1.3 Cartesian Vectors

So far the discussion has been in **symbolic notation**[1], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. The vectors exist independently of any coordinate system. It turns out that much of vector (tensor) mathematics is more concise and easier to manipulate in such notation than in terms of corresponding component notations. However, there are many circumstances in which use of the component forms of vectors (and tensors) is more helpful – or essential. In this section, vectors are discussed in terms of components – **component form**.

1.3.1 The Cartesian Basis

Consider three dimensional (Euclidean) space. In this space, consider the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \tag{1.3.1}$$

so that they are mutually perpendicular (mutually **orthogonal**), and

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \tag{1.3.2}$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 1.3.1. Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

This set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1.3.3}$$

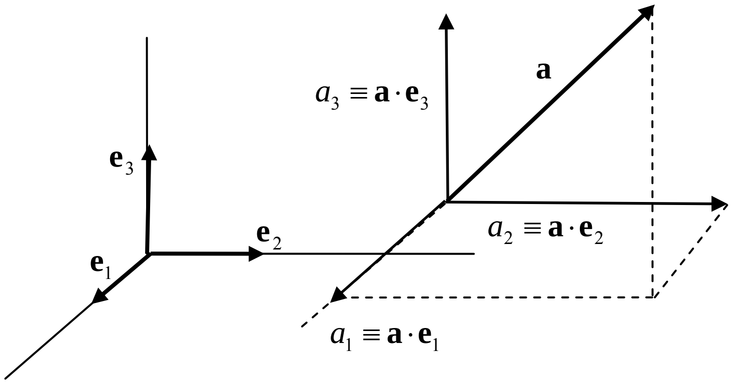

Figure 1.3.1: an orthonormal set of base vectors and Cartesian components

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[1] or **absolute** or **invariant** or **direct** or **vector** notation

By repeated application of Eqn. 1.1.2 to a vector **a**, and using 1.3.2, the scalars in 1.3.3 can be expressed as (see Fig. 1.3.1)

$$a_1 = \mathbf{a}\cdot\mathbf{e}_1, \quad a_2 = \mathbf{a}\cdot\mathbf{e}_2, \quad a_3 = \mathbf{a}\cdot\mathbf{e}_3 \tag{1.3.4}$$

The scalars a_1, a_2 and a_3 are called the **Cartesian components** of **a** in the given basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The unit vectors are called **base vectors** when used for this purpose.

Note that it is not necessary to have three mutually orthogonal vectors, or vectors of unit size, or a right-handed system, to form a basis – only that the three vectors are not co-planar. The right-handed orthonormal set is often the easiest basis to use in practice, but this is not always the case – for example, when one wants to describe a body with curved boundaries (e.g., see §1.6.10).

The dot product of two vectors **u** and **v**, referred to the above basis, can be written as

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1\cdot\mathbf{e}_1) + u_1v_2(\mathbf{e}_1\cdot\mathbf{e}_2) + u_1v_3(\mathbf{e}_1\cdot\mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2\cdot\mathbf{e}_1) + u_2v_2(\mathbf{e}_2\cdot\mathbf{e}_2) + u_2v_3(\mathbf{e}_2\cdot\mathbf{e}_3) \\
&\quad\quad + u_3v_1(\mathbf{e}_3\cdot\mathbf{e}_1) + u_3v_2(\mathbf{e}_3\cdot\mathbf{e}_2) + u_3v_3(\mathbf{e}_3\cdot\mathbf{e}_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned} \tag{1.3.5}$$

Similarly, the cross product is

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\times(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1v_1(\mathbf{e}_1\times\mathbf{e}_1) + u_1v_2(\mathbf{e}_1\times\mathbf{e}_2) + u_1v_3(\mathbf{e}_1\times\mathbf{e}_3) \\
&\quad + u_2v_1(\mathbf{e}_2\times\mathbf{e}_1) + u_2v_2(\mathbf{e}_2\times\mathbf{e}_2) + u_2v_3(\mathbf{e}_2\times\mathbf{e}_3) \\
&\quad\quad + u_3v_1(\mathbf{e}_3\times\mathbf{e}_1) + u_3v_2(\mathbf{e}_3\times\mathbf{e}_2) + u_3v_3(\mathbf{e}_3\times\mathbf{e}_3) \\
&= (u_2v_3 - u_3v_2)\mathbf{e}_1 - (u_1v_3 - u_3v_1)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3
\end{aligned} \tag{1.3.6}$$

This is often written in the form

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \tag{1.3.7}$$

that is, the cross product is equal to the determinant of the 3×3 matrix

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

1.3.2 The Index Notation

The expression for the cross product in terms of components, Eqn. 1.3.6, is quite lengthy – for more complicated quantities things get unmanageably long. Thus a short-hand notation is used for these component equations, and this **index notation**[2] is described here.

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{1.3.8}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (*i* in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i\mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_i v_i} \qquad \textbf{Dot Product} \tag{1.3.9}$$

The repeated index *i* is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u}\cdot\mathbf{v} = u_j v_j$ or $u_k v_k$.

For the purpose of writing the vector cross product in index notation, the **permutation symbol** (or **alternating symbol**) ε_{ijk} can be introduced:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if two or more indices are equal} \end{cases} \tag{1.3.10}$$

For example (see Fig. 1.3.2),

$$\begin{aligned} \varepsilon_{123} &= +1 \\ \varepsilon_{132} &= -1 \\ \varepsilon_{122} &= 0 \end{aligned}$$

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[2] or **indicial** or **subscript** or **suffix** notation

[3] 2 in the case of a two-dimensional space/analysis

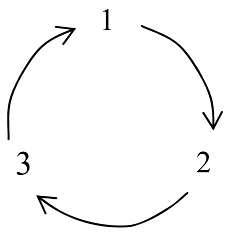

Figure 1.3.2: schematic for the permutation symbol (clockwise gives +1)

Note that

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj} \tag{1.3.11}$$

and that, in terms of the base vectors {▲Problem 7},

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k \tag{1.3.12}$$

and {▲Problem 7}

$$\varepsilon_{ijk} = \left(\mathbf{e}_i \times \mathbf{e}_j\right) \cdot \mathbf{e}_k . \tag{1.3.13}$$

The cross product can now be written concisely as {▲Problem 8}

$$\boxed{\mathbf{u} \times \mathbf{v} = \varepsilon_{ijk} u_i v_j \mathbf{e}_k} \qquad \textbf{Cross Product} \tag{1.3.14}$$

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{1.3.15}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (1.3.1, 1.3.2) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \qquad \textbf{Orthonormal Basis Rule} \tag{1.3.16}$$

The triple scalar product (1.1.4) can now be written as

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left(\varepsilon_{ijk} u_i v_j \mathbf{e}_k\right) \cdot w_m \mathbf{e}_m \\ &= \varepsilon_{ijk} u_i v_j w_m \delta_{km} \\ &= \varepsilon_{ijk} u_i v_j w_k \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned} \tag{1.3.17}$$

Note that, since the determinant of a matrix is equal to the determinant of the transpose of a matrix, this is equivalent to

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=\begin{vmatrix}u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\\ u_3 & v_3 & w_3\end{vmatrix} \tag{1.3.18}$$

Here follow some useful formulae involving the permutation and Kronecker delta symbol {▲Problem 13}:

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{kpq} &= \delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\\ \varepsilon_{ijk}\varepsilon_{ijp} &= 2\delta_{pk}\end{aligned} \tag{1.3.19}$$

Finally, here are some other important identities involving vectors; the third of these is called **Lagrange's identity**[4] {▲Problem 15}:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) &= |\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2\\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}\\ (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})]\mathbf{c}-[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{d}\\ [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{d} &= [\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}+[\mathbf{a}\cdot(\mathbf{d}\times\mathbf{c})]\mathbf{b}+[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{d})]\mathbf{c}\end{aligned} \tag{1.3.20}$$

1.3.3 Matrix Notation for Vectors

The symbolic notation $\mathbf{v}$ and index notation $v_i\mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector **v** can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the **element form** v_i. The element form for a matrix is essentially the same as the index notation for the vector it represents.

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[4] to be precise, the special case of 1.3.20c, 1.3.20a, is Lagrange's identity

Formally, a vector can be represented by the ordered triplet of real numbers, (v_1, v_2, v_3). The set of all vectors can be represented by R^3, the set of all ordered triplets of real numbers:

$$R^3 = \{(v_1, v_2, v_3) \mid v_1, v_2, v_3 \in R\} \tag{1.3.21}$$

It is important to *note the distinction between a vector and a matrix*: the former is a mathematical object independent of any basis, the latter is a representation of the vector with respect to a particular basis – use a different set of basis vectors and the elements of the matrix will change, but the matrix is still describing the same vector. Said another way, there is a difference between an element (vector) **v** of Euclidean vector space and an ordered triplet $v_i \in R^3$. This notion will be discussed more fully in the next section.

As an example, the dot product can be written in the matrix notation as

$$\underset{\substack{\uparrow \\ \text{“short”} \\ \text{matrix notation}}}{[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]} = \underset{\substack{\uparrow \\ \text{“full”} \\ \text{matrix notation}}}{[u_1 \quad u_2 \quad u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}$$

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, i.e. a scalar, in element form $u_i v_i$.

1.3.4 Cartesian Coordinates

Thus far, the notion of an origin has not been used. Choose a point **o** in Euclidean (point) space, to be called the **origin**. An origin together with a right-handed orthonormal basis $\{\mathbf{e}_i\}$ constitutes a (**rectangular**) **Cartesian coordinate system**, Fig. 1.3.3.

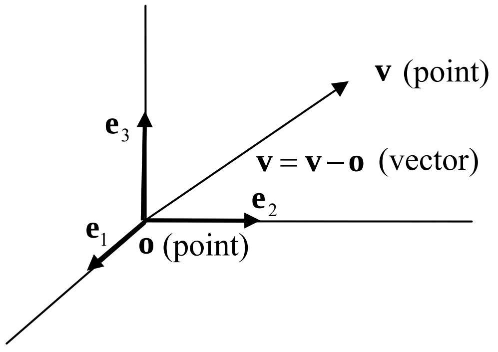

Figure 1.3.3: a Cartesian coordinate system

A second point **v** then defines a **position vector** $\mathbf{v}-\mathbf{o}$, Fig. 1.3.3. The components of the vector $\mathbf{v}-\mathbf{o}$ are called the (**rectangular**) **Cartesian coordinates** of the point **v** [5]. For brevity, the vector $\mathbf{v}-\mathbf{o}$ is simply labelled **v**, that is, one uses the same symbol for both the position vector and associated point.

1.3.5 Problems

1. Evaluate $\mathbf{u}\cdot\mathbf{v}$ where $\mathbf{u}=\mathbf{e}_1+3\mathbf{e}_2-2\mathbf{e}_3$, $\mathbf{v}=4\mathbf{e}_1-2\mathbf{e}_2+4\mathbf{e}_3$.
2. Prove that for any vector **u**, $\mathbf{u}=(\mathbf{u}\cdot\mathbf{e}_1)\mathbf{e}_1+(\mathbf{u}\cdot\mathbf{e}_2)\mathbf{e}_2+(\mathbf{u}\cdot\mathbf{e}_3)\mathbf{e}_3$. [Hint: write **u** in component form.]
3. Find the projection of the vector $\mathbf{u}=\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3$ on the vector $\mathbf{v}=4\mathbf{e}_1-4\mathbf{e}_2+7\mathbf{e}_3$.
4. Find the angle between $\mathbf{u}=3\mathbf{e}_1+2\mathbf{e}_2-6\mathbf{e}_3$ and $\mathbf{v}=4\mathbf{e}_1-3\mathbf{e}_2+\mathbf{e}_3$.
5. Write down an expression for a unit vector parallel to the resultant of two vectors **u** and **v** (in symbolic notation). Find this vector when $\mathbf{u}=2\mathbf{e}_1+4\mathbf{e}_2-5\mathbf{e}_3$, $\mathbf{v}=\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$ (in component form). Check that your final vector is indeed a unit vector.
6. Evaluate $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}=-\mathbf{e}_1-2\mathbf{e}_2+2\mathbf{e}_3$, $\mathbf{v}=2\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3$.
7. Verify that $\mathbf{e}_i\times\mathbf{e}_j=\varepsilon_{ijm}\mathbf{e}_m$. Hence, by dotting each side with $\mathbf{e}_k$, show that $\varepsilon_{ijk}=\left(\mathbf{e}_i\times\mathbf{e}_j\right)\cdot\mathbf{e}_k$.
8. Show that $\mathbf{u}\times\mathbf{v}=\varepsilon_{ijk}u_iv_j\mathbf{e}_k$.
9. The triple scalar product is given by $\left(\mathbf{u}\times\mathbf{v}\right)\cdot\mathbf{w}=\varepsilon_{ijk}u_iv_jw_k$. Expand this equation and simplify, so as to express the triple scalar product in full (non-index) component form.
10. Write the following in index notation: $\left|\mathbf{v}\right|$, $\mathbf{v}\cdot\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{e}_k$.
11. Show that $\delta_{ij}a_ib_j$ is equivalent to $\mathbf{a}\cdot\mathbf{b}$.
12. Verify that $\varepsilon_{ijk}\varepsilon_{ijk}=6$.
13. Verify that $\varepsilon_{ijk}\varepsilon_{kpq}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}$ and hence show that $\varepsilon_{ijk}\varepsilon_{ijp}=2\delta_{pk}$.
14. Evaluate or simplify the following expressions:
    (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$ (d) $\varepsilon_{1jk}\delta_{3j}v_k$
15. Prove Lagrange's identity 1.3.20c.
16. If **e** is a unit vector and **a** an arbitrary vector, show that
$$\mathbf{a}=\left(\mathbf{a}\cdot\mathbf{e}\right)\mathbf{e}+\mathbf{e}\times\left(\mathbf{a}\times\mathbf{e}\right)$$
which is another representation of Eqn. 1.1.2, where **a** can be resolved into components parallel and perpendicular to **e**.

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[5] That is, "components" are used for vectors and "coordinates" are used for points