1.7 Vector Calculus 2 - Integration

1.7.1 Ordinary Integrals of a Vector

A vector can be integrated in the ordinary way to produce another vector, for example

$$\int_1^2 \left\{ \left(t - t^2\right)\mathbf{e}_1 + 2t^2\mathbf{e}_2 - 3\mathbf{e}_3 \right\} dt = -\frac{5}{6}\mathbf{e}_1 + \frac{15}{2}\mathbf{e}_2 - 3\mathbf{e}_3$$

1.7.2 Line Integrals

Discussed here is the notion of a definite integral involving a vector function that generates a scalar.

Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ be a position vector tracing out the curve C between the points p_1 and p_2. Let **f** be a vector field. Then

$$\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x} = \int_C \mathbf{f} \cdot d\mathbf{x} = \int_C \left\{ f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \right\}$$

is an example of a line integral.

Example (of a Line Integral)

A particle moves along a path C from the point $(0,0,0)$ to $(1,1,1)$, where C is the straight line joining the points, Fig. 1.7.1. The particle moves in a force field given by

$$\mathbf{f} = \left(3x_1^2 + 6x_2\right)\mathbf{e}_1 - 14x_2x_3\mathbf{e}_2 + 20x_1x_3^2\mathbf{e}_3$$

What is the work done on the particle?

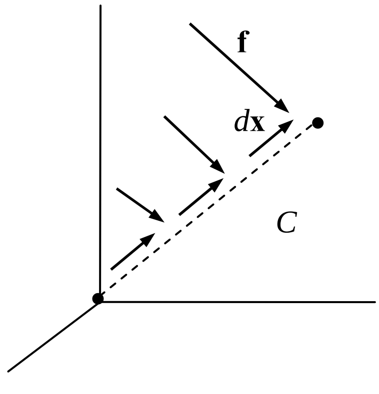

Figure 1.7.1: a particle moving in a force field

Solution
The work done is

$$W = \int_C \mathbf{f} \cdot d\mathbf{x} = \int_C \left\{ \left(3x_1^2 + 6x_2\right) dx_1 - 14x_2 x_3 dx_2 + 20x_1 x_3^2 dx_3 \right\}$$

The straight line can be written in the parametric form $x_1 = t,\, x_2 = t,\, x_3 = t$, so that

$$W = \int_0^1 \left(20t^3 - 11t^2 + 6t\right) dt = \frac{13}{3} \quad \text{or} \quad W = \int_C \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} dt = \int_C \mathbf{f} \cdot \left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\right) dt = \frac{13}{3}$$

■

If C is a closed curve, i.e. a loop, the line integral is often denoted $\oint_C \mathbf{v} \cdot d\mathbf{x}$.

Note: in fluid mechanics and aerodynamics, when **v** is the velocity field, this integral $\oint_C \mathbf{v} \cdot d\mathbf{x}$ is called the **circulation** of **v** about C.

1.7.3 Conservative Fields

If for a vector **f** one can find a scalar ϕ such that

$$\mathbf{f} = \nabla \phi \tag{1.7.1}$$

then

(1) $\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x}$ is independent of the path C joining p_1 and p_2

(2) $\oint_C \mathbf{f} \cdot d\mathbf{x} = 0$ around any closed curve C

In such a case, **f** is called a **conservative vector field** and ϕ is its **scalar potential**[1]. For example, the work done by a conservative force field **f** is

$$\int_{p_1}^{p_2} \mathbf{f} \cdot d\mathbf{x} = \int_{p_1}^{p_2} \nabla \phi \cdot d\mathbf{x} = \int_{p_1}^{p_2} \frac{\partial \phi}{\partial x_i} dx_i = \int_{p_1}^{p_2} d\phi = \phi(p_2) - \phi(p_1)$$

which clearly depends only on the values at the end-points p_1 and p_2, and not on the path taken between them.

It can be shown that a vector **f** is conservative if and only if $\mathrm{curl}\mathbf{f} = \mathbf{o}$ {▲Problem 3}.

[1] in general, of course, there does not exist a scalar field ϕ such that $\mathbf{f} = \nabla \phi$; this is not surprising since a vector field has three scalar components whereas $\nabla \phi$ is determined from just one

Example (of a Conservative Force Field)

The gravitational force field $\mathbf{f} = -mg\mathbf{e}_3$ is an example of a conservative vector field. Clearly, $\mathrm{curl}\mathbf{f} = \mathbf{o}$, and the gravitational scalar potential is $\phi = -mgx_3$:

$$W = -\int_{p_1}^{p_2} mg\mathbf{e}_3 \cdot d\mathbf{x} = -mg\int_{p_1}^{p_2} dx_3 = -mg[x_3(p_2) - x_3(p_1)] = \phi(p_2) - \phi(p_1)$$

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Example (of a Conservative Force Field)

Consider the force field

$$\mathbf{f} = (2x_1x_2 + x_3^3)\mathbf{e}_1 + x_1^2\mathbf{e}_2 + 3x_1x_3^2\mathbf{e}_3$$

Show that it is a conservative force field, find its scalar potential and find the work done in moving a particle in this field from $(1,-2,1)$ to $(3,1,4)$.

Solution

One has

$$\mathrm{curl}\mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ 2x_1x_2 + x_3^3 & x_1^2 & 3x_1x_3^2 \end{vmatrix} = \mathbf{o}$$

so the field is conservative.

To determine the scalar potential, let

$$f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3 = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3.$$

Equating coefficients and integrating leads to

$$\begin{aligned} \phi &= x_1^2x_2 + x_1x_3^3 + p(x_2, x_3) \\ \phi &= x_1^2x_2 \qquad\quad\; + q(x_1, x_3) \\ \phi &= \qquad\quad x_1x_3^3 + r(x_1, x_1) \end{aligned}$$

which agree if one chooses $p = 0,\ q = x_1x_3^3,\ r = x_1^2x_2$, so that $\phi = x_1^2x_2 + x_1x_3^3$, to which may be added a constant.

The work done is

$$W = \phi(3,1,4) - \phi(1,-2,1) = 202$$

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Helmholtz Theory

As mentioned, a conservative vector field which is irrotational, i.e. $\mathbf{f} = \nabla\phi$, implies $\nabla \times \mathbf{f} = \mathbf{o}$, and *vice versa*. Similarly, it can be shown that if one can find a vector **a** such that $\mathbf{f} = \nabla \times \mathbf{a}$, where **a** is called the **vector potential**, then **f** is solenoidal, i.e. $\nabla \cdot \mathbf{f} = 0$ {▲Problem 4}.

Helmholtz showed that a vector can always be represented in terms of a scalar potential ϕ and a vector potential **a**:[2]

Type of Vector	Condition	Representation
General		$\mathbf{f} = \nabla\phi + \nabla \times \mathbf{a}$
Irrotational (conservative)	$\nabla \times \mathbf{f} = \mathbf{o}$	$\mathbf{f} = \nabla\phi$
Solenoidal	$\nabla \cdot \mathbf{f} = 0$	$\mathbf{f} = \nabla \times \mathbf{a}$

1.7.4 Double Integrals

The most elementary type of two-dimensional integral is that over a plane region. For example, consider the integral over a region *R* in the $x_1 - x_2$ plane, Fig. 1.7.2. The integral

$$\iint_R dx_1 dx_2$$

then gives the area of *R* and, just as the one dimensional integral of a function gives the area under the curve, the integral

$$\iint_R f(x_1, x_2) dx_1 dx_2$$

gives the volume under the (in general, curved) surface $x_3 = f(x_1, x_2)$. These integrals are called **double integrals**.

[2] this decomposition can be made unique by requiring that $\mathbf{f} \to \mathbf{0}$ as $\mathbf{x} \to \infty$; in general, if one is given **f**, then ϕ and **a** can be obtained by solving a number of differential equations

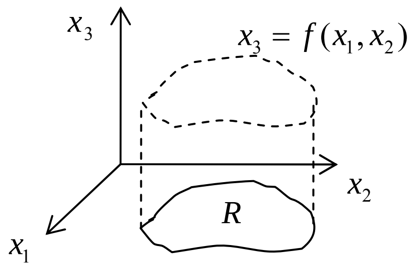

Figure 1.7.2: integration over a region

Change of variables in Double Integrals

To evaluate integrals of the type $\iint_R f(x_1, x_2) dx_1 dx_2$, it is often convenient to make a change of variable. To do this, one must find an elemental surface area in terms of the new variables, t_1, t_2 say, equivalent to that in the x_1, x_2 coordinate system, $dS = dx_1 dx_2$.

The region *R* over which the integration takes place is the plane surface $g(x_1, x_2) = 0$. Just as a curve can be represented by a position vector of one single parameter *t* (*cf.* §1.6.2), this surface can be represented by a position vector with two parameters[3], t_1 and t_2:

$$\mathbf{x} = x_1(t_1, t_2)\mathbf{e}_1 + x_2(t_1, t_2)\mathbf{e}_2$$

Parameterising the plane surface in this way, one can calculate the element of surface dS in terms of t_1, t_2 by considering curves of constant t_1, t_2, as shown in Fig. 1.7.3. The vectors bounding the element are

$$d\mathbf{x}^{(1)} = d\mathbf{x}\big|_{t_2 \text{ const}} = \frac{\partial \mathbf{x}}{\partial t_1} dt_1, \qquad d\mathbf{x}^{(2)} = d\mathbf{x}\big|_{t_1 \text{ const}} = \frac{\partial \mathbf{x}}{\partial t_2} dt_2 \tag{1.7.2}$$

so the area of the element is given by

$$dS = \left| d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \right| = \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2 = J\, dt_1 dt \tag{1.7.3}$$

where *J* is the **Jacobian** of the transformation,

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[3] for example, the unit circle $x_1^2 + x_2^2 - 1 = 0$ can be represented by $\mathbf{x} = t_1 \cos t_2 \mathbf{e}_1 + t_1 \sin t_2 \mathbf{e}_2$, $0 < t_1 \le 1$, $0 < t_2 \le 2\pi$ (t_1, t_2 being in this case the polar coordinates r, θ, respectively)

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} \\ \dfrac{\partial x_1}{\partial t_2} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} \quad \text{or} \quad J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} \\ \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} \tag{1.7.4}$$

The Jacobian is also often written using the notation

$$dx_1 dx_2 = J dt_1 dt_2, \qquad J = \left| \frac{\partial(x_1, x_2)}{\partial(t_1, t_2)} \right|$$

The integral can now be written as

$$\iint_R f(t_1, t_2) J dt_1 dt_2$$

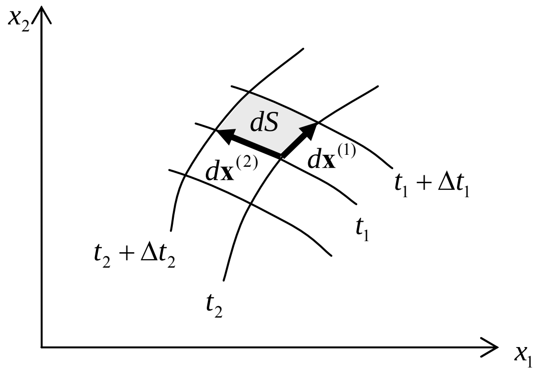

Figure 1.7.3: a surface element

Example

Consider a region R, the quarter unit-circle in the first quadrant, $0 \le x_2 \le \sqrt{1 - x_1^2}$, $0 \le x_1 \le 1$. The moment of inertia about the x_1 – axis is defined by

$$I_{x_1} \equiv \iint_R x_2^2 dx_1 dx_2$$

Transform the integral into the new coordinate system t_1, t_2 by making the substitutions[4] $x_1 = t_1 \cos t_2$, $x_2 = t_1 \sin t_2$. Then

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_1}{\partial t_2} \\ \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_2}{\partial t_2} \end{vmatrix} = \begin{vmatrix} \cos t_2 & -t_1 \sin t_2 \\ \sin t_2 & t_1 \cos t_2 \end{vmatrix} = t_1$$

[4] these are the polar coordinates, t_1, t_2 equal to r, θ, respectively

so

$$I_{x_1} = \int_0^{\pi/2}\int_0^1 t_1^3 \sin^2 t_2 dt_1 dt_2 = \frac{\pi}{16}$$

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1.7.5 Surface Integrals

Up to now, double integrals over a plane region have been considered. In what follows, consideration is given to integrals over more complex, curved, surfaces in space, such as the surface of a sphere.

Surfaces

Again, a curved surface can be parameterized by t_1, t_2, now by the position vector

$$\mathbf{x} = x_1(t_1,t_2)\mathbf{e}_1 + x_2(t_1,t_2)\mathbf{e}_2 + x_3(t_1,t_2)\mathbf{e}_3$$

One can generate a curve C on the surface S by taking $t_1 = t_1(s)$, $t_2 = t_2(s)$ so that C has position vector, Fig. 1.7.4,

$$\mathbf{x}(s) = \mathbf{x}(t_1(s), t_2(s))$$

A vector tangent to C at a point p on S is, from Eqn. 1.6.3,

$$\frac{d\mathbf{x}}{ds} = \frac{\partial \mathbf{x}}{\partial t_1}\frac{dt_1}{ds} + \frac{\partial \mathbf{x}}{\partial t_2}\frac{dt_2}{ds}$$

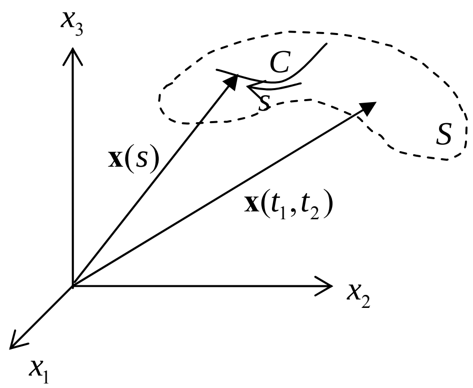

Figure 1.7.4: a curved surface

Many different curves C pass through p, and hence there are many different tangents, with different corresponding values of dt_1/ds, dt_2/ds. Thus the partial derivatives $\partial\mathbf{x}/\partial t_1$, $\partial\mathbf{x}/\partial t_2$ must also both be tangential to C and so a normal to the surface at p is given by their cross-product, and a unit normal is

$$\mathbf{n} = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2}\right) \bigg/ \left|\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2}\right| \tag{1.7.5}$$

In some cases, it is possible to use a non-parametric form for the surface, for example $g(x_1, x_2, x_3) = c$, in which case the normal can be obtained simply from $\mathbf{n} = \operatorname{grad} g / |\operatorname{grad} g|$.

Example (Parametric Representation and the Normal to a Sphere)

The surface of a sphere of radius a can be parameterised as[5]

$$\mathbf{x} = a\{\sin t_1 \cos t_2 \mathbf{e}_1 + \sin t_1 \sin t_2 \mathbf{e}_2 + \cos t_1 \mathbf{e}_3\}, \qquad 0 \le t_1 \le \pi,\ 0 \le t_2 \le 2\pi$$

Here, lines of $t_1 = \text{const}$ are parallel to the $x_1 - x_2$ plane ("parallels"), whereas lines of $t_2 = \text{const}$ are "meridian" lines, Fig. 1.7.5. If one takes the simple expressions $t_1 = s, t_2 = \pi/2 - s$, over $0 \le s \le \pi/2$, one obtains a curve C_1 joining $(0,0,1)$ and $(1,0,0)$, and passing through $(1/2, 1/2, 1/\sqrt{2})$, as shown.

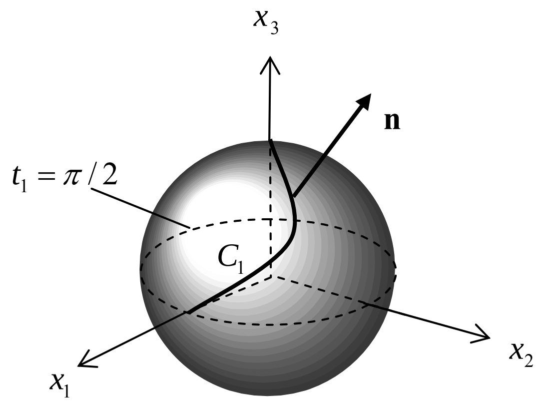

Figure 1.7.5: a sphere

The partial derivatives with respect to the parameters are

$$\frac{\partial \mathbf{x}}{\partial t_1} = a\{\cos t_1 \cos t_2 \mathbf{e}_1 + \cos t_1 \sin t_2 \mathbf{e}_2 - \sin t_1 \mathbf{e}_3\}$$
$$\frac{\partial \mathbf{x}}{\partial t_2} = a\{-\sin t_1 \sin t_2 \mathbf{e}_1 + \sin t_1 \cos t_2 \mathbf{e}_2\}$$

so that

$$\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} = a^2\{\sin^2 t_1 \cos t_2 \mathbf{e}_1 + \sin^2 t_1 \sin t_2 \mathbf{e}_2 + \sin t_1 \cos t_1 \mathbf{e}_3\}$$

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[5] these are the **spherical coordinates** (see §1.6.10); $t_1 = \theta, t_2 = \phi$

and a unit normal to the spherical surface is

$$\mathbf{n} = \sin t_1 \cos t_2 \mathbf{e}_1 + \sin t_1 \sin t_2 \mathbf{e}_2 + \cos t_1 \mathbf{e}_3$$

For example, at $t_1 = t_2 = \pi / 4$ (this is on the curve C_1), one has

$$\mathbf{n}(\pi / 4, \pi / 4) = \tfrac{1}{2}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + \tfrac{1}{\sqrt{2}}\mathbf{e}_3$$

and, as expected, it is in the same direction as **r**.

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Surface Integrals

Consider now the integral $\iint_S \mathbf{f} dS$ where **f** is a vector function and *S* is some curved surface. As for the integral over the plane region,

$$dS = \left| d\mathbf{x}\big|_{t_2 \text{ const}} \times d\mathbf{x}\big|_{t_1 \text{ const}} \right| = \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2 ,$$

only now dS is not "flat" and **x** is three dimensional. The integral can be evaluated if one parameterises the surface with t_1, t_2 and then writes

$$\iint_S \mathbf{f} \left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right| dt_1 dt_2$$

One way to evaluate this cross product is to use the relation (**Lagrange's identity**, Problem 15, §1.3)

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \qquad (1.7.6)$$

so that

$$\left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right|^2 = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) \cdot \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) = \left(\frac{\partial \mathbf{x}}{\partial t_1} \cdot \frac{\partial \mathbf{x}}{\partial t_1} \right)\left(\frac{\partial \mathbf{x}}{\partial t_2} \cdot \frac{\partial \mathbf{x}}{\partial t_2} \right) - \left(\frac{\partial \mathbf{x}}{\partial t_1} \cdot \frac{\partial \mathbf{x}}{\partial t_2} \right)^2 \qquad (1.7.7)$$

Example (Surface Area of a Sphere)

Using the parametric form for a sphere given above, one obtains

$$\left| \frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right|^2 = a^4 \sin^2 t_1$$

so that

$$\text{area} = \iint_S dS = a^2 \int_0^{2\pi} \int_0^{\pi} \sin t_1 dt_1 dt_2 = 4\pi a^2$$

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Flux Integrals

Surface integrals often involve the normal to the surface, as in the following example.

Example

If $\mathbf{f} = 4x_1 x_3 \mathbf{e}_1 - x_2^2 \mathbf{e}_2 + x_2 x_3 \mathbf{e}_3$, evaluate $\iint_S \mathbf{f} \cdot \mathbf{n} dS$, where S is the surface of the cube bounded by $x_1 = 0,1;\ x_2 = 0,1;\ x_3 = 0,1$, and $\mathbf{n}$ is the unit outward normal, Fig. 1.7.6.

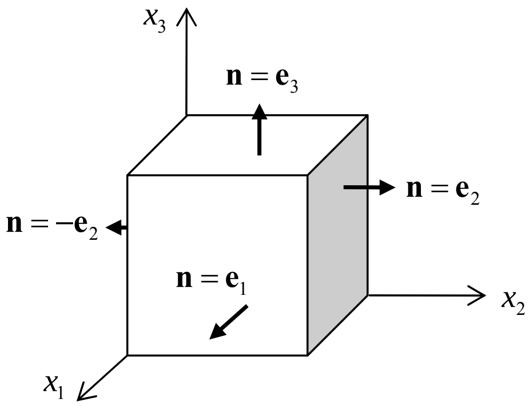

Figure 1.7.6: the unit cube

<u>Solution</u>

The integral needs to be evaluated over the six faces. For the face with $\mathbf{n} = +\mathbf{e}_1$, $x_1 = 1$ and

$$\iint_S \mathbf{f} \cdot \mathbf{n} dS = \int_0^1 \int_0^1 \left(4x_3 \mathbf{e}_1 - x_2^2 \mathbf{e}_2 + x_2 x_3 \mathbf{e}_3\right) \cdot \mathbf{e}_1 dx_2 dx_3 = 4 \int_0^1 \int_0^1 x_3 dx_2 dx_3 = 2$$

Similarly for the other five sides, whence $\iint_S \mathbf{f} \cdot \mathbf{n} dS = \frac{3}{2}$.

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Integrals of the form $\iint_S \mathbf{f} \cdot \mathbf{n} dS$ are known as **flux integrals** and arise quite often in applications. For example, consider a material flowing with velocity **v**, in particular the flow through a small surface element dS with outward unit normal **n**, Fig. 1.7.7. The volume of material flowing through the surface in time dt is equal to the volume of the slanted cylinder shown, which is the base dS times the height. The slanted height is (=

velocity × time) is $|\mathbf{v}|dt$, and the vertical height is then $\mathbf{v}\cdot\mathbf{n}dt$. Thus the *rate* of flow is the **volume flux** (volume per unit time) through the surface element: $\mathbf{v}\cdot\mathbf{n}dS$.

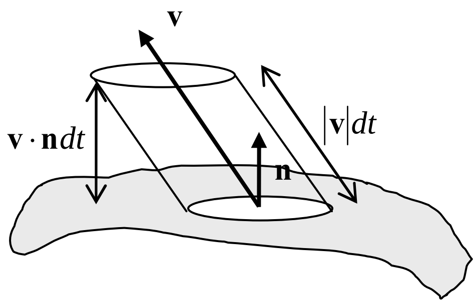

Figure 1.7.7: flow through a surface element

The total (volume) flux *out* of a surface S is then[6]

$$\textit{volume flux:}\quad \iint_S \mathbf{v}\cdot\mathbf{n}dS \tag{1.7.8}$$

Similarly, the **mass flux** is given by

$$\textit{mass flux:}\quad \iint_S \rho\mathbf{v}\cdot\mathbf{n}dS \tag{1.7.9}$$

For more complex surfaces, one can write using Eqn. 1.7.3, 1.7.5,

$$\iint_S \mathbf{f}\cdot\mathbf{n}dS = \iint_S \mathbf{f}\cdot\left(\frac{\partial\mathbf{x}}{\partial t_1}\times\frac{\partial\mathbf{x}}{\partial t_2}\right)dt_1 dt_2$$

Example (of a Flux Integral)

Compute the flux integral $\iint_S \mathbf{f}\cdot\mathbf{n}dS$, where S is the parabolic cylinder represented by

$$x_2 = x_1^2,\quad 0\le x_1\le 2,\quad 0\le x_3\le 3$$

and $\mathbf{f} = x_2\mathbf{e}_1 + 2\mathbf{e}_2 + x_1x_3\mathbf{e}_3$, Fig. 1.7.8.

<u>Solution</u>

Making the substitutions $x_1 = t_1$, $x_3 = t_2$, so that $x_2 = t_1^2$, the surface can be represented by the position vector

[6] if **v** acts in the same direction as **n**, i.e. pointing outward, the dot product is positive and this integral is positive; if, on the other hand, material is flowing *in* through the surface, **v** and **n** are in opposite directions and the dot product is negative, so the integral is negative

$$\mathbf{x} = t_1\mathbf{e}_1 + t_1^2\mathbf{e}_2 + t_2\mathbf{e}_3, \qquad 0 \le t_1 \le 2,\ 0 \le t_2 \le 3$$

Then $\partial\mathbf{x}/\partial t_1 = \mathbf{e}_1 + 2t_1\mathbf{e}_2$, $\partial\mathbf{x}/\partial t_2 = \mathbf{e}_3$ and

$$\frac{\partial\mathbf{x}}{\partial t_1} \times \frac{\partial\mathbf{x}}{\partial t_2} = 2t_1\mathbf{e}_1 - \mathbf{e}_2$$

so the integral becomes

$$\int_0^3\int_0^2 \left(t_1^2\mathbf{e}_1 + 2\mathbf{e}_2 + t_1t_2\mathbf{e}_3\right)\cdot\left(2t_1\mathbf{e}_1 - \mathbf{e}_2\right)dt_1dt_2 = 12$$

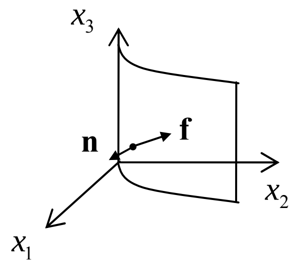

Figure 1.7.8: flux through a parabolic cylinder

Note: in this example, the value of the integral depends on the choice of **n**. If one chooses $-\mathbf{n}$ instead of **n**, one would obtain -12. The normal in the opposite direction (on the "other side" of the surface) can be obtained by simply switching t_1 and t_2, since $\partial\mathbf{x}/\partial t_1 \times \partial\mathbf{x}/\partial t_2 = -\partial\mathbf{x}/\partial t_2 \times \partial\mathbf{x}/\partial t_1$.

■

Surface flux integrals can also be evaluated by first converting them into double integrals over a plane region. For example, if a surface *S* has a projection *R* on the $x_1 - x_2$ plane, then an element of surface dS is related to the projected element dx_1dx_2 through (see Fig. 1.7.9)

$$\cos\theta dS = \left(\mathbf{n}\cdot\mathbf{e}_3\right)dS = dx_1dx_2$$

and so

$$\iint_S \mathbf{f}\cdot\mathbf{n}dS = \iint_R \mathbf{f}\cdot\mathbf{n}\frac{1}{\left|\mathbf{n}\cdot\mathbf{e}_3\right|}dx_1dx_2$$

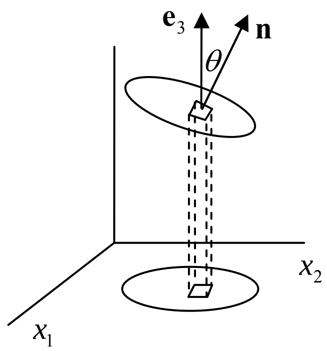

Figure 1.7.9: projection of a surface element onto a plane region

The Normal and Surface Area Elements

It is sometimes convenient to associate a special vector $d\mathbf{S}$ with a differential element of surface area dS, where

$$d\mathbf{S} = \mathbf{n}\, dS$$

so that $d\mathbf{S}$ is the vector with magnitude dS and direction of the unit normal to the surface. Flux integrals can then be written as

$$\iint_S \mathbf{f} \cdot \mathbf{n} dS = \iint_S \mathbf{f} \cdot d\mathbf{S}$$

1.7.6 Volume Integrals

The volume integral, or triple integral, is a generalisation of the double integral.

Change of Variable in Volume Integrals

For a volume integral, it is often convenient to make the change of variables $(x_1, x_2, x_3) \to (t_1, t_2, t_3)$. The volume of an element dV is given by the triple scalar product (Eqns. 1.1.5, 1.3.17)

$$dV = \left(\frac{\partial \mathbf{x}}{\partial t_1} \times \frac{\partial \mathbf{x}}{\partial t_2} \right) \cdot \frac{\partial \mathbf{x}}{\partial t_3} dt_1 dt_2 dt_3 = J dt_1 dt_2 dt_3 \tag{1.7.10}$$

where the Jacobian is now

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \frac{\partial x_3}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_3}{\partial t_2} \\ \frac{\partial x_1}{\partial t_3} & \frac{\partial x_2}{\partial t_3} & \frac{\partial x_3}{\partial t_3} \end{vmatrix} \quad \text{or} \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \frac{\partial x_1}{\partial t_3} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_2}{\partial t_3} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3} \end{vmatrix} \tag{1.7.11}$$

so that

$$\iiint_V \mathbf{f}(x_1, x_2, x_3)\,dxdydz = \iiint_V \mathbf{f}\big(x_1(t_1,t_2,t_3), x_2(t_1,t_2,t_3), x_3(t_1,t_2,t_3)\big) J\, dt_1 dt_2 dt_3$$

1.7.7 Integral Theorems

A number of integral theorems and relations are presented here (without proof), the most important of which is the divergence theorem. These theorems can be used to simplify the evaluation of line, double, surface and triple integrals. They can also be used in various proofs of other important results.

The Divergence Theorem

Consider an arbitrary differentiable vector field $\mathbf{v}(\mathbf{x}, t)$ defined in some finite region of physical space. Let V be a volume in this space with a closed surface S bounding the volume, and let the outward normal to this bounding surface be $\mathbf{n}$. The **divergence theorem of Gauss** states that (in symbolic and index notation)

$$\boxed{\int_S \mathbf{v}\cdot\mathbf{n}\,dS = \int_V \operatorname{div}\mathbf{v}\,dV \qquad \int_S v_i n_i\,dS = \int_V \frac{\partial v_i}{\partial x_i}\,dV} \quad \textbf{Divergence Theorem} \tag{1.7.12}$$

and one has the following useful identities {▲Problem 10}

$$\begin{aligned} \int_S \phi\mathbf{u}\cdot\mathbf{n}\,dS &= \int_V \operatorname{div}(\phi\mathbf{u})\,dV \\ \int_S \phi\mathbf{n}\,dS &= \int_V \operatorname{grad}\phi\,dV \\ \int_S \mathbf{n}\times\mathbf{u}\,dS &= \int_V \operatorname{curl}\mathbf{u}\,dV \end{aligned} \tag{1.7.13}$$

By applying the divergence theorem to a very small volume, one finds that

$$\operatorname{div}\mathbf{v} = \lim_{V\to 0}\frac{\int_S \mathbf{v}\cdot\mathbf{n}\,dS}{V}$$

that is, the divergence is equal to the outward flux per unit volume, the result 1.6.18.

Stoke's Theorem

Stoke's theorem transforms line integrals into surface integrals and *vice versa*. It states that

$$\iint_S (\operatorname{curl}\mathbf{f}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{f} \cdot \boldsymbol{\tau}\, ds \tag{1.7.14}$$

Here C is the boundary of the surface S, $\mathbf{n}$ is the unit outward normal and $\boldsymbol{\tau} = d\mathbf{r}/ds$ is the unit tangent vector.

As has been seen, Eqn. 1.6.24, the curl of the velocity field is a measure of how much a fluid is rotating. The direction of this vector is along the direction of the local axis of rotation and its magnitude measures the local angular velocity of the fluid. Stoke's theorem then states that the amount of rotation of a fluid can be measured by integrating the tangential velocity around a curve (the line integral), or by integrating the amount of vorticity "moving through" a surface bounded by the same curve.

Green's Theorem and Related Identities

Green's theorem relates a line integral to a double integral, and states that

$$\oint_C \{\psi_1 dx_1 + \psi_2 dx_2\} = \iint_R \left(\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \right) dx_1 dx_2 \,, \tag{1.7.15}$$

where R is a region in the $x_1 - x_2$ plane bounded by the curve C. In vector form, Green's theorem reads as

$$\oint_C \mathbf{f} \cdot d\mathbf{x} = \iint_R \operatorname{curl}\mathbf{f} \cdot \mathbf{e}_3 dx_1 dx_2 \quad \text{where} \quad \mathbf{f} = \psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2 \tag{1.7.16}$$

from which it can be seen that Green's theorem is a special case of Stoke's theorem, for the case of a plane surface (region) in the $x_1 - x_2$ plane.

It can also be shown that (this is **Green's first identity**)

$$\iint_S \psi (\mathbf{n} \cdot \operatorname{grad}\phi) dS = \iiint_V \{\psi \nabla^2 \phi + \operatorname{grad}\psi \cdot \operatorname{grad}\phi\} dV \tag{1.7.17}$$

Note that the term $\mathbf{n} \cdot \operatorname{grad}\phi$ is the directional derivative of ϕ in the direction of the outward unit normal. This is often denoted as $\partial\phi/\partial n$. Green's first identity can be regarded as a multi-dimensional "integration by parts" – compare the rule $\int u dv = uv - \int v du$ with the identity re-written as

$$\iiint_V \psi (\nabla \cdot \nabla\phi) dV = \iint_S \psi (\nabla\phi \cdot \mathbf{n}) dS - \iiint_V (\nabla\psi) \cdot (\nabla\phi) dV \tag{1.7.18}$$

or

$$\iiint_V \psi(\nabla \cdot \mathbf{u})dV = \iint_S \psi(\mathbf{u} \cdot \mathbf{n})dS - \iiint_V (\nabla \psi) \cdot \mathbf{u} dV \qquad (1.7.18)$$

One also has the relation (this is **Green's second identity**)

$$\iint_S \{\psi(\mathbf{n} \cdot \operatorname{grad}\phi) - \phi(\mathbf{n} \cdot \operatorname{grad}\psi)\}dS = \iiint_V \{\psi\nabla^2\phi - \phi\nabla^2\psi\}dV \qquad (1.7.19)$$

1.7.8 Problems

1. Find the work done in moving a particle in a force field given by $\mathbf{f} = 3x_1x_2\mathbf{e}_1 - 5x_3\mathbf{e}_2 + 10x_1\mathbf{e}_1$ along the curve $x_1 = t^2 + 1$, $x_2 = 2t^2$, $x_3 = t^3$, from $t = 1$ to $t = 2$. (Plot the curve.)
2. Show that the following vectors are conservative and find their scalar potentials:
   (i) $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$
   (ii) $\mathbf{v} = e^{x_1x_2}(x_2\mathbf{e}_1 + x_1\mathbf{e}_2)$
   (iii) $\mathbf{u} = (1/x_2)\mathbf{e}_1 - (x_1/x_2^2)\mathbf{e}_2 + x_3\mathbf{e}_3$
3. Show that if $\mathbf{f} = \nabla\phi$ then $\operatorname{curl}\mathbf{f} = \mathbf{o}$.
4. Show that if $\mathbf{f} = \nabla \times \mathbf{a}$ then $\nabla \cdot \mathbf{f} = 0$.
5. Find the volume beneath the surface $x_1^2 + x_2^2 - x_3 = 0$ and above the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ in the $x_1 - x_2$ plane.
6. Find the Jacobian (and sketch lines of constant t_1, t_2) for the rotation
$$x_1 = t_1\cos\theta - t_2\sin\theta$$
$$x_2 = t_1\sin\theta + t_2\cos\theta$$
7. Find a unit normal to the circular cylinder with parametric representation
$$\mathbf{x}(t_1, t_2) = a\cos t_1\mathbf{e}_1 + a\sin t_1\mathbf{e}_2 + t_2\mathbf{e}_3, \quad 0 \le t_1 \le 2\pi, \quad 0 \le t_1 \le 1$$
8. Evaluate $\int_S \psi dS$ where $\psi = x_1 + x_2 + x_3$ and *S* is the plane surface $x_3 = x_1 + x_2$, $0 \le x_2 \le x_1$, $0 \le x_1 \le 1$.
9. Evaluate the flux integral $\int_S \mathbf{f} \cdot \mathbf{n} dS$ where $\mathbf{f} = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$ and *S* is the cone $x_3 = a(x_1^2 + x_2^2)$, $x_3 \le a$ [Hint: first parameterise the surface with t_1, t_2.]
10. Prove the relations in (1.7.13). [Hint: first write the expressions in index notation.]
11. Use the divergence theorem to show that
$$\int_S \mathbf{x} \cdot \mathbf{n} dS = 3V,$$
where *V* is the volume enclosed by *S* (and **x** is the position vector).
12. Verify the divergence theorem for $\mathbf{v} = x_1^3\mathbf{e}_1 + x_2^3\mathbf{e}_2 + x_3^3\mathbf{e}_3$ where *S* is the surface of the sphere $x_1^2 + x_2^2 + x_3^2 = a^2$.
13. Interpret the divergence theorem (1.7.12) for the case when **v** is the velocity field. See (1.6.18, 1.7.8). Interpret also the case of $\operatorname{div}\mathbf{v} = 0$.

14. Verify Stoke's theorem for $\mathbf{f} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_1\mathbf{e}_3$ where *S* is $x_3 = 1 - x_1^2 - x_2^2 \geq 0$ (so that *C* is the circle of radius 1 in the $x_1 - x_2$ plane).
15. Verify Green's theorem for the case of $\psi_1 = x_1^2 - 2x_2, \psi_2 = x_1 + x_2$, with *C* the unit circle $x_1^2 + x_2^2 = 1$. The following relations might be useful:
$$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi, \quad \int_0^{2\pi} \sin\theta\cos\theta d\theta = \int_0^{2\pi} \sin\theta\cos^2\theta d\theta = 0$$
16. Evaluate $\oint_C \mathbf{f} \cdot d\mathbf{x}$ using Green's theorem, where $\mathbf{f} = -x_2^3\mathbf{e}_1 + x_1^3\mathbf{e}_2$ and *C* is the circle $x_1^2 + x_2^2 = 4$.
17. Use Green's theorem to show that the double integral of the Laplacian of *p* over a region *R* is equivalent to the integral of $\partial p / \partial n = \text{grad}p \cdot \mathbf{n}$ around the curve *C* bounding the region:
$$\iint_R \nabla^2 p dx_1 dx_2 = \oint_C \frac{\partial p}{\partial n} ds$$
[Hint: Let $\psi_1 = -\partial p / \partial x_2, \psi_2 = +\partial p / \partial x_1$. Also, show that
$$\mathbf{n} = \frac{dx_2}{ds}\mathbf{e}_1 - \frac{dx_1}{ds}\mathbf{e}_2$$
is a unit normal to *C*, Fig. 1.7.10]

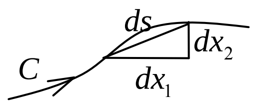

Figure 1.7.10: projection of a surface element onto a plane region