1.11 The Eigenvalue Problem and Polar Decomposition

1.11.1 Eigenvalues, Eigenvectors and Invariants of a Tensor

Consider a second-order tensor **A**. Suppose that one can find a scalar λ and a (non-zero) normalised, i.e. unit, vector $\hat{\mathbf{n}}$ such that

$$\mathbf{A}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \tag{1.11.1}$$

In other words, **A** transforms the vector $\hat{\mathbf{n}}$ into a vector parallel to itself, Fig. 1.11.1. If this transformation is possible, the scalars are called the **eigenvalues** (or **principal values**) of the tensor, and the vectors are called the **eigenvectors** (or **principal directions** or **principal axes**) of the tensor. It will be seen that there are *three* vectors $\hat{\mathbf{n}}$ (to each of which corresponds some scalar λ) for which the above holds.

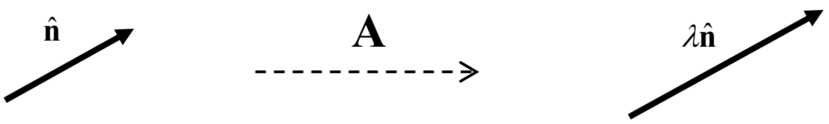

Figure 1.11.1: the action of a tensor A on a unit vector

Equation 1.11.1 can be solved for the eigenvalues and eigenvectors by rewriting it as

$$(\mathbf{A} - \lambda\mathbf{I})\hat{\mathbf{n}} = 0 \tag{1.11.2}$$

or, in terms of a Cartesian coordinate system,

$$\begin{aligned}
&A_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\hat{n}_k\mathbf{e}_k - \lambda\delta_{pq}(\mathbf{e}_p \otimes \mathbf{e}_q)\hat{n}_r\mathbf{e}_r = 0 \\
&\rightarrow A_{ij}\hat{n}_j\mathbf{e}_i - \lambda\hat{n}_r\mathbf{e}_r = 0 \\
&\rightarrow (A_{ij}\hat{n}_j - \lambda\hat{n}_i)\mathbf{e}_i = 0
\end{aligned}$$

In full,

$$\begin{aligned}
&[(A_{11} - \lambda)\hat{n}_1 + A_{12}\hat{n}_2 + A_{13}\hat{n}_3]\mathbf{e}_1 = 0 \\
&[A_{21}\hat{n}_1 + (A_{22} - \lambda)\hat{n}_2 + A_{23}\hat{n}_3]\mathbf{e}_2 = 0 \\
&[A_{31}\hat{n}_1 + A_{32}\hat{n}_2 + (A_{33} - \lambda)\hat{n}_3]\mathbf{e}_3 = 0
\end{aligned} \tag{1.11.3}$$

Dividing out the base vectors, this is a set of three homogeneous equations in three unknowns (if one treats λ as known). From basic linear algebra, this system has a solution (apart from $\hat{n}_i = 0$) if and only if the determinant of the coefficient matrix is zero, i.e. if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{bmatrix} = 0 \qquad (1.11.4)$$

Evaluating the determinant, one has the following cubic **characteristic equation** of **A**,

$$\boxed{\lambda^3 - \mathrm{I}_{\mathbf{A}}\lambda^2 + \mathrm{II}_{\mathbf{A}}\lambda - \mathrm{III}_{\mathbf{A}} = 0} \quad \textbf{Tensor Characteristic Equation} \quad (1.11.5)$$

where

$$\begin{aligned} \mathrm{I}_{\mathbf{A}} &= A_{ii} \\ &= \mathrm{tr}\mathbf{A} \\ \mathrm{II}_{\mathbf{A}} &= \tfrac{1}{2}\left(A_{ii}A_{jj} - A_{ji}A_{ij}\right) \\ &= \tfrac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right] \\ \mathrm{III}_{\mathbf{A}} &= \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} \\ &= \det\mathbf{A} \end{aligned} \qquad (1.11.6)$$

It can be seen that there are three roots $\lambda_1, \lambda_2, \lambda_3$, to the characteristic equation. Solving for λ, one finds that

$$\begin{aligned} \mathrm{I}_{\mathbf{A}} &= \lambda_1 + \lambda_2 + \lambda_3 \\ \mathrm{II}_{\mathbf{A}} &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ \mathrm{III}_{\mathbf{A}} &= \lambda_1\lambda_2\lambda_3 \end{aligned} \qquad (1.11.7)$$

The eigenvalues (principal values) λ_i must be independent of any coordinate system and, from Eqn. 1.11.5, it follows that the functions $\mathrm{I}_{\mathbf{A}}, \mathrm{II}_{\mathbf{A}}, \mathrm{III}_{\mathbf{A}}$ are also independent of any coordinate system. They are called the **principal scalar invariants** (or simply **invariants**) of the tensor.

Once the eigenvalues are found, the eigenvectors (principal directions) can be found by solving

$$\begin{aligned} (A_{11} - \lambda)\hat{n}_1 + A_{12}\hat{n}_2 + A_{13}\hat{n}_3 &= 0 \\ A_{21}\hat{n}_1 + (A_{22} - \lambda)\hat{n}_2 + A_{23}\hat{n}_3 &= 0 \\ A_{31}\hat{n}_1 + A_{32}\hat{n}_2 + (A_{33} - \lambda)\hat{n}_3 &= 0 \end{aligned} \qquad (1.11.8)$$

for the three components of the principal direction vector $\hat{n}_1, \hat{n}_2, \hat{n}_3$, in addition to the condition that $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = \hat{n}_i\hat{n}_i = 1$. There will be three vectors $\hat{\mathbf{n}} = \hat{n}_i\mathbf{e}_i$, one corresponding to each of the three principal values.

Note: a unit eigenvector $\hat{\mathbf{n}}$ has been used in the above discussion, but *any* vector parallel to $\hat{\mathbf{n}}$, for example $\alpha\hat{\mathbf{n}}$, is also an eigenvector (with the same eigenvalue λ):

$$\mathbf{A}(\alpha\hat{\mathbf{n}}) = \alpha(\mathbf{A}\hat{\mathbf{n}}) = \alpha(\lambda\hat{\mathbf{n}}) = \lambda(\alpha\hat{\mathbf{n}})$$

Example (of Eigenvalues and Eigenvectors of a Tensor)

A second order tensor **T** is given with respect to the axes $Ox_1x_2x_3$ by the values

$$\mathbf{T} = [\mathbf{T}]_{ij} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}.$$

Determine (a) the principal values, (b) the principal directions (and sketch them).

<u>Solution:</u>
(a)
The principal values are the solution to the characteristic equation

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -6-\lambda & -12 \\ 0 & -12 & 1-\lambda \end{vmatrix} = (-10+\lambda)(5-\lambda)(15+\lambda) = 0$$

which yields the three principal values $\lambda_1 = 10,\ \lambda_2 = 5,\ \lambda_3 = -15$.
(b)
The eigenvectors are now obtained from $(T_{ij} - \delta_{ij}\lambda)n_j = 0$. First, for $\lambda_1 = 10$,

$$\begin{aligned} -5n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 16n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 9n_3 &= 0 \end{aligned}$$

and using also the equation $n_i n_i = 1$ leads to $\hat{\mathbf{n}}_1 = -(3/5)\mathbf{e}_2 + (4/5)\mathbf{e}_3$. Similarly, for $\lambda_2 = 5$ and $\lambda_3 = -15$, one has, respectively,

$$\begin{aligned} 0n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 11n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 4n_3 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 20n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 + 9n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 + 16n_3 &= 0 \end{aligned}$$

which yield $\hat{\mathbf{n}}_2 = \mathbf{e}_1$ and $\hat{\mathbf{n}}_3 = (4/5)\mathbf{e}_2 + (3/5)\mathbf{e}_3$. The principal directions are sketched in Fig. 1.11.2.

Note: the three components of a principal direction, n_1, n_2, n_3, are the direction cosines between that direction and the three coordinate axes respectively. For example, for λ_1 with $n_1 = 0, n_2 = -3/5, n_3 = 4/5$, the angles made with the coordinate axes x_1, x_2, x_3, are 0, 127° and 37°.

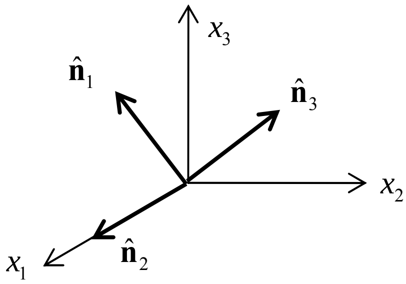

Figure 1.11.2: eigenvectors of the tensor T

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1.11.2 Real Symmetric Tensors

Suppose now that **A** is a *real symmetric* tensor (real meaning that its components are real). In that case it can be proved (see below) that[1]

(i) the eigenvalues are real
(ii) the three eigenvectors form an orthonormal basis $\{\hat{\mathbf{n}}_i\}$.

In that case, the components of **A** can be written relative to the basis of principal directions as (see Fig. 1.11.3)

$$\mathbf{A} = A_{ij}\left(\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j\right) \tag{1.11.9}$$

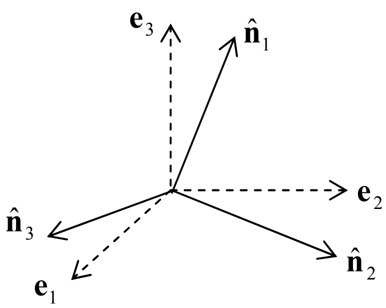

Figure 1.11.3: eigenvectors forming an orthonormal set

The components of **A** in this new basis can be obtained from Eqn. 1.9.4,

$$\begin{aligned} A_{ij} &= \hat{\mathbf{n}}_i \cdot \mathbf{A}\hat{\mathbf{n}}_j \\ &= \hat{\mathbf{n}}_i \cdot \left(\lambda_j \hat{\mathbf{n}}_j\right) \qquad \text{(no summation over } j\text{)} \\ &= \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases} \end{aligned} \tag{1.11.10}$$

where λ_i is the eigenvalue corresponding to the basis vector $\hat{\mathbf{n}}_i$. Thus[2]

[1] this was the case in the previous example – the tensor is real symmetric and the principal directions are orthogonal

$$\boxed{\mathbf{A} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i} \qquad \textbf{Spectral Decomposition} \qquad (1.11.11)$$

This is called the **spectral decomposition** (or **spectral representation**) of **A**. In matrix form,

$$[\mathbf{A}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad (1.11.12)$$

For example, the tensor used in the previous example can be written in terms of the basis vectors in the principal directions as

$$\mathbf{T} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -15 \end{bmatrix}, \qquad \text{basis: } \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_j$$

To prove that real symmetric tensors have real eigenvalues and orthonormal eigenvectors, take $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$ to be the eigenvectors of an arbitrary tensor **A**, with components $\hat{n}_{1i}, \hat{n}_{2i}, \hat{n}_{3i}$, which are solutions of (the 9 equations – see Eqn. 1.11.2)

$$\begin{aligned} (\mathbf{A} - \lambda_1 \mathbf{I})\hat{\mathbf{n}}_1 &= 0 \\ (\mathbf{A} - \lambda_2 \mathbf{I})\hat{\mathbf{n}}_2 &= 0 \\ (\mathbf{A} - \lambda_3 \mathbf{I})\hat{\mathbf{n}}_3 &= 0 \end{aligned} \qquad (1.11.13)$$

Dotting the first of these by $\hat{\mathbf{n}}_1$ and the second by $\hat{\mathbf{n}}_1$, leads to

$$\begin{aligned} (\mathbf{A}\mathbf{n}_1) \cdot \mathbf{n}_2 - \lambda_1 \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 &= 0 \\ (\mathbf{A}\mathbf{n}_2) \cdot \mathbf{n}_1 - \lambda_2 \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 &= 0 \end{aligned}$$

Using the fact that $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, subtracting these equations leads to

$$(\lambda_2 - \lambda_1)\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = 0 \qquad (1.11.14)$$

Assume now that the eigenvalues are not all real. Since the coefficients of the characteristic equation are all real, this implies that the eigenvalues come in a complex conjugate pair, say λ_1 and λ_2, and one real eigenvalue λ_3. It follows from Eqn. 1.11.13 that the components of $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are conjugates of each other, say $\hat{\mathbf{n}}_1 = \mathbf{a} + \mathbf{b}i$, $\hat{\mathbf{n}}_2 = \mathbf{a} - \mathbf{b}i$, and so

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[2] it is necessary to introduce the summation sign here, because the summation convention is only used when *two* indices are the same – it cannot be used when there are more than two indices the same

$$\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = (\mathbf{a} + \mathbf{b}i) \cdot (\mathbf{a} - \mathbf{b}i) = |\mathbf{a}|^2 + |\mathbf{b}|^2 > 0$$

It follows from 1.11.14 that $\lambda_2 - \lambda_1 = 0$ which is a contradiction, since this cannot be true for conjugate pairs. Thus the original assumption regarding complex roots must be false and the eigenvalues are all real. With three distinct eigenvalues, Eqn. 1.11.14 (and similar) show that the eigenvectors form an orthonormal set. When the eigenvalues are not distinct, more than one set of eigenvectors may be taken to form an orthonormal set (see the next subsection).

Equal Eigenvalues

There are some special tensors for which two or three of the principal directions are equal. When all three are equal, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, one has $\mathbf{A} = \lambda\mathbf{I}$, and the tensor is spherical: every direction is a principal direction, since $\mathbf{A}\hat{\mathbf{n}} = \lambda\mathbf{I}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}}$ for all $\hat{\mathbf{n}}$. When two of the eigenvalues are equal, one of the eigenvectors will be unique but the other two directions will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, in order to form an orthonormal set.

Eigenvalues and Positive Definite Tensors

Since $\mathbf{A}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}}$, then $\hat{\mathbf{n}} \cdot \mathbf{A}\hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \lambda\hat{\mathbf{n}} = \lambda$. Thus if **A** is positive definite, Eqn. 1.10.38, the eigenvalues are all *positive*.

In fact, it can be shown that a tensor is positive definite if and only if its symmetric part has all positive eigenvalues.

Note: if there exists a non-zero eigenvector corresponding to a zero eigenvalue, then the tensor is singular. This is the case for the skew tensor **W**, which is singular. Since $\mathbf{W}\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{o} = 0\boldsymbol{\omega}$ (see , §1.10.11), the axial vector $\boldsymbol{\omega}$ is an eigenvector corresponding to a zero eigenvalue of **W**.

1.11.3 Maximum and Minimum Values

The diagonal components of a tensor **A**, A_{11}, A_{22} A_{33}, have different values in different coordinate systems. However, the three eigenvalues include the extreme (maximum and minimum) possible values that any of these three components can take, in any coordinate system. To prove this, consider an arbitrary set of unit base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, other than the eigenvectors. From Eqn. 1.9.4, the components of **A** in a new coordinate system with these base vectors are $A'_{ij} = \mathbf{e}_i \mathbf{A} \mathbf{e}_j$ Express $\mathbf{e}_1$ using the eigenvectors as a basis,

$$\mathbf{e}_1 = \alpha\hat{\mathbf{n}}_1 + \beta\hat{\mathbf{n}}_2 + \gamma\hat{\mathbf{n}}_3$$

Then

$$A'_{11} = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha^2 \lambda_1 + \beta^2 \lambda_2 + \gamma^2 \lambda_3$$

Without loss of generality, let $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then, with $\alpha^2 + \beta^2 + \gamma^2 = 1$, one has

$$\begin{aligned} \lambda_1 &= \lambda_1 \left(\alpha^2 + \beta^2 + \gamma^2 \right) \geq \alpha^2 \lambda_1 + \beta^2 \lambda_2 + \gamma^2 \lambda_3 = A'_{11} \\ \lambda_3 &= \lambda_3 \left(\alpha^2 + \beta^2 + \gamma^2 \right) \leq \alpha^2 \lambda_1 + \beta^2 \lambda_2 + \gamma^2 \lambda_3 = A'_{11} \end{aligned}$$

which proves that the eigenvalues include the largest and smallest possible diagonal element of **A**.

1.11.4 The Cayley-Hamilton Theorem

The **Cayley-Hamilton theorem** states that a tensor **A** (not necessarily symmetric) satisfies its own characteristic equation 1.11.5:

$$\mathbf{A}^3 - \mathrm{I}_{\mathbf{A}} \mathbf{A}^2 + \mathrm{II}_{\mathbf{A}} \mathbf{A} - \mathrm{III}_{\mathbf{A}} \mathbf{I} = \mathbf{0} \qquad (1.11.15)$$

This can be proved as follows: one has $\mathbf{A}\hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$, where λ is an eigenvalue of **A** and $\hat{\mathbf{n}}$ is the corresponding eigenvector. A repeated application of **A** to this equation leads to $\mathbf{A}^n \hat{\mathbf{n}} = \lambda^n \hat{\mathbf{n}}$. Multiplying 1.11.5 by $\hat{\mathbf{n}}$ then leads to 1.11.15.

The third invariant in Eqn. 1.11.6 can now be written in terms of traces by a double contraction of the Cayley-Hamilton equation with **I**, and by using the definition of the trace, Eqn.1.10.6:

$$\begin{aligned} & \mathbf{A}^3 : \mathbf{I} - \mathrm{I}_{\mathbf{A}} \mathbf{A}^2 : \mathbf{I} + \mathrm{II}_{\mathbf{A}} \mathbf{A} : \mathbf{I} - \mathrm{III}_{\mathbf{A}} \mathbf{I} : \mathbf{I} = 0 \\ \rightarrow \quad & \mathrm{tr}\, \mathbf{A}^3 - \mathrm{I}_{\mathbf{A}} \mathrm{tr} \mathbf{A}^2 + \mathrm{II}_{\mathbf{A}} \mathrm{tr}\, \mathbf{A} - 3\mathrm{III}_{\mathbf{A}} = 0 \\ \rightarrow \quad & \mathrm{tr}\, \mathbf{A}^3 - \mathrm{tr}\, \mathbf{A} \mathrm{tr} \mathbf{A}^2 + \tfrac{1}{2} \left[(\mathrm{tr}\, \mathbf{A})^2 - \mathrm{tr}\, \mathbf{A}^2 \right] \mathrm{tr}\, \mathbf{A} - 3\mathrm{III}_{\mathbf{A}} = 0 \\ \rightarrow \quad & \mathrm{III}_{\mathbf{A}} = \tfrac{1}{3} \left[\mathrm{tr}\, \mathbf{A}^3 - \tfrac{3}{2} \mathrm{tr}\, \mathbf{A}\, \mathrm{tr} \mathbf{A}^2 + \tfrac{1}{2} (\mathrm{tr}\, \mathbf{A})^3 \right] \end{aligned} \qquad (1.11.16)$$

The three invariants of a tensor can now be listed as

$$\boxed{\begin{aligned} \mathrm{I}_{\mathbf{A}} &= \mathrm{tr} \mathbf{A} \\ \mathrm{II}_{\mathbf{A}} &= \tfrac{1}{2} \left[(\mathrm{tr} \mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2) \right] \\ \mathrm{III}_{\mathbf{A}} &= \tfrac{1}{3} \left[\mathrm{tr}\, \mathbf{A}^3 - \tfrac{3}{2} \mathrm{tr}\, \mathbf{A}\, \mathrm{tr} \mathbf{A}^2 + \tfrac{1}{2} (\mathrm{tr}\, \mathbf{A})^3 \right] \end{aligned}} \qquad \textbf{Invariants of a Tensor} \quad (1.11.17)$$

The Deviatoric Tensor

Denote the eigenvalues of the deviatoric tensor dev**A**, Eqn. 1.10.36, s_1, s_2, s_3 and the principal scalar invariants by J_1, J_2, J_3. The characteristic equation analogous to Eqn. 1.11.5 is then

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \tag{1.11.18}$$

and the deviatoric invariants are[3]

$$\begin{aligned} J_1 &= \mathrm{tr}(\mathrm{dev}\mathbf{A}) = s_1 + s_2 + s_3 \\ J_2 &= -\tfrac{1}{2}\left[(\mathrm{tr}(\mathrm{dev}\mathbf{A}))^2 - \mathrm{tr}\left((\mathrm{dev}\mathbf{A})^2 \right) \right] = -(s_1 s_2 + s_2 s_3 + s_3 s_1) \\ J_3 &= \det(\mathrm{dev}\mathbf{A}) = s_1 s_2 s_3 \end{aligned} \tag{1.11.19}$$

From Eqn. 1.10.37,

$$J_1 = 0 \tag{1.11.20}$$

The second invariant can also be expressed in the useful forms {▲Problem 4}

$$J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2 \right), \tag{1.11.21}$$

and, in terms of the eigenvalues of **A**, {▲Problem 5}

$$J_2 = \frac{1}{6}\left[(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2 \right]. \tag{1.11.22}$$

Further, the deviatoric invariants are related to the tensor invariants through {▲Problem 6}

$$J_2 = \tfrac{1}{3}\left(\mathrm{I}_\mathbf{A}^2 - 3\mathrm{II}_\mathbf{A} \right), \quad J_3 = \tfrac{1}{27}\left(2\mathrm{I}_\mathbf{A}^3 - 9\mathrm{I}_\mathbf{A}\mathrm{II}_\mathbf{A} + 27\mathrm{III}_\mathbf{A} \right) \tag{1.11.23}$$

1.11.5 Coaxial Tensors

Two tensors are **coaxial** if they have the same eigenvectors. It can be shown that a necessary and sufficient condition that two tensors **A** and **B** be coaxial is that their simple contraction is commutative, $\mathbf{AB} = \mathbf{BA}$.

Since for a tensor **T**, $\mathbf{T}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{T}$, a tensor and its inverse are coaxial and have the same eigenvectors.

[3] there is a convention (adhered to by most authors) to write the characteristic equation for a general tensor with a $+II_\mathbf{A}\lambda$ term and that for a deviatoric tensor with a $-J_2 s$ term (which ensures that $J_2 > 0$ - see 1.11.22 below) ; this means that the formulae for J_2 in Eqn. 1.11.19 are the negative of those for $II_\mathbf{A}$ in Eqn. 1.11.6

1.11.6 Fractional Powers of Tensors

Integer powers of tensors were defined in §1.9.2. Fractional powers of tensors can be defined provided the tensor is real, symmetric and positive definite (so that the eigenvalues are all positive).

Contracting both sides of $\mathbf{T}\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}}$ with **T** repeatedly gives $\mathbf{T}^n\hat{\mathbf{n}} = \lambda^n\hat{\mathbf{n}}$. It follows that, if **T** has eigenvectors $\hat{\mathbf{n}}_i$ and corresponding eigenvalues λ_i, then $\mathbf{T}^n$ is coaxial, having the same eigenvectors, but corresponding eigenvalues λ_i^n. Because of this, fractional powers of tensors are defined as follows: $\mathbf{T}^m$, where *m* is any real number, is that tensor which has the same eigenvectors as **T** but which has corresponding eigenvalues λ_i^m. For example, the square root of the positive definite tensor $\mathbf{T} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$ is

$$\mathbf{T}^{1/2} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \qquad (1.11.24)$$

and the inverse is

$$\mathbf{T}^{-1} = \sum_{i=1}^{3} (1/\lambda_i)\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i \qquad (1.11.25)$$

These new tensors are also positive definite.

1.11.7 Polar Decomposition of Tensors

Any (non-singular second-order) tensor **F** can be split up multiplicatively into an arbitrary proper orthogonal tensor **R** ($\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, $\det\mathbf{R} = 1$) and a tensor **U** as follows:

$$\boxed{\mathbf{F} = \mathbf{R}\mathbf{U}} \qquad \textbf{Polar Decomposition} \qquad (1.11.26)$$

The consequence of this is that any transformation of a vector **a** according to $\mathbf{Fa}$ can be decomposed into two transformations, one involving a transformation **U**, followed by a rotation **R**.

The decomposition is not, in general, unique; one can often find more than one orthogonal tensor **R** which will satisfy the above relation. In practice, **R** is chosen such that **U** is symmetric. To this end, consider $\mathbf{F}^{\mathrm{T}}\mathbf{F}$. Since

$$\mathbf{v} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{v} = \mathbf{F}\mathbf{v} \cdot \mathbf{F}\mathbf{v} = |\mathbf{F}\mathbf{v}|^2 > 0,$$

$\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is positive definite. Further, $\mathbf{F}^{\mathrm{T}}\mathbf{F} \equiv F_{ji}F_{jk}$ is clearly symmetric, i.e. the same result is obtained upon an interchange of *i* and *k*. Thus the square-root of $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ can be taken: let **U** in 1.11.26 be given by

$$\mathbf{U} = \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)^{1/2} \tag{1.11.27}$$

and **U** is also symmetric positive definite. Then, with 1.10.3e,

$$\begin{aligned} \mathbf{R}^{\mathrm{T}}\mathbf{R} &= \left(\mathbf{F}\mathbf{U}^{-1}\right)^{\mathrm{T}}\left(\mathbf{F}\mathbf{U}^{-1}\right) \\ &= \mathbf{U}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{U}^{-1} \\ &= \mathbf{U}^{-\mathrm{T}}\mathbf{U}\mathbf{U}\mathbf{U}^{-1} \\ &= \mathbf{I} \end{aligned} \tag{1.11.28}$$

Thus if **U** is symmetric, **R** is orthogonal. Further, from (1.10.16a,b) and (1.100.18d), $\det\mathbf{U} = \det\mathbf{F}$ and $\det\mathbf{R} = \det\mathbf{F}/\det\mathbf{U} = 1$ so that **R** is proper orthogonal. It can also be proved that this decomposition is unique.

An alternative decomposition is given by

$$\mathbf{F} = \mathbf{V}\mathbf{R} \tag{1.11.29}$$

Again, this decomposition is unique and **R** is proper orthogonal, this time with

$$\mathbf{V} = \left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{1/2} \tag{1.11.30}$$

1.11.8 Problems

1. Find the eigenvalues, (normalised) eigenvectors and principal invariants of
$$\mathbf{T} = \mathbf{I} + \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$$
2. Derive the spectral decomposition 1.11.11 by writing the identity tensor as $\mathbf{I} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, and writing $\mathbf{A} = \mathbf{A}\mathbf{I}$. [Hint: $\hat{\mathbf{n}}_i$ is an eigenvector.]
3. Derive the characteristic equation and Cayley-Hamilton equation for a 2-D space. Let **A** be a second order tensor with square root $\mathbf{S} = \sqrt{\mathbf{A}}$. By using the Cayley-Hamilton equation for **S**, and relating $\det\mathbf{S}$, $\mathrm{tr}\mathbf{S}$ to $\det\mathbf{A}$, $\mathrm{tr}\mathbf{A}$ through the corresponding eigenvalues, show that $\sqrt{\mathbf{A}} = \dfrac{\mathbf{A} + \sqrt{\det\mathbf{A}}\mathbf{I}}{\sqrt{\mathrm{tr}\mathbf{A} + 2\sqrt{\det\mathbf{A}}}}$.
4. The second invariant of a deviatoric tensor is given by Eqn. 1.11.19b,
$$J_2 = -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right)$$
By squaring the relation $J_1 = s_1 + s_2 + s_3 = 0$, derive Eqn. 1.11.21,
$J_2 = \frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right)$
5. Use Eqns. 1.11.21 (and your work from Problem 4) and the fact that $\lambda_1 - \lambda_2 = s_1 - s_2$, etc. to derive Eqn. 1.11.22.
6. Use the fact that $s_1 + s_2 + s_3 = 0$ to show that

$$\begin{aligned} \mathrm{I}_{\mathbf{A}} &= 3\lambda_m \\ \mathrm{II}_{\mathbf{A}} &= (s_1 s_2 + s_2 s_3 + s_3 s_1) + 3\lambda_m^2 \\ \mathrm{III}_{\mathbf{A}} &= s_1 s_2 s_3 + \sigma_m (s_1 s_2 + s_2 s_3 + s_3 s_1) + \lambda_m^3 \end{aligned}$$

where $\lambda_m = \frac{1}{3} A_{ii}$. Hence derive Eqns. 1.11.23.

7. Consider the tensor

$$\mathbf{F} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Verify that the polar decomposition for **F** is $\mathbf{F} = \mathbf{RU}$ where

$$\mathbf{R} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{U} = \begin{bmatrix} 3/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 3/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(verify that **R** is proper orthogonal).

(b) Evaluate $\mathbf{Fa}, \mathbf{Fb}$, where $\mathbf{a} = [1,1,0]^{\mathrm{T}}$, $\mathbf{b} = [0,1,0]^{\mathrm{T}}$ by evaluating the individual transformations $\mathbf{Ua}, \mathbf{Ub}$ followed by $\mathbf{R}(\mathbf{Ua}), \mathbf{R}(\mathbf{Ub})$. Sketch the vectors and their images. Note how **R** rotates the vectors into their final positions. Why does **U** only stretch **a** but stretches *and* rotates **b**?

(c) Evaluate the eigenvalues λ_i and eigenvectors $\hat{\mathbf{n}}_i$ of the tensor $\mathbf{F}^{\mathrm{T}}\mathbf{F}$. Hence determine the spectral decomposition (diagonal matrix representation) of $\mathbf{F}^{\mathrm{T}}\mathbf{F}$. Hence evaluate $\mathbf{U} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$ with respect to the basis $\{\hat{\mathbf{n}}_i\}$ – again, this will be a diagonal matrix.