1.6 Vector Calculus 1 - Differentiation

Calculus involving vectors is discussed in this section, rather intuitively at first and more formally toward the end of this section.

1.6.1 The Ordinary Calculus

Consider a **scalar-valued function of a scalar**, for example the time-dependent density of a material $\rho = \rho(t)$. The calculus of scalar valued functions of scalars is just the ordinary calculus. Some of the important concepts of the ordinary calculus are reviewed in Appendix B to this Chapter, §1.B.2.

1.6.2 Vector-valued Functions of a scalar

Consider a **vector-valued function of a scalar**, for example the time-dependent displacement of a particle $\mathbf{u} = \mathbf{u}(t)$. In this case, the derivative is defined in the usual way,

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t},$$

which turns out to be simply the derivative of the coefficients[1],

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3 \equiv \frac{du_i}{dt}\mathbf{e}_i$$

Partial derivatives can also be defined in the usual way. For example, if **u** is a function of the coordinates, $\mathbf{u}(x_1, x_2, x_3)$, then

$$\frac{\partial \mathbf{u}}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{\mathbf{u}(x_1 + \Delta x_1, x_2, x_3) - \mathbf{u}(x_1, x_2, x_3)}{\Delta x_1}$$

Differentials of vectors are also defined in the usual way, so that when u_1, u_2, u_3 undergo increments $du_1 = \Delta u_1$, $du_2 = \Delta u_2$, $du_3 = \Delta u_3$, the differential of **u** is

$$d\mathbf{u} = du_1\mathbf{e}_1 + du_2\mathbf{e}_2 + du_3\mathbf{e}_3$$

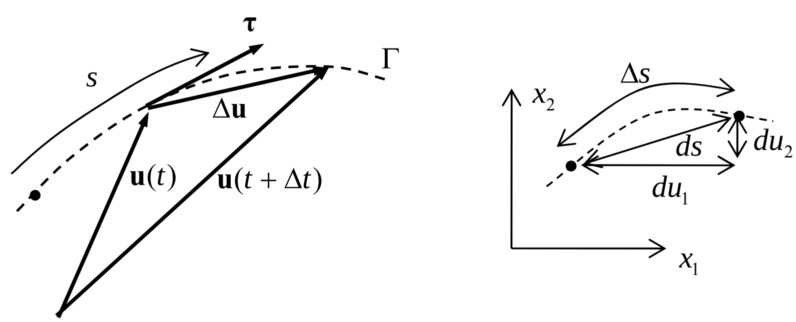
and the differential and actual increment $\Delta\mathbf{u}$ approach one another as $\Delta u_1, \Delta u_2, \Delta u_3 \to 0$.

[1] assuming that the base vectors do not depend on *t*

Space Curves

The derivative of a vector can be interpreted geometrically as shown in Fig. 1.6.1: $\Delta\mathbf{u}$ is the increment in $\mathbf{u}$ consequent upon an increment Δt in t. As t changes, the end-point of the vector $\mathbf{u}(t)$ traces out the dotted curve Γ shown – it is clear that as $\Delta t \to 0$, $\Delta\mathbf{u}$ approaches the tangent to Γ, so that $d\mathbf{u}/dt$ is tangential to Γ. The unit vector tangent to the curve is denoted by $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{|d\mathbf{u}/dt|} \tag{1.6.1}$$



(a) (b)

Figure 1.6.1: a space curve; (a) the tangent vector, (b) increment in arc length

Let s be a measure of the length of the curve Γ, measured from some fixed point on Γ. Let Δs be the increment in arc-length corresponding to increments in the coordinates, $\Delta\mathbf{u} = [\Delta u_1, \Delta u_2, \Delta u_3]^{\mathrm{T}}$, Fig. 1.6.1b. Then, from the ordinary calculus (see Appendix 1.B),

$$(ds)^2 = (du_1)^2 + (du_2)^2 + (du_3)^2$$

so that

$$\frac{ds}{dt} = \sqrt{\left(\frac{du_1}{dt}\right)^2 + \left(\frac{du_2}{dt}\right)^2 + \left(\frac{du_3}{dt}\right)^2}$$

But

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\mathbf{e}_1 + \frac{du_2}{dt}\mathbf{e}_2 + \frac{du_3}{dt}\mathbf{e}_3$$

so that

$$\left|\frac{d\mathbf{u}}{dt}\right| = \frac{ds}{dt} \tag{1.6.2}$$

Thus the unit vector tangent to the curve can be written as

$$\boldsymbol{\tau} = \frac{d\mathbf{u}/dt}{ds/dt} = \frac{d\mathbf{u}}{ds} \tag{1.6.3}$$

If $\mathbf{u}$ is interpreted as the position vector of a particle and t is interpreted as time, then $\mathbf{v} = d\mathbf{u}/dt$ is the velocity vector of the particle as it moves with speed ds/dt along Γ.

Example (of particle motion)

A particle moves along a curve whose parametric equations are $x_1 = 2t^2$, $x_2 = t^2 - 4t$, $x_3 = 3t - 5$ where t is time. Find the component of the velocity at time $t = 1$ in the direction $\mathbf{a} = \mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

Solution

The velocity is

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left\{2t^2\mathbf{e}_1 + \left(t^2 - 4t\right)\mathbf{e}_2 + \left(3t - 5\right)\mathbf{e}_3\right\} \\ &= 4\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 \quad \text{at } t = 1\end{aligned}$$

The component in the given direction is $\mathbf{v} \cdot \hat{\mathbf{a}}$, where $\hat{\mathbf{a}}$ is a unit vector in the direction of $\mathbf{a}$, giving $8\sqrt{14}/7$.

■

Curvature

The scalar **curvature** $\kappa(s)$ of a space curve is defined to be the magnitude of the rate of change of the unit tangent vector:

$$\kappa(s) = \left|\frac{d\boldsymbol{\tau}}{ds}\right| = \left|\frac{d^2\mathbf{u}}{ds^2}\right| \tag{1.6.4}$$

Note that $\Delta\boldsymbol{\tau}$ is in a direction perpendicular to $\boldsymbol{\tau}$, Fig. 1.6.2. In fact, this can be proved as follows: since $\boldsymbol{\tau}$ is a unit vector, $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ is a constant ($= 1$), and so $d(\boldsymbol{\tau} \cdot \boldsymbol{\tau})/ds = 0$, but also,

$$\frac{d}{ds}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}) = 2\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds}$$

and so $\boldsymbol{\tau}$ and $d\boldsymbol{\tau}/ds$ are perpendicular. The unit vector defined in this way is called the **principal normal vector**:

$$\mathbf{v} = \frac{1}{\kappa}\frac{d\boldsymbol{\tau}}{ds} \tag{1.6.5}$$

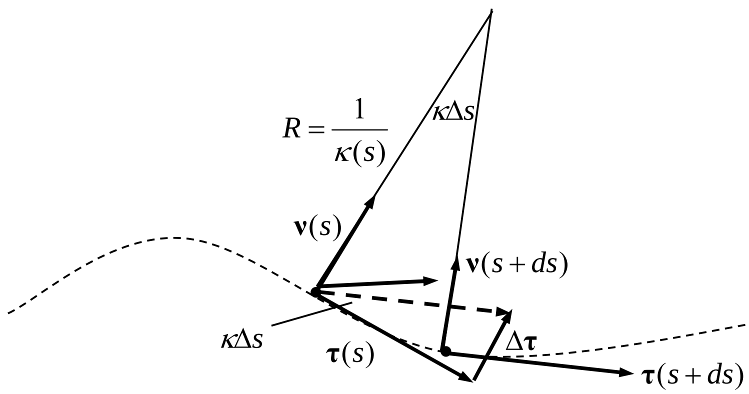


Figure 1.6.2: the curvature

This can be seen geometrically in Fig. 1.6.2: from Eqn. 1.6.5, $\Delta\boldsymbol{\tau}$ is a vector of magnitude $\kappa\Delta s$ in the direction of the vector normal to $\boldsymbol{\tau}$. The **radius of curvature** *R* is defined as the reciprocal of the curvature; it is the radius of the circle which just touches the curve at *s*, Fig. 1.6.2.

Finally, the unit vector perpendicular to both the tangent vector and the principal normal vector is called the **unit binormal vector**:

$$\mathbf{b} = \boldsymbol{\tau} \times \mathbf{v} \tag{1.6.6}$$

The planes defined by these vectors are shown in Fig. 1.6.3; they are called the **rectifying plane**, the **normal plane** and the **osculating plane**.

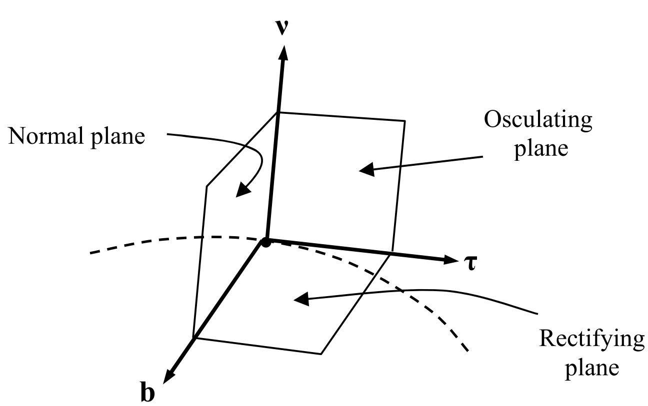


Figure 1.6.3: the unit tangent, principal normal and binormal vectors and associated planes

Rules of Differentiation

The derivative of a vector is also a vector and the usual rules of differentiation apply,

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$
$$\frac{d}{dt}(\alpha(t)\mathbf{v}) = \alpha\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{d\alpha}{dt} \tag{1.6.7}$$

Also, it is straight forward to show that {▲Problem 2}

$$\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a}) = \mathbf{v}\cdot\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt}\cdot\mathbf{a} \qquad \frac{d}{dt}(\mathbf{v}\times\mathbf{a}) = \mathbf{v}\times\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt}\times\mathbf{a} \tag{1.6.8}$$

(The order of the terms in the cross-product expression is important here.)

1.6.3 Fields

In many applications of vector calculus, a scalar or vector can be associated with each point in space $\mathbf{x}$. In this case they are called **scalar** or **vector fields**. For example

$\theta(\mathbf{x})$	temperature	a scalar field (a scalar-valued function of position)
$\mathbf{v}(\mathbf{x})$	velocity	a vector field (a vector valued function of position)

These quantities will in general depend also on time, so that one writes $\theta(\mathbf{x},t)$ or $\mathbf{v}(\mathbf{x},t)$. Partial differentiation of scalar and vector fields with respect to the variable *t* is symbolised by $\partial/\partial t$. On the other hand, partial differentiation with respect to the coordinates is symbolised by $\partial/\partial x_i$. The notation can be made more compact by introducing the **subscript comma** to denote partial differentiation with respect to the coordinate variables, in which case $\phi_{,i} = \partial\phi/\partial x_i$, $u_{i,jk} = \partial^2 u_i/\partial x_j \partial x_k$, and so on.

1.6.4 The Gradient of a Scalar Field

Let $\phi(\mathbf{x})$ be a scalar field. The **gradient** of ϕ is a vector field defined by (see Fig. 1.6.4)

$$\boxed{\begin{aligned}\nabla\phi &= \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \\ &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \\ &\equiv \frac{\partial\phi}{\partial\mathbf{x}}\end{aligned}} \qquad \textbf{Gradient of a Scalar Field} \tag{1.6.9}$$

The gradient $\nabla\phi$ is of considerable importance because if one takes the dot product of $\nabla\phi$ with $d\mathbf{x}$, it gives the increment in ϕ:

$$\begin{aligned}\nabla\phi\cdot d\mathbf{x} &= \frac{\partial\phi}{\partial x_i}\mathbf{e}_i\cdot dx_j\mathbf{e}_j \\ &= \frac{\partial\phi}{\partial x_i}dx_i \\ &= d\phi \\ &= \phi(\mathbf{x}+d\mathbf{x})-\phi(d\mathbf{x})\end{aligned} \tag{1.6.10}$$

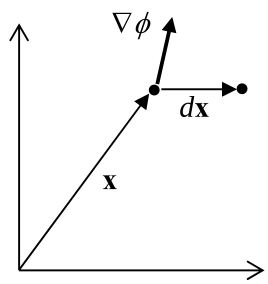


Figure 1.6.4: the gradient of a vector

If one writes $d\mathbf{x}$ as $|d\mathbf{x}|\mathbf{e} = dx\mathbf{e}$, where **e** is a unit vector in the direction of $d\mathbf{x}$, then

$$\nabla\phi\cdot\mathbf{e} = \left(\frac{d\phi}{dx}\right)_{\text{in } \mathbf{e} \text{ direction}} \equiv \frac{d\phi}{dn} \tag{1.6.11}$$

This quantity is called the **directional derivative** of ϕ, in the direction of **e**, and will be discussed further in §1.6.11.

The gradient of a scalar field is also called the **scalar gradient**, to distinguish it from the **vector gradient** (see later)[2], and is also denoted by

$$\text{grad}\,\phi \equiv \nabla\phi \tag{1.6.12}$$

Example (of the Gradient of a Scalar Field)

Consider a two-dimensional temperature field $\theta = x_1^2 + x_2^2$. Then

$$\nabla\theta = 2x_1\mathbf{e}_1 + 2x_2\mathbf{e}_2$$

For example, at $(1,0)$, $\theta = 1$, $\nabla\theta = 2\mathbf{e}_1$ and at $(1,1)$, $\theta = 2$, $\nabla\theta = 2\mathbf{e}_1 + 2\mathbf{e}_2$, Fig. 1.6.5. Note the following:

(i) $\nabla\theta$ points in the direction *normal* to the curve $\theta =$ const.
(ii) the direction of *maximum* rate of change of θ is in the direction of $\nabla\theta$

---

[2] in this context, a *gradient* is a derivative with respect to a position vector, but the term gradient is used more generally than this, e.g. see §1.14

(iii) the direction of zero $d\theta$ is in the direction *perpendicular* to $\nabla\theta$

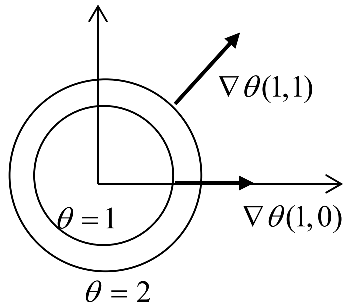


Figure 1.6.5: gradient of a temperature field

The curves $\theta(x_1, x_2) = \text{const.}$ are called **isotherms** (curves of constant temperature). In general, they are called **iso-curves** (or **iso-surfaces** in three dimensions).

■

Many physical laws are given in terms of the gradient of a scalar field. For example, **Fourier's law** of heat conduction relates the heat flux **q** (the rate at which heat flows through a surface of unit area[3]) to the temperature gradient through

$$\mathbf{q} = -k\nabla\theta \tag{1.6.13}$$

where *k* is the **thermal conductivity** of the material, so that heat flows along the direction normal to the isotherms.

The Normal to a Surface

In the above example, it was seen that $\nabla\theta$ points in the direction normal to the curve $\theta = \text{const.}$ Here it will be seen generally how and why the gradient can be used to obtain a normal vector to a surface.

Consider a surface represented by the scalar function $f(x_1, x_2, x_3) = c$, c a constant[4], and also a space curve *C* lying on the surface, defined by the position vector $\mathbf{r} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$. The components of **r** must satisfy the equation of the surface, so $f(x_1(t), x_2(t), x_3(t)) = c$. Differentiation gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} = 0$$

---

[3] the **flux** is the rate of flow of fluid, particles or energy through a given surface; the **flux density** is the flux per unit area but, as here, this is more commonly referred to simply as the flux

[4] a surface can be represented by the equation $f(x_1, x_2, x_3) = c$; for example, the expression $x_1^2 + x_2^2 + x_3^2 = 4$ is the equation for a sphere of radius 2 (with centre at the origin). Alternatively, the surface can be written in the form $x_3 = g(x_1, x_2)$, for example $x_3 = \sqrt{4 - x_1^2 - x_2^2}$

which is equivalent to the equation $\operatorname{grad} f \cdot (d\mathbf{r}/dt) = 0$ and, as seen in §1.6.2, $d\mathbf{r}/dt$ is a vector tangential to the surface. Thus $\operatorname{grad} f$ is normal to the tangent vector; $\operatorname{grad} f$ must be normal to all the tangents to all the curves through *p*, so it must be normal to the plane tangent to the surface.

Taylor's Series

Writing ϕ as a function of three variables (omitting time *t*), so that $\phi = \phi(x_1, x_2, x_3)$, then ϕ can be expanded in a three-dimensional Taylor's series:

$$\phi(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) = \phi(x_1, x_2, x_3) + \left\{ \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 \right\} + \frac{1}{2} \left\{ \frac{\partial^2 \phi}{\partial x_1^2} (dx_1)^2 + \cdots \right\}$$

Neglecting the higher order terms, this can be written as

$$\phi(\mathbf{x} + d\mathbf{x}) = \phi(\mathbf{x}) + \frac{\partial \phi}{\partial \mathbf{x}} \cdot d\mathbf{x}$$

which is equivalent to 1.6.9, 1.6.10.

1.6.5 The Nabla Operator

The symbolic vector operator ∇ is called the **Nabla operator**[5]. One can write this in component form as

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} = \mathbf{e}_i \frac{\partial}{\partial x_i} \tag{1.6.14}$$

One can generalise the idea of the gradient of a scalar field by defining the dot product and the cross product of the vector operator ∇ with a vector field $(\bullet)$, according to the rules

$$\nabla \cdot (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot (\bullet), \quad \nabla \times (\bullet) = \mathbf{e}_i \frac{\partial}{\partial x_i} \times (\bullet) \tag{1.6.15}$$

The following terminology is used:

$$\begin{aligned} \operatorname{grad} \phi &= \nabla \phi \\ \operatorname{div} \mathbf{u} &= \nabla \cdot \mathbf{u} \\ \operatorname{curl} \mathbf{u} &= \nabla \times \mathbf{u} \end{aligned} \tag{1.6.16}$$

[5] or **del** or the **Gradient operator**

These latter two are discussed in the following sections.

1.6.6 The Divergence of a Vector Field

From the definition (1.6.15), the **divergence** of a vector field $\mathbf{a}(\mathbf{x})$ is the scalar field

$$\boxed{\begin{aligned}\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a} &= \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(a_j\mathbf{e}_j\right) = \frac{\partial a_i}{\partial x_i} \\ &= \frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}+\frac{\partial a_3}{\partial x_3}\end{aligned}} \quad \textbf{Divergence of a Vector Field} \quad (1.6.17)$$

Differential Elements & Physical interpretations of the Divergence

Consider a flowing compressible[6] material with velocity field $\mathbf{v}(x_1, x_2, x_3)$. Consider now a **differential element** of this material, with dimensions $\Delta x_1, \Delta x_2, \Delta x_3$, with bottom left-hand corner at (x_1, x_2, x_3), fixed in space and through which the material flows[7], Fig. 1.6.6.

The component of the velocity in the x_1 direction, v_1, will vary over a face of the element but, *if the element is small*, the velocities will vary linearly as shown; only the components at the four corners of the face are shown for clarity.

Since [distance = time $\times$ velocity], the volume of material flowing through the right-hand face in time Δt is Δt times the "volume" bounded by the four corner velocities (between the right-hand face and the plane surface denoted by the dotted lines); it is straightforward to show that this volume is equal to the volume shown to the right, Fig. 1.6.6b, with constant velocity equal to the average velocity v_{ave}, which occurs at the centre of the face. Thus the volume of material flowing out is[8] $\Delta x_2 \Delta x_3 v_{ave}\Delta t$ and the **volume flux**, i.e. the *rate* of volume flow, is $\Delta x_2 \Delta x_3 v_{ave}$. Now

$$v_{ave} = v_1(x_1 + \Delta x_1, x_2 + \tfrac{1}{2}\Delta x_2, x_3 + \tfrac{1}{2}\Delta x_3)$$

Using a Taylor's series expansion, and neglecting higher order terms,

$$v_{ave} \approx v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial v_1}{\partial x_3}$$

---

[6] that is, it can be compressed or expanded

[7] this type of fixed volume in space, used in analysis, is called a **control volume**

[8] the velocity will change by a small amount during the time interval Δt. One could use the average velocity in the calculation, i.e. $\frac{1}{2}\left(v_1(\mathbf{x}, t) + v_1(\mathbf{x}, t+\Delta t)\right)$, but in the limit as $\Delta t \to 0$, this will reduce to $v_1(\mathbf{x}, t)$

with the partial derivatives evaluated at (x_1, x_2, x_3), so the volume flux out is

$$\Delta x_2 \Delta x_3 \left\{ v_1(x_1, x_2, x_3) + \Delta x_1 \frac{\partial v_1}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial v_1}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial v_1}{\partial x_3} \right\}$$

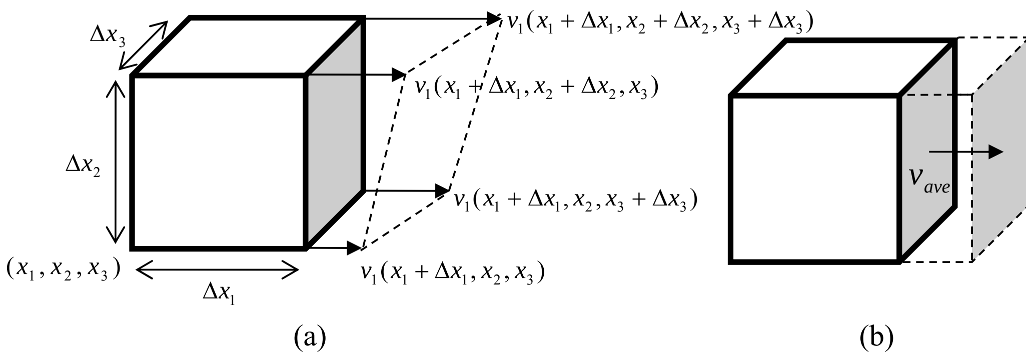


Figure 1.6.6: a differential element; (a) flow through a face, (b) volume of material flowing through the face

The net volume flux out (rate of volume flow out through the right-hand face minus the rate of volume flow in through the left-hand face) is then $\Delta x_1 \Delta x_2 \Delta x_3 (\partial v_1 / \partial x_1)$ and the net volume flux per unit volume is $\partial v_1 / \partial x_1$. Carrying out a similar calculation for the other two coordinate directions leads to

$$\textit{net unit volume flux out of an elemental volume}: \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \equiv \operatorname{div}\mathbf{v} \qquad (1.6.18)$$

which is the physical meaning of the divergence of the velocity field.

If $\operatorname{div}\mathbf{v} > 0$, there is a net flow out and the density of material is decreasing. On the other hand, if $\operatorname{div}\mathbf{v} = 0$, the inflow equals the outflow and the density remains constant – such a material is called **incompressible**[9]. A flow which is divergence free is said to be **isochoric**. A vector $\mathbf{v}$ for which $\operatorname{div}\mathbf{v} = 0$ is said to be **solenoidal**.

Notes:
- The above result holds only in the limit when the element shrinks to zero size – so that the extra terms in the Taylor series tend to zero and the velocity field varies in a linear fashion over a face
- consider the velocity at a fixed point in space, $\mathbf{v}(\mathbf{x}, t)$. The velocity at a later time, $\mathbf{v}(\mathbf{x}, t + \Delta t)$, actually gives the velocity of a different material particle. This is shown in Fig. 1.6.7 below: the material particles $1, 2, 3$ are moving through space and whereas $\mathbf{v}(\mathbf{x}, t)$ represents the velocity of particle 2, $\mathbf{v}(\mathbf{x}, t + \Delta t)$ now represents the velocity of particle 1, which has moved into position $\mathbf{x}$. This point is important in the consideration of the kinematics of materials, to be discussed in Chapter 2

[9] a **liquid**, such as water, is a material which is incompressible

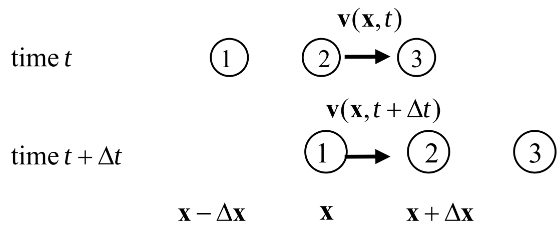


Figure 1.6.7: moving material particles

Another example would be the divergence of the heat flux vector **q**. This time suppose also that there is some generator of heat inside the element (a **source**), generating at a rate of *r* per unit volume, *r* being a scalar field. Again, assuming the element to be small, one takes *r* to be acting at the mid-point of the element, and one considers $r(x_1 + \frac{1}{2}\Delta x_1, \cdots)$. Assume a **steady-state** heat flow, so that the (heat) energy within the elemental volume remains constant with time – the law of balance of (heat) energy then requires that the net flow of heat out must equal the heat generated within, so

$$
\begin{aligned}
&\Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_1}{\partial x_1} + \Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_2}{\partial x_2} + \Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial q_3}{\partial x_3} \\
&\qquad = \Delta x_1 \Delta x_2 \Delta x_3 \left\{ r(x_1, x_2, x_3) + \tfrac{1}{2}\Delta x_1 \frac{\partial r}{\partial x_1} + \tfrac{1}{2}\Delta x_2 \frac{\partial r}{\partial x_2} + \tfrac{1}{2}\Delta x_3 \frac{\partial r}{\partial x_3} \right\}
\end{aligned}
$$

Dividing through by $\Delta x_1 \Delta x_2 \Delta x_3$ and taking the limit as $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$, one obtains

$$
\operatorname{div}\mathbf{q} = r \tag{1.6.19}
$$

Here, the divergence of the heat flux vector field can be interpreted as the heat generated (or absorbed) per unit volume per unit time in a temperature field. If the divergence is zero, there is no heat being generated (or absorbed) and the heat leaving the element is equal to the heat entering it.

1.6.7 The Laplacian

Combining Fourier's law of heat conduction (1.6.13), $\mathbf{q} = -k\nabla\theta$, with the energy balance equation (1.6.19), $\operatorname{div}\mathbf{q} = r$, and assuming the conductivity is constant, leads to $-k\,\nabla \cdot \nabla\theta = r$. Now

$$
\begin{aligned}
\nabla \cdot \nabla\theta &= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \left(\frac{\partial\theta}{\partial x_j} \mathbf{e}_j \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial\theta}{\partial x_j} \right) \delta_{ij} = \frac{\partial^2\theta}{\partial x_i^2} \\
&= \frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} + \frac{\partial^2\theta}{\partial x_3^2}
\end{aligned}
\tag{1.6.20}
$$

This expression is called the **Laplacian** of θ. By introducing the Laplacian operator $\nabla^2 \equiv \nabla \cdot \nabla$, one has

$$\nabla^2 \theta = -\frac{r}{k} \tag{1.6.21}$$

This equation governs the steady state heat flow for constant conductivity. In general, the equation $\nabla^2 \phi = a$ is called **Poisson's equation**. When there are no heat sources (or sinks), one has **Laplace's equation**, $\nabla^2 \theta = 0$. Laplace's and Poisson's equation arise in many other mathematical models in mechanics, electromagnetism, etc.

1.6.8 The Curl of a Vector Field

From the definition 1.6.15 and 1.6.14, the **curl** of a vector field $\mathbf{a}(\mathbf{x})$ is the vector field

$$\boxed{\begin{aligned} \operatorname{curl} \mathbf{a} = \nabla \times \mathbf{a} &= \mathbf{e}_i \frac{\partial}{\partial x_i} \times \left(a_j \mathbf{e}_j\right) \\ &= \frac{\partial a_j}{\partial x_i} \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k \end{aligned}} \qquad \textbf{Curl of a Vector Field} \tag{1.6.22}$$

It can also be expressed in the form

$$\begin{aligned} \operatorname{curl} \mathbf{a} = \nabla \times \mathbf{a} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \mathbf{e}_k = \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j} \mathbf{e}_i = \varepsilon_{ijk} \frac{\partial a_i}{\partial x_k} \mathbf{e}_j \end{aligned} \tag{1.6.23}$$

Note: the divergence and curl of a vector field are independent of any coordinate system (for example, the divergence of a vector and the length and direction of $\operatorname{curl} \mathbf{a}$ are independent of the coordinate system in use) – these will be re-defined without reference to any particular coordinate system when discussing tensors (see §1.14).

Physical interpretation of the Curl

Consider a particle with position vector **r** and moving with velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, that is, with an angular velocity ω about an axis in the direction of $\boldsymbol{\omega}$. Then {▲Problem 7}

$$\operatorname{curl} \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega} \tag{1.6.24}$$

Thus the curl of a vector field is associated with rotational properties. In fact, if **v** is the velocity of a moving fluid, then a small paddle wheel placed in the fluid would tend to rotate in regions where $\operatorname{curl} \mathbf{v} \neq 0$, in which case the velocity field **v** is called a **vortex**

field. The paddle wheel would remain stationary in regions where $\text{curl}\,\mathbf{v} = 0$, in which case the velocity field $\mathbf{v}$ is called **irrotational**.

1.6.9 Identities

Here are some important identities of vector calculus {▲Problem 8}:

$$\begin{aligned} \text{grad}(\phi + \psi) &= \text{grad}\,\phi + \text{grad}\,\psi \\ \text{div}(\mathbf{u} + \mathbf{v}) &= \text{div}\,\mathbf{u} + \text{div}\,\mathbf{v} \\ \text{curl}(\mathbf{u} + \mathbf{v}) &= \text{curl}\,\mathbf{u} + \text{curl}\,\mathbf{v} \end{aligned} \tag{1.6.25}$$

$$\begin{aligned} \text{grad}(\phi\psi) &= \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi \\ \text{div}(\phi\mathbf{u}) &= \phi\,\text{div}\,\mathbf{u} + \text{grad}\,\phi \cdot \mathbf{u} \\ \text{curl}(\phi\mathbf{u}) &= \phi\,\text{curl}\,\mathbf{u} + \text{grad}\,\phi \times \mathbf{u} \\ \text{div}(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \\ \text{curl}(\text{grad}\,\phi) &= \mathbf{o} \\ \text{div}(\text{curl}\,\mathbf{u}) &= 0 \\ \text{div}(\lambda\,\text{grad}\,\phi) &= \lambda\nabla^2\phi + \text{grad}\,\lambda \cdot \text{grad}\,\phi \end{aligned} \tag{1.6.26}$$

1.6.10 Cylindrical and Spherical Coordinates

Cartesian coordinates have been used exclusively up to this point. In many practical problems, it is easier to carry out an analysis in terms of cylindrical or spherical coordinates. Differentiation in these coordinate systems is discussed in what follows[10].

Cylindrical Coordinates

Cartesian and cylindrical coordinates are related through (see Fig. 1.6.8)

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2 + y^2} \\ y &= r\sin\theta\,, & \theta &= \tan^{-1}(y/x) \\ z &= z & z &= z \end{aligned} \tag{1.6.27}$$

Then the Cartesian partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{1.6.28}$$

[10] this section also serves as an introduction to the more general topic of **Curvilinear Coordinates** covered in §1.16-§1.19

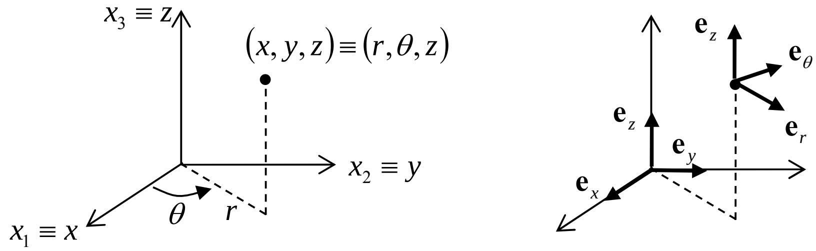


Figure 1.6.8: cylindrical coordinates

The base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \qquad & \mathbf{e}_r &= \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta, \qquad & \mathbf{e}_\theta &= -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta \\ \mathbf{e}_z &= \mathbf{e}_z & \mathbf{e}_z &= \mathbf{e}_z \end{aligned} \tag{1.6.29}$$

so that from Eqn. 1.6.14, after some algebra, the Nabla operator in cylindrical coordinates reads as {▲Problem 9}

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.6.30}$$

which allows one to take the gradient of a scalar field in cylindrical coordinates:

$$\nabla\phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \phi}{\partial z}\mathbf{e}_z \tag{1.6.31}$$

Cartesian base vectors are independent of position. However, the cylindrical base vectors, although they are always of unit magnitude, change direction with position. In particular, the directions of the base vectors $\mathbf{e}_r, \mathbf{e}_\theta$ depend on θ, and so these base vectors have derivatives with respect to θ: from Eqn. 1.6.29,

$$\begin{aligned} \frac{\partial}{\partial \theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial \theta}\mathbf{e}_\theta &= -\mathbf{e}_r \end{aligned} \tag{1.6.32}$$

with all other derivatives of the base vectors with respect to r, θ, z equal to zero.

The divergence can now be evaluated:

$$\nabla \cdot \mathbf{v} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right)$$
$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \qquad (1.6.33)$$

Similarly the curl of a vector and the Laplacian of a scalar are {▲Problem 10}

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right)\mathbf{e}_\theta + \left[\frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta} \right) \right]\mathbf{e}_z$$
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \qquad (1.6.34)$$

Spherical Coordinates

Cartesian and spherical coordinates are related through (see Fig. 1.6.9)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta \end{aligned} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi &= \tan^{-1}(y/x) \end{aligned} \qquad (1.6.35)$$

and the base vectors are related through

$$\begin{aligned} \mathbf{e}_x &= \mathbf{e}_r \sin\theta\cos\phi + \mathbf{e}_\theta \cos\theta\cos\phi - \mathbf{e}_\phi \sin\phi \\ \mathbf{e}_y &= \mathbf{e}_r \sin\theta\sin\phi + \mathbf{e}_\theta \cos\theta\sin\phi + \mathbf{e}_\phi \cos\phi \\ \mathbf{e}_z &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \\ \\ \mathbf{e}_r &= \mathbf{e}_x \sin\theta\cos\phi + \mathbf{e}_y \sin\theta\sin\phi + \mathbf{e}_z \cos\theta \\ \mathbf{e}_\theta &= \mathbf{e}_x \cos\theta\cos\phi + \mathbf{e}_y \cos\theta\sin\phi - \mathbf{e}_z \sin\theta \\ \mathbf{e}_\phi &= -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi \end{aligned} \qquad (1.6.36)$$

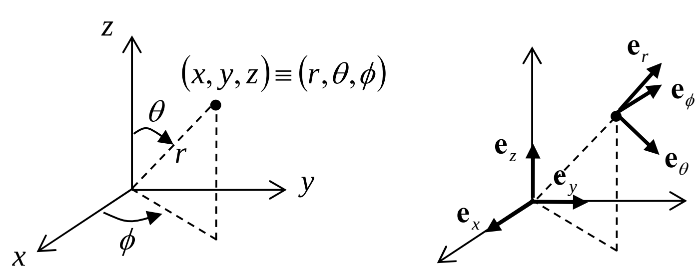


Figure 1.6.9: spherical coordinates

In this case the non-zero derivatives of the base vectors are

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathbf{e}_r &= \mathbf{e}_\theta \\ \frac{\partial}{\partial\theta}\mathbf{e}_\theta &= -\mathbf{e}_r\end{aligned}, \quad \begin{aligned}\frac{\partial}{\partial\phi}\mathbf{e}_r &= \sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\theta &= \cos\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial\phi}\mathbf{e}_\phi &= -\sin\theta\mathbf{e}_r - \cos\theta\mathbf{e}_\theta\end{aligned} \tag{1.6.37}$$

and it can then be shown that {▲Problem 11}

$$\begin{aligned}\nabla\varphi &= \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\phi \\ \nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \\ \nabla^2\varphi &= \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\end{aligned} \tag{1.6.38}$$

1.6.11 The Directional Derivative

Consider a function $\phi(\mathbf{x})$. The directional derivative of φ in the direction of some vector $\mathbf{w}$ is the change in φ in that direction. Now the difference between its values at position $\mathbf{x}$ and $\mathbf{x}+\mathbf{w}$ is, Fig. 1.6.10,

$$d\phi = \phi(\mathbf{x}+\mathbf{w}) - \phi(\mathbf{x}) \tag{1.6.39}$$

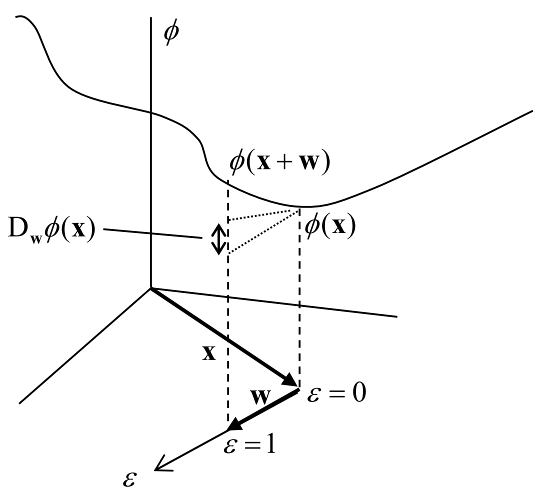


Figure 1.6.10: the directional derivative

An approximation to $d\phi$ can be obtained by introducing a parameter ε and by considering the function $\phi(\mathbf{x}+\varepsilon\mathbf{w})$; one has $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=0} = \phi(\mathbf{x})$ and $\phi(\mathbf{x}+\varepsilon\mathbf{w})_{\varepsilon=1} = \phi(\mathbf{x}+\mathbf{w})$.

If one treats ϕ as a function of ε, a Taylor's series about $\varepsilon = 0$ gives

$$\phi(\varepsilon) = \phi(0) + \varepsilon \left.\frac{d\phi(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left.\frac{d^2\phi(\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} + \cdots$$

or, writing it as a function of $\mathbf{x}+\varepsilon\mathbf{w}$,

$$\phi(\mathbf{x}+\varepsilon\mathbf{w}) = \phi(\mathbf{x}) + \varepsilon \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w}) + \cdots$$

By setting $\varepsilon = 1$, the derivative here can be seen to be a linear approximation to the increment $d\phi$, Eqn. 1.6.39. This is defined as the **directional derivative** of the function $\phi(\mathbf{x})$ at the point **x** in the direction of **w**, and is denoted by

$$\boxed{\partial_{\mathbf{x}}\phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{x}+\varepsilon\mathbf{w})} \qquad \textbf{The Directional Derivative} \qquad (1.6.40)$$

The directional derivative is also written as $\mathrm{D}_{\mathbf{w}}\phi(\mathbf{x})$.

The power of the directional derivative as defined by Eqn. 1.6.40 is its generality, as seen in the following example.

Example (the Directional Derivative of the Determinant)

Consider the directional derivative of the determinant of the 2×2 matrix **A**, in the direction of a second matrix **T** (the word "direction" is obviously used loosely in this context). One has

$$\begin{aligned}
\partial_{\mathbf{A}}(\det\mathbf{A})[\mathbf{T}] &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \det(\mathbf{A}+\varepsilon\mathbf{T}) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left[(A_{11}+\varepsilon T_{11})(A_{22}+\varepsilon T_{22}) - (A_{12}+\varepsilon T_{12})(A_{21}+\varepsilon T_{21})\right] \\
&= A_{11}T_{22} + A_{22}T_{11} - A_{12}T_{21} - A_{21}T_{12}
\end{aligned}$$

■

The Directional Derivative and The Gradient

Consider a scalar-valued function ϕ of a vector **z**. Let **z** be a function of a parameter ε, $\phi \equiv \phi(z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon))$. Then

$$\frac{d\phi}{d\varepsilon} = \frac{\partial \phi}{\partial z_i}\frac{dz_i}{d\varepsilon} = \frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}$$

Thus, with $\mathbf{z} = \mathbf{x} + \varepsilon\mathbf{w}$,

$$\partial_{\mathbf{x}}\phi[\mathbf{w}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi(\mathbf{z}(\varepsilon)) = \left(\frac{\partial \phi}{\partial \mathbf{z}} \cdot \frac{d\mathbf{z}}{d\varepsilon}\right)_{\varepsilon=0} = \frac{\partial \phi}{\partial \mathbf{x}} \cdot \mathbf{w} \qquad (1.6.41)$$

which can be compared with Eqn. 1.6.11. Note that for Eqns. 1.6.11 and 1.6.41 to be consistent definitions of the directional derivative, **w** here should be a *unit* vector.

1.6.12 Formal Treatment of Vector Calculus

The calculus of vectors is now treated more formally in what follows, following on from the introductory section in §1.2. Consider a vector **h**, an element of the Euclidean vector space *E*, $\mathbf{h} \in E$. In order to be able to speak of limits as elements become "small" or "close" to each other in this space, one requires a norm. Here, take the standard Euclidean norm on *E*, Eqn. 1.2.8,

$$\|\mathbf{h}\| \equiv \sqrt{\langle \mathbf{h}, \mathbf{h} \rangle} = \sqrt{\mathbf{h} \cdot \mathbf{h}} \qquad (1.6.42)$$

Consider next a scalar function $f : E \to R$. If there is a constant $M > 0$ such that $|f(\mathbf{h})| \le M\|\mathbf{h}\|$ as $\mathbf{h} \to \mathbf{o}$, then one writes

$$f(\mathbf{h}) = O(\|\mathbf{h}\|) \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \qquad (1.6.43)$$

This is called the **Big Oh** (or **Landau**) notation. Eqn. 1.6.43 states that $|f(\mathbf{h})|$ goes to zero at least as fast as $\|\mathbf{h}\|$. An expression such as

$$f(\mathbf{h}) = g(\mathbf{h}) + O(\|\mathbf{h}\|) \qquad (1.6.44)$$

then means that $|f(\mathbf{h}) - g(\mathbf{h})|$ is smaller than $\|\mathbf{h}\|$ for $\mathbf{h}$ sufficiently close to **o**.

Similarly, if

$$\frac{f(\mathbf{h})}{\|\mathbf{h}\|} \to 0 \quad \text{as} \quad \mathbf{h} \to \mathbf{o} \qquad (1.6.45)$$

then one writes $f(\mathbf{h}) = o(\|\mathbf{h}\|)$ as $\mathbf{h} \to \mathbf{o}$. This implies that $|f(\mathbf{h})|$ goes to zero faster than $\|\mathbf{h}\|$.

A **field** is a function which is defined in a Euclidean (point) space E^3. A **scalar field** is then a function $f : E^3 \to R$. A scalar field is **differentiable** at a point $\mathbf{x} \in E^3$ if there exists a vector $Df(\mathbf{x}) \in E$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + Df(\mathbf{x}) \cdot \mathbf{h} + o(\|\mathbf{h}\|) \quad \text{for all} \quad \mathbf{h} \in E \tag{1.6.46}$$

In that case, the vector $Df(\mathbf{x})$ is called the **derivative** (or **gradient**) of *f* at $\mathbf{x}$ (and is given the symbol $\nabla f(\mathbf{x})$).

Now setting $\mathbf{h} = \varepsilon \mathbf{w}$ in 1.6.46, where $\mathbf{w} \in E$ is a unit vector, dividing through by ε and taking the limit as $\varepsilon \to 0$, one has the equivalent statement

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(\mathbf{x}+\varepsilon \mathbf{w}) \quad \text{for all} \quad \mathbf{w} \in E \tag{1.6.47}$$

which is 1.6.41. In other words, for the derivative to exist, the scalar field must have a directional derivative in all directions at $\mathbf{x}$.

Using the chain rule as in §1.6.11, Eqn. 1.6.47 can be expressed in terms of the Cartesian basis $\{\mathbf{e}_i\}$,

$$\nabla f(\mathbf{x}) \cdot \mathbf{w} = \frac{\partial f}{\partial x_i} w_i = \frac{\partial f}{\partial x_i} \mathbf{e}_i \cdot w_j \mathbf{e}_j \tag{1.6.48}$$

This must be true for all $\mathbf{w}$ and so, in a Cartesian basis,

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x_i} \mathbf{e}_i \tag{1.6.49}$$

which is Eqn. 1.6.9.

1.6.13 Problems

1. A particle moves along a curve in space defined by
$$\mathbf{r} = (t^3 - 4t)\mathbf{e}_1 + (t^2 + 4t)\mathbf{e}_2 + (8t^2 - 3t^3)\mathbf{e}_3$$
   Here, *t* is time. Find
   (i) a unit tangent vector at $t = 2$
   (ii) the magnitudes of the tangential and normal components of acceleration at $t = 2$
2. Use the index notation (1.3.12) to show that $\frac{d}{dt}(\mathbf{v} \times \mathbf{a}) = \mathbf{v} \times \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{v}}{dt} \times \mathbf{a}$. Verify this result for $\mathbf{v} = 3t\mathbf{e}_1 - t^2\mathbf{e}_3$, $\mathbf{a} = t^2\mathbf{e}_1 + t\mathbf{e}_2$. [Note: the permutation symbol and the unit vectors are independent of *t*; the components of the vectors are scalar functions of *t* which can be differentiated in the usual way, for example by using the product rule of differentiation.]

3. The density distribution throughout a material is given by $\rho = 1 + \mathbf{x} \cdot \mathbf{x}$.
   (i) what sort of function is this?
   (ii) the density is given in symbolic notation - write it in index notation
   (iii) evaluate the gradient of ρ
   (iv) give a unit vector in the direction in which the density is increasing the most
   (v) give a unit vector in *any* direction in which the density is not increasing
   (vi) take any unit vector other than the base vectors and the other vectors you used above and calculate $d\rho / dx$ in the direction of this unit vector
   (vii) evaluate and sketch all these quantities for the point (2,1).
   In parts (iii-iv), give your answer in (a) symbolic, (b) index, and (c) full notation.
4. Consider the scalar field defined by $\phi = x^2 + 3yx + 2z$.
   (i) find the unit normal to the surface of constant ϕ at the origin (0,0,0)
   (ii) what is the maximum value of the directional derivative of ϕ at the origin?
   (iii) evaluate $d\phi / dx$ at the origin if $d\mathbf{x} = ds(\mathbf{e}_1 + \mathbf{e}_3)$.
5. If $\mathbf{u} = x_1 x_2 x_3 \mathbf{e}_1 + x_1 x_2 \mathbf{e}_2 + x_1 \mathbf{e}_3$, determine $\operatorname{div}\mathbf{u}$ and $\operatorname{curl}\mathbf{u}$.
6. Determine the constant *a* so that the vector
$$\mathbf{v} = (x_1 + 3x_2)\mathbf{e}_1 + (x_2 - 2x_3)\mathbf{e}_2 + (x_1 + ax_3)\mathbf{e}_3$$
   is solenoidal.
7. Show that $\operatorname{curl}\mathbf{v} = 2\boldsymbol{\omega}$ (see also Problem 9 in §1.1).
8. Verify the identities (1.6.25-26).
9. Use (1.6.14) to derive the Nabla operator in cylindrical coordinates (1.6.30).
10. Derive Eqn. (1.6.34), the curl of a vector and the Laplacian of a scalar in the cylindrical coordinates.
11. Derive (1.6.38), the gradient, divergence and Laplacian in spherical coordinates.
12. Show that the directional derivative $\mathrm{D}_{\mathbf{v}}\phi(\mathbf{u})$ of the scalar-valued function of a vector $\phi(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}$, in the direction **v**, is $2\mathbf{u} \cdot \mathbf{v}$.
13. Show that the directional derivative of the functional
$$U(v(x)) = \frac{1}{2}\int_0^l EI\left(\frac{d^2 v}{dx^2}\right)^2 dx - \int_0^l p(x)v(x)dx$$
   in the direction of $\omega(x)$ is given by
$$\int_0^l EI \frac{d^2 v(x)}{dx^2}\frac{d^2 \omega(x)}{dx^2} dx - \int_0^l p(x)\omega(x)dx .$$