1.5 Coordinate Transformation of Vector Components

Very often in practical problems, the components of a vector are known in one coordinate system but it is necessary to find them in some other coordinate system.

For example, one might know that the force **f** acting "in the x_1 direction" has a certain value, Fig. 1.5.1 – this is equivalent to knowing the x_1 component of the force, in an $x_1 - x_2$ coordinate system. One might then want to know what force is "acting" in some other direction – for example in the x_1' direction shown – this is equivalent to asking what the x_1' component of the force is in a new $x_1' - x_2'$ coordinate system.

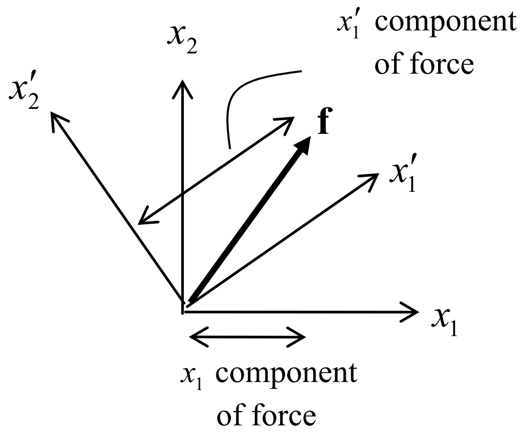

Figure 1.5.1: a vector represented using two different coordinate systems

The relationship between the components in one coordinate system and the components in a second coordinate system are called the **transformation equations**. These transformation equations are derived and discussed in what follows.

1.5.1 Rotations and Translations

Any change of Cartesian coordinate system will be due to a **translation** of the base vectors and a **rotation** of the base vectors. A translation of the base vectors does not change the components of a vector. Mathematically, this can be expressed by saying that the components of a vector **a** are $\mathbf{e}_i \cdot \mathbf{a}$, and these three quantities do not change under a translation of base vectors. Rotation of the base vectors is thus what one is concerned with in what follows.

1.5.2 Components of a Vector in Different Systems

Vectors are mathematical objects which exist *independently of any coordinate system*. Introducing a coordinate system for the purpose of analysis, one could choose, for example, a certain Cartesian coordinate system with base vectors $\mathbf{e}_i$ and origin *o*, Fig.

1.5.2. In that case the vector can be written as $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, and u_1, u_2, u_3 are its components.

Now a second coordinate system can be introduced (with the same origin), this time with base vectors $\mathbf{e}'_i$. In that case, the vector can be written as $\mathbf{u} = u'_1\mathbf{e}'_1 + u'_2\mathbf{e}'_2 + u'_3\mathbf{e}'_3$, where u'_1, u'_2, u'_3 are its components in this second coordinate system, as shown in the figure. Thus the *same vector* can be written in more than one way:

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = u'_1\mathbf{e}'_1 + u'_2\mathbf{e}'_2 + u'_3\mathbf{e}'_3 \tag{1.5.1}$$

The first coordinate system is often referred to as "the $ox_1x_2x_3$ system" and the second as "the $ox'_1x'_2x'_3$ system".

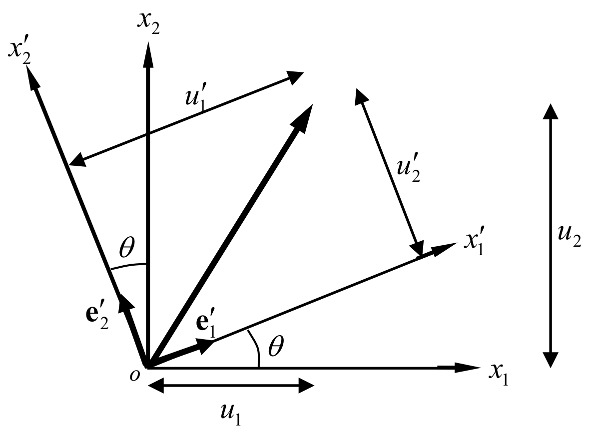

Figure 1.5.2: a vector represented using two different coordinate systems

Note that the new coordinate system is obtained from the first one by a *rotation* of the base vectors. The figure shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Two Dimensions

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 1.5.3),

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \\ &= \left[|OB| - |AB|\right]\mathbf{e}_1 + \left[|BD| + |CP|\right]\mathbf{e}_2 \\ &= \left[\cos\theta\, u'_1 - \sin\theta\, u'_2\right]\mathbf{e}_1 + \left[\sin\theta\, u'_1 + \cos\theta\, u'_2\right]\mathbf{e}_2 \end{aligned} \tag{1.5.2}$$

and so

$$
\begin{aligned}
u_1 &= \cos\theta\, u_1' - \sin\theta\, u_2' \\
u_2 &= \sin\theta\, u_1' + \cos\theta\, u_2'
\end{aligned}
$$

vector components in first coordinate system

vector components in second coordinate system

In matrix form, these transformation equations can be written as

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \tag{1.5.3}
$$

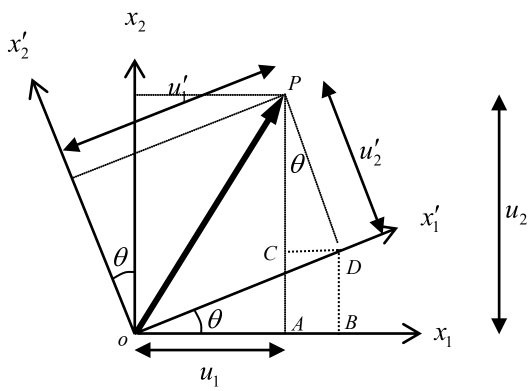

Figure 1.5.3: geometry of the 2D coordinate transformation

The 2×2 matrix is called the **transformation** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$
\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.5.4}
$$

An important property of the transformation matrix is that it is **orthogonal**, by which is meant that

$$
\boxed{[\mathbf{Q}^{-1}] = [\mathbf{Q}^{\mathrm{T}}]} \quad \textbf{Orthogonality of Transformation/Rotation Matrix} \tag{1.5.5}
$$

Three Dimensions

The three dimensional case is shown in Fig. 1.5.4a. In this more general case, note that

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{u} \\ \mathbf{e}_2 \cdot \mathbf{u} \\ \mathbf{e}_3 \cdot \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \left(u_1'\mathbf{e}_1' + u_2'\mathbf{e}_2' + u_3'\mathbf{e}_3' \right) \\ \mathbf{e}_2 \cdot \left(u_1'\mathbf{e}_1' + u_2'\mathbf{e}_2' + u_3'\mathbf{e}_3' \right) \\ \mathbf{e}_3 \cdot \left(u_1'\mathbf{e}_1' + u_2'\mathbf{e}_2' + u_3'\mathbf{e}_3' \right) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{e}_1' & \mathbf{e}_1 \cdot \mathbf{e}_2' & \mathbf{e}_1 \cdot \mathbf{e}_3' \\ \mathbf{e}_2 \cdot \mathbf{e}_1' & \mathbf{e}_2 \cdot \mathbf{e}_2' & \mathbf{e}_2 \cdot \mathbf{e}_3' \\ \mathbf{e}_3 \cdot \mathbf{e}_1' & \mathbf{e}_3 \cdot \mathbf{e}_2' & \mathbf{e}_3 \cdot \mathbf{e}_3' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} \quad (1.5.6)$$

The dot products of the base vectors from the two different coordinate systems can be seen to be the cosines of the angles between the coordinate axes. This is illustrated in Fig. 1.5.4b for the case of $\mathbf{e}_1' \cdot \mathbf{e}_j$. In general:

$$\mathbf{e}_i \cdot \mathbf{e}_j' = \cos\left(x_i, x_j' \right) \quad (1.5.7)$$

The nine quantities $\cos(x_i, x_j')$ are called the **direction cosines**, and Eqn. 1.5.6 can be expressed alternatively as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} \quad (1.5.8)$$

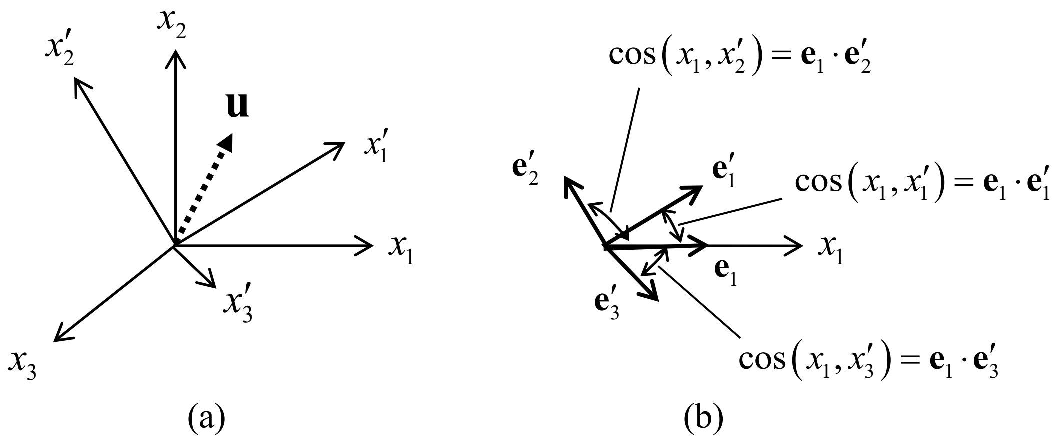

Figure 1.5.4: a 3D space: (a) two different coordinate systems, (b) direction cosines

Again denoting the components of this transformation matrix by the letter Q, $Q_{11} = \cos(x_1, x_1'), Q_{12} = \cos(x_1, x_2')$, etc., so that

$$Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j' . \quad (1.5.9)$$

One has the general 3D transformation matrix equations

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} \quad (1.5.10)$$

or, in element form and short-hand matrix notation,

$$u_i = Q_{ij} u'_j \qquad \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \tag{1.5.11}$$

Note: some authors define the matrix of direction cosines to consist of the components $Q_{ij} = \cos(x'_i, x_j)$, so that the subscript *i* refers to the new coordinate system and the *j* to the old coordinate system, rather than the other way around as used here.

Formal Derivation of the Transformation Equations

The above derivation of the transformation equations Eqns. 1.5.11, $u_i = Q_{ij} u'_j$, is here carried out again using the index notation in a concise manner: start with the relations $\mathbf{u} = u_k \mathbf{e}_k = u'_j \mathbf{e}'_j$ and post-multiply both sides by $\mathbf{e}_i$ to get (the corresponding matrix representation is to the right (also, see Problem 3 in §1.4.3)):

$$\begin{aligned}
& u_k \mathbf{e}_k \cdot \mathbf{e}_i = u'_j \mathbf{e}'_j \cdot \mathbf{e}_i \\
& \to u_k \delta_{ki} = u'_j Q_{ij} \\
& \to \quad u_i = u'_j Q_{ij} \qquad \ldots \quad \left[\mathbf{u}^{\mathrm{T}}\right] = \left[\mathbf{u}'^{\mathrm{T}}\right]\left[\mathbf{Q}^{\mathrm{T}}\right] \\
& \to \quad u_i = Q_{ij} u'_j \qquad \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}']
\end{aligned} \tag{1.5.12}$$

The inverse equations are {▲Problem 3}

$$u'_i = Q_{ji} u_j \qquad \ldots \quad [\mathbf{u}'] = \left[\mathbf{Q}^{\mathrm{T}}\right][\mathbf{u}] \tag{1.5.13}$$

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

As in the two dimensional case, the transformation matrix is orthogonal, $\left[\mathbf{Q}^{\mathrm{T}}\right] = \left[\mathbf{Q}^{-1}\right]$. This follows from 1.5.11, 1.5.13.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\begin{aligned}
\mathbf{e}'_1 &= n_1^{(1)} \mathbf{e}_1 + n_2^{(1)} \mathbf{e}_2 + n_3^{(1)} \mathbf{e}_3 \\
\mathbf{e}'_2 &= n_1^{(2)} \mathbf{e}_1 + n_2^{(2)} \mathbf{e}_2 + n_3^{(2)} \mathbf{e}_3 \\
\mathbf{e}'_3 &= n_1^{(3)} \mathbf{e}_1 + n_2^{(3)} \mathbf{e}_2 + n_3^{(3)} \mathbf{e}_3
\end{aligned}$$

What is the transformation matrix?

<u>Solution</u>

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x'_j) = \mathbf{e}_i \cdot \mathbf{e}'_j$, so

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix}$$

■

1.5.3 Problems

1. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

   Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.

2. The $ox_1'x_2'x_3'$ coordinate system is obtained from the $ox_1x_2x_3$ coordinate system by a positive (counterclockwise) rotation of θ about the x_3 axis. Find the (full three dimensional) transformation matrix $[\mathbf{Q}]$. A further positive rotation β about the x_2 axis is then made to give the $ox_1''x_2''x_3''$ coordinate system. Find the corresponding transformation matrix $[\mathbf{P}]$. Then construct the transformation matrix $[\mathbf{R}]$ for the complete transformation from the $ox_1x_2x_3$ to the $ox_1''x_2''x_3''$ coordinate system.

3. Beginning with the expression $u_j \mathbf{e}_j \cdot \mathbf{e}_i' = u_k' \mathbf{e}_k' \cdot \mathbf{e}_i'$, formally derive the relation $u_i' = Q_{ji} u_j$ ($[\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}]$).