonal tensors in general, i.e. having the possibility of being reflection tensors, with $\det\mathbf{Q} = -1$, are not used in the definition of isotropy, otherwise one would have the less desirable $\varepsilon'_{ijk} = -\varepsilon_{ijk}$. Note also that this issue does not arise with the second order tensor (or the fourth order tensor –see below), since the above result, that $\alpha\mathbf{I}$ is the only isotropic second order tensor, holds regardless of whether **Q** is proper orthogonal or not.

There are three independent fourth-order isotropic tensors – these are the tensors encountered in §1.12.1, Eqns. 1.12.4-5,

$$\boldsymbol{\mathsf{I}},\quad \bar{\boldsymbol{\mathsf{I}}},\quad \mathbf{I}\otimes\mathbf{I}$$

For example,

$$Q_{ip}Q_{jq}Q_{kr}Q_{ls}(\mathbf{I}\otimes\mathbf{I})_{pqrs} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}\delta_{pq}\delta_{rs} = (Q_{ip}Q_{jp})(Q_{kr}Q_{lr}) = \delta_{ij}\delta_{kl} = (\mathbf{I}\otimes\mathbf{I})_{ijkl}$$

The most general isotropic fourth order tensor is then a linear combination of these tensors:

$$\boxed{\boldsymbol{\mathsf{C}} = \lambda\mathbf{I}\otimes\mathbf{I} + \mu\boldsymbol{\mathsf{I}} + \gamma\bar{\boldsymbol{\mathsf{I}}}} \quad \textbf{Most General Isotropic Fourth-Order Tensor} \qquad (1.13.11)$$

1.13.4 Invariance of Tensor Components

The components of (non-isotropic) tensors will change upon a rotation of base vectors. However, certain combinations of these components are the same in *every* coordinate system. Such quantities are called **invariants**. For example, the following are examples of **scalar invariants** {▲Problem 2}

$$\begin{aligned} \mathbf{a}\cdot\mathbf{a} &= a_i a_i \\ \mathbf{a}\cdot\mathbf{T}\mathbf{a} &= T_{ij}a_i a_j \\ \operatorname{tr}\mathbf{A} &= A_{ii} \end{aligned} \qquad (1.13.12)$$

The first of these is the only independent scalar invariant of a vector. A second-order tensor has three independent scalar invariants, the first, second and third principal scalar invariants, defined by Eqn. 1.11.17 (or linear combinations of these).

1.13.5 Problems

1. Consider a coordinate system $ox_1x_2x_3$ with base vectors $\mathbf{e}_i$. Let a second coordinate system be represented by the set $\{\mathbf{e}'_i\}$ with the transformation law
$$\mathbf{e}'_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$$
$$\mathbf{e}'_3 = \mathbf{e}_3$$
(a) find $\mathbf{e}'_1$ in terms of the old set $\{\mathbf{e}_i\}$ of basis vectors
(b) find the orthogonal matrix $[\mathbf{Q}]$ and express the old coordinates in terms of the new ones
(c) express the vector $\mathbf{u} = -6\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ in terms of the new set $\{\mathbf{e}'_i\}$ of basis vectors.
2. Show that
(a) the trace of a tensor **A**, $\mathrm{tr}\mathbf{A} = A_{ii}$, is an invariant.
(b) $\mathbf{a} \cdot \mathbf{T}\mathbf{a} = T_{ij}a_i a_j$ is an invariant.
3. Consider Problem 7 in §1.11. Take the tensor $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$ with respect to the basis $\{\hat{\mathbf{n}}_i\}$ and carry out a coordinate transformation of its tensor components so that it is given with respect to the original $\{\mathbf{e}_i\}$ basis – in which case the matrix representation for **U** given in Problem 7, §1.11, should be obtained.