8.1 Introduction to Plasticity

8.1.1 Introduction

The theory of linear elasticity is useful for modelling materials which undergo small deformations and which return to their original configuration upon removal of load. Almost all real materials will undergo some **permanent** deformation, which remains after removal of load. With metals, significant permanent deformations will usually occur when the stress reaches some critical value, called the **yield stress**, a material property.

Elastic deformations are termed **reversible;** the energy expended in deformation is stored as elastic strain energy and is completely recovered upon load removal. Permanent deformations involve the dissipation of energy; such processes are termed **irreversible**, in the sense that the original state can be achieved only by the expenditure of more energy.

The **classical theory of plasticity** grew out of the study of metals in the late nineteenth century. It is concerned with materials which initially deform elastically, but which deform **plastically** upon reaching a yield stress. In metals and other crystalline materials the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level. In sands and other granular materials plastic flow is due both to the irreversible rearrangement of individual particles and to the irreversible crushing of individual particles. Similarly, compression of bone to high stress levels will lead to particle crushing. The deformation of micro-voids and the development of micro-cracks is also an important cause of plastic deformations in materials such as rocks.

A good part of the discussion in what follows is concerned with the plasticity of metals; this is the 'simplest' type of plasticity and it serves as a good background and introduction to the modelling of plasticity in other material-types. There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst. The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material **failure**[1].

The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. These problems occur in the analysis of metals manufacturing and forming processes, which can involve extrusion, drawing, forging, rolling and so on. In these latter-type problems, a simplified model known as **perfect plasticity** is usually employed (see below), and use is made of special **limit theorems** which hold for such models.

Plastic deformations are normally **rate independent,** that is, the stresses induced are independent of the rate of deformation (or rate of loading). This is in marked

[1] two other types of failure, *brittle fracture*, due to dynamic crack growth, and the *buckling* of some structural components, can be modelled reasonably accurately using elasticity theory (see, for example, Part I, §6.1, Part II, §5.3)

contrast to classical **Newtonian fluids** for example, where the stress levels are governed by the *rate* of deformation through the viscosity of the fluid.

Materials commonly known as “plastics” are not plastic in the sense described here. They, like other polymeric materials, exhibit **viscoelastic** behaviour where, as the name suggests, the material response has both elastic and viscous components. Due to their viscosity, their response is, unlike the plastic materials, **rate-dependent**. Further, although the viscoelastic materials can suffer irrecoverable deformation, they do not have any critical yield or threshold stress, which is the characteristic property of plastic behaviour. When a material undergoes plastic deformations, i.e. irrecoverable and at a critical yield stress, and these effects *are* rate dependent, the material is referred to as being **viscoplastic**.

Plasticity theory began with Tresca in 1864, when he undertook an experimental program into the extrusion of metals and published his famous yield criterion discussed later on. Further advances with yield criteria and plastic flow rules were made in the years which followed by Saint-Venant, Levy, Von Mises, Hencky and Prandtl. The 1940s saw the advent of the classical theory; Prager, Hill, Drucker and Koiter amongst others brought together many fundamental aspects of the theory into a single framework. The arrival of powerful computers in the 1980s and 1990s provided the impetus to develop the theory further, giving it a more rigorous foundation based on thermodynamics principles, and brought with it the need to consider many numerical and computational aspects to the plasticity problem.

8.1.2 Observations from Standard Tests

In this section, a number of phenomena observed in the material testing of metals will be noted. Some of these phenomena are simplified or ignored in some of the standard plasticity models discussed later on.

At issue here is the fact that any model of a component with complex geometry, loaded in a complex way and undergoing plastic deformation, must involve material parameters which can be obtained in a straight forward manner from simple laboratory tests, such as the tension test described next.

The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching (see Part I, §5.2.1). The force required to hold the specimen at a given stretch is recorded, Fig. 8.1.1. If the material is a metal, the deformation remains elastic up to a certain force level, the yield point of the material. Beyond this point, permanent plastic deformations are induced. On unloading only the elastic deformation is recovered and the specimen will have undergone a permanent elongation (and consequent lateral contraction).

In the elastic range the force-displacement behaviour for most engineering materials (metals, rocks, plastics, but not soils) is linear. After passing the elastic limit (point *A* in Fig. 8.1.1), the material “gives” and is said to undergo plastic **flow**. Further increases in load are usually required to maintain the plastic flow and an increase in displacement; this phenomenon is known as **work-hardening** or **strain-hardening**. In some cases, after an initial plastic flow and hardening, the force-displacement curve decreases, as in some soils; the material is said to be **softening**. If the specimen is unloaded from a plastic state (*B*) it will return along the path *BC* shown, parallel to the original elastic line. This is **elastic recovery**. The strain which remains upon unloading is the permanent plastic deformation. If the material is now loaded again, the force-displacement curve will re-trace the unloading path *CB* until it again reaches the plastic state. Further increases in stress will cause the curve to follow *BD*.

Two important observations concerning the above tension test (on most metals) are the following:
(1) after the onset of plastic deformation, the material will be seen to undergo negligible volume change, that is, it is **incompressible**.
(2) the force-displacement curve is more or less the same regardless of the rate at which the specimen is stretched (at least at moderate temperatures).

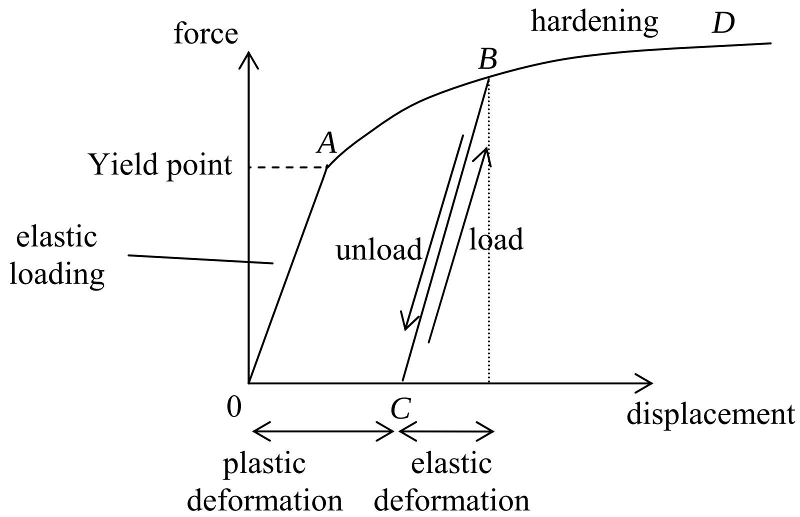


Figure 8.1.1: force/displacement curve for the tension test

Nominal and True Stress and Strain

There are two different ways of describing the force *F* which acts in a tension test. First, normalising with respect to the *original* cross sectional area of the tension test specimen A_0, one has the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{8.1.1}$$

Alternatively, one can normalise with respect to the *current* cross-sectional area *A*, leading to the **true stress**,

$$\sigma = \frac{F}{A} \tag{8.1.2}$$

in which F and A are both changing with time. For very small elongations, within the elastic range say, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. Denoting the original specimen length by l_0 and the current length by l, one has the **engineering strain**

$$\varepsilon = \frac{l - l_0}{l_0} \tag{8.1.3}$$

Alternatively, the **true strain** is based on the fact that the "original length" is continually changing; a small change in length dl leads to a **strain increment** $d\varepsilon = dl / l$ and the total strain is *defined* as the accumulation of these increments:

$$\varepsilon_t = \int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right) \tag{8.1.4}$$

The true strain is also called the **logarithmic strain** or **Hencky strain**. Again, at small deformations, the difference between these two strain measures is negligible. The true strain and engineering strain are related through

$$\varepsilon_t = \ln(1 + \varepsilon) \tag{8.1.5}$$

Using the assumption of constant volume for plastic deformation and ignoring the very small elastic volume changes, one has also {▲Problem 3}

$$\sigma = \sigma_n \frac{l}{l_0}. \tag{8.1.6}$$

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 8.1.2. The shape of the nominal stress/strain diagram, Fig. 8.1.2a, is of course the same as the graph of force versus displacement (change in length) in Fig. 8.1.1. A here denotes the point at which the maximum force the specimen can withstand has been reached. The *nominal stress* at A is called the **Ultimate Tensile Strength** (UTS) of the material. After this point, the specimen "necks", with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen ruptures, as indicated by the asterisk.

Note that, during loading into the plastic region, *the yield stress increases*. For example, if one unloads and re-loads (as in Fig. 8.1.1), the material stays elastic up until a stress higher than the original yield stress Y. In this respect, the stress-strain curve can be regarded as a yield stress versus strain curve.

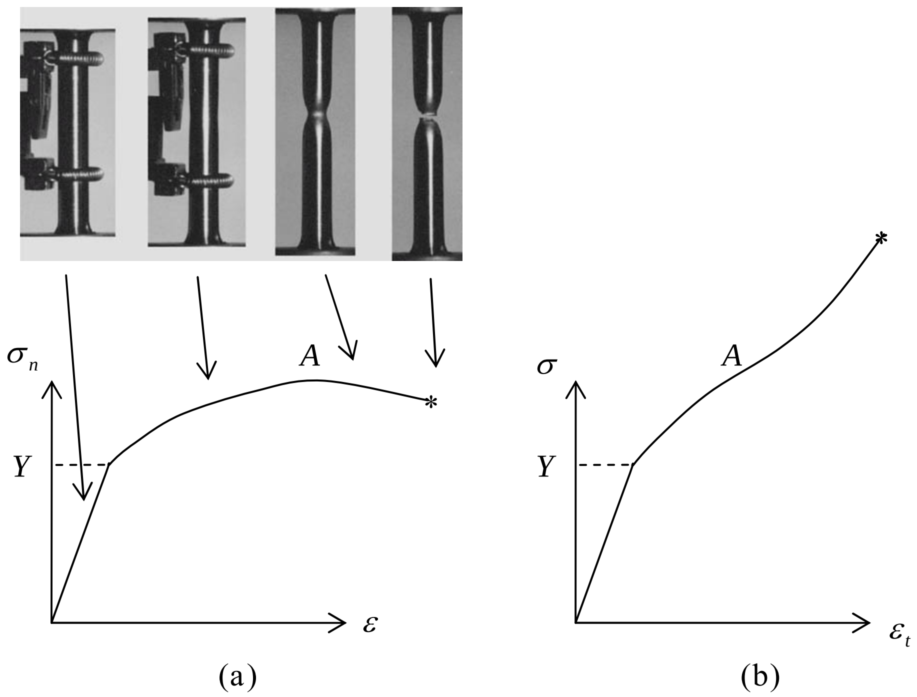


Figure 8.1.2: typical stress/strain curves; (a) engineering stress and strain, (b) true stress and strain

Compression Test

A compression test will lead to similar results as the tensile stress. The yield stress in compression will be approximately the same as (the negative of) the yield stress in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), the two curves will more or less coincide. This would indicate that the behaviour of the material under compression is broadly similar to that under tension. If one were to use the nominal stress and strain, then the two curves would not coincide; this is one of a number of good reasons for using the *true* definitions.

The Bauschinger Effect

If one takes a virgin sample and loads it in tension into the plastic range, and *then* unloads it and continues on into compression, one finds that the yield stress in compression is *not* the same as the yield strength in tension, as it would have been if the specimen had not first been loaded in tension. In fact the yield point in this case will be significantly *less* than the corresponding yield stress in tension. This reduction in yield stress is known as the **Bauschinger effect**. The effect is illustrated in Fig. 8.1.3. The solid line depicts the response of a real material. The dotted lines are two extreme cases which are used in plasticity models; the first is the **isotropic hardening** model, in which the yield stress in tension and compression are maintained equal, the second being **kinematic hardening**, in which the total elastic range is maintained constant throughout the deformation.

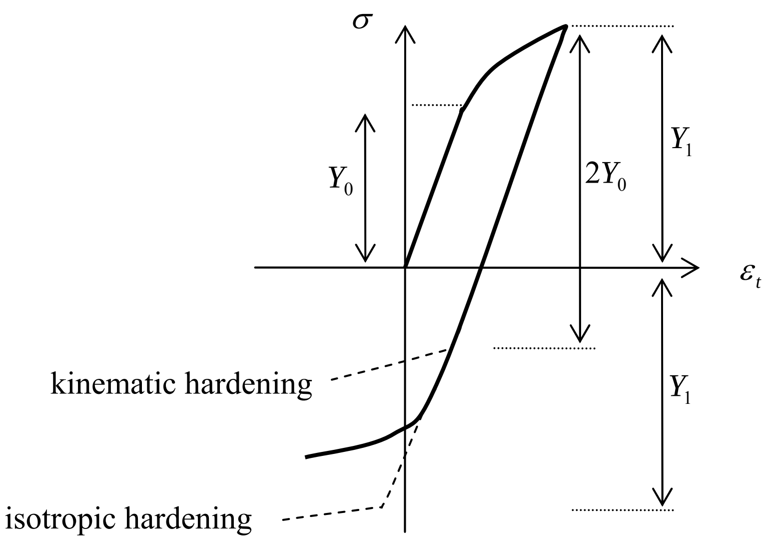


Figure 8.1.3: The Bauschinger effect

The presence of the Bauschinger effect complicates any plasticity theory. However, it is not an issue provided there are no reversals of stress in the problem under study.

Hydrostatic Pressure

Careful experiments show that, for metals, the yield behaviour is independent of hydrostatic pressure. That is, a stress state $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ has negligible effect on the yield stress of a material, right up to very high pressures. Note however that this is not true for soils or rocks.

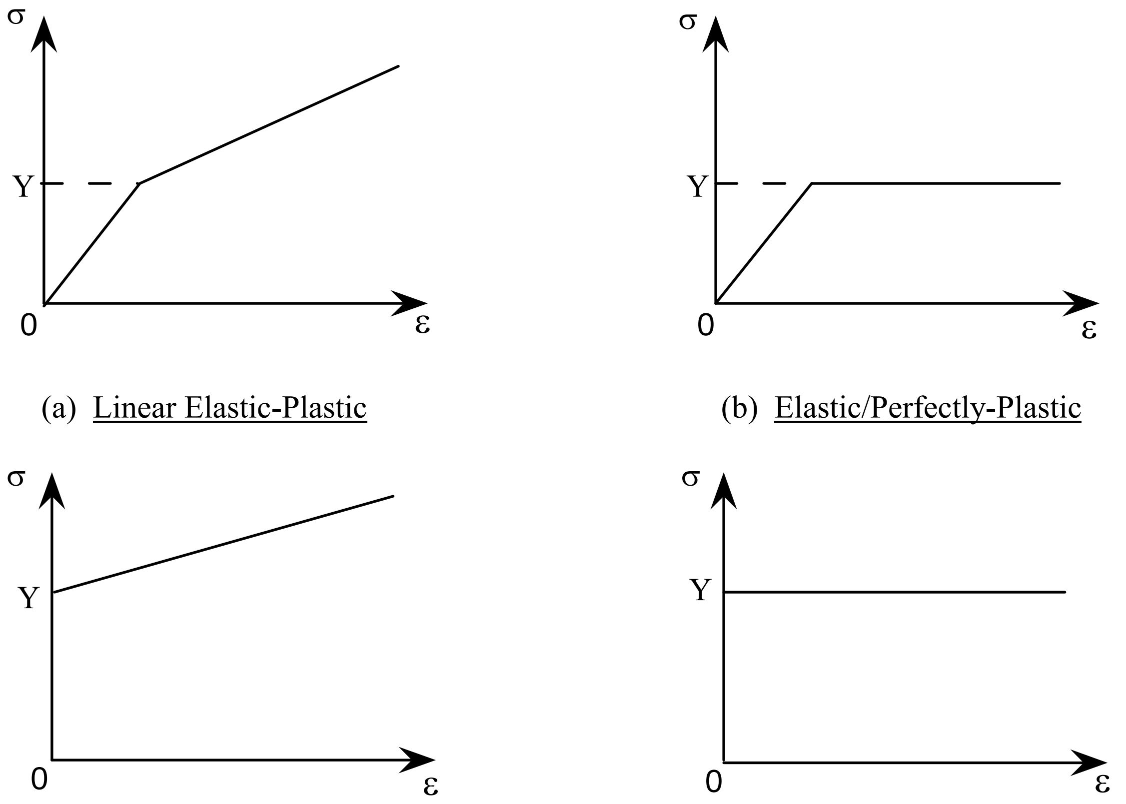
8.1.3 Assumptions of Plasticity Theory

Regarding the above test results then, in formulating a basic plasticity theory with which to begin, the following assumptions are usually made:

(1) the response is independent of rate effects
(2) the material is incompressible in the plastic range
(3) there is no Bauschinger effect
(4) the yield stress is independent of hydrostatic pressure
(5) the material is isotropic

The first two of these will usually be very good approximations, the other three may or may not be, depending on the material and circumstances. For example, most metals can be regarded as isotropic. After large plastic deformation however, for example in rolling, the material will have become anisotropic: there will be distinct material directions and asymmetries.

Together with these, assumptions can be made on the type of hardening and on whether elastic deformations are significant. For example, consider the hierarchy of models illustrated in Fig. 8.1.4 below, commonly used in theoretical analyses. In (a) both the elastic and plastic curves are assumed linear. In (b) work-hardening is neglected and the yield stress is constant after initial yield. Such **perfectly-plastic** models are particularly appropriate for studying processes where the metal is worked at a high temperature – such as hot rolling – where work hardening is small. In many areas of applications the strains involved are large, e.g. in metal working processes such as extrusion, rolling or drawing, where up to 50% reduction ratios are common. In such cases the elastic strains can be neglected altogether as in the two models (c) and (d). The **rigid/perfectly-plastic** model (d) is the crudest of all – and hence in many ways the most useful. It is widely used in analysing metal forming processes, in the design of steel and concrete structures and in the analysis of soil and rock stability.



(a) Linear Elastic-Plastic (b) Elastic/Perfectly-Plastic (c) Rigid/Linear Hardening (d) Rigid-Perfectly-Plastic

Figure 8.1.4: Simple models of elastic and plastic deformation

8.1.4 The Tangent and Plastic Modulus

Stress and strain are related through $\sigma = E\varepsilon$ in the elastic region, E being the Young's modulus, Fig. 8.1.5. The **tangent modulus** K is the slope of the stress-strain curve in the plastic region and will in general change during a deformation. At any instant of strain, the *increment* in stress $d\sigma$ is related to the *increment* in strain $d\varepsilon$ through[2]

$$d\sigma = K d\varepsilon \tag{8.1.7}$$

[2] the symbol ε here represents the true strain (the subscript t has been dropped for clarity); as mentioned, when the strains are small, it is not necessary to specify which strain is in use since all strain measures are then equivalent

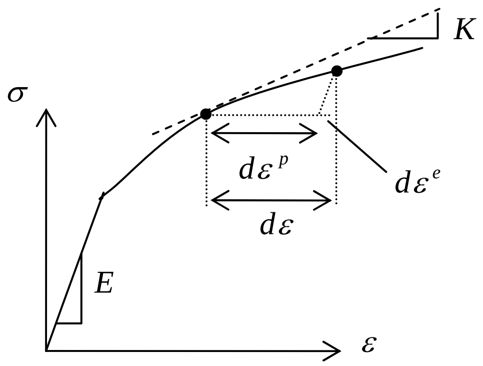


Figure 8.1.5: The tangent modulus

After yield, the strain increment consists of both elastic, ε^e, and plastic, $d\varepsilon^p$, strains:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{8.1.8}$$

The stress and plastic strain increments are related by the **plastic modulus** *H*:

$$d\sigma = H\, d\varepsilon^p \tag{8.1.9}$$

and it follows that {▲Problem 4}

$$\frac{1}{K} = \frac{1}{E} + \frac{1}{H} \tag{8.1.10}$$

8.1.5 Friction Block Models

Some additional insight into the way plastic materials respond can be obtained from friction block models. The rigid perfectly plastic model can be simulated by a Coulomb friction block, Fig. 8.1.6. No strain occurs until σ reaches the yield stress *Y*. Then there is movement – although the amount of movement or plastic strain cannot be determined without more information being available. The stress cannot exceed the yield stress in this model:

$$|\sigma| \le Y \tag{8.1.11}$$

If unloaded, the block stops moving and the stress returns to zero, leaving a permanent strain, Fig. 8.1.6b.

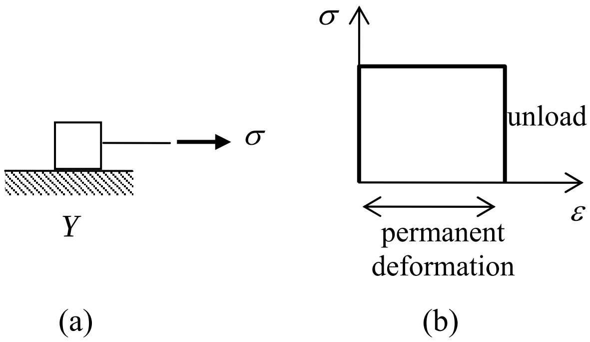


Figure 8.1.6: (a) Friction block model for the rigid perfectly plastic material, (b) response of the rigid-perfectly plastic model

The linear elastic perfectly plastic model incorporates a free spring with modulus *E* in series with a friction block, Fig. 8.1.7. The spring stretches when loaded and the block also begins to move when the stress reaches *Y*, at which time the spring stops stretching, the maximum possible stress again being *Y*. Upon unloading, the block stops moving and the spring contracts.

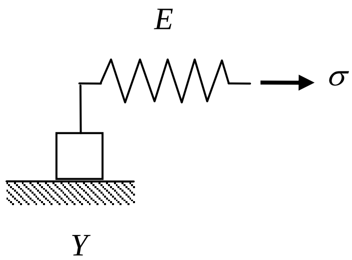


Figure 8.1.7: Friction block model for the elastic perfectly plastic material

The linear elastic plastic model with linear strain hardening incorporates a second, hardening, spring with stiffness *H*, in parallel with the friction block, Fig. 8.1.8. Once the yield stress is reached, an ever increasing stress needs to be applied in order to keep the block moving – and elastic strain continues to occur due to further elongation of the free spring. The stress is then split into the yield stress, which is carried by the moving block, and an **overstress** $\sigma - Y$ carried by the hardening spring.

Upon unloading, the block "locks" – the stress in the hardening spring remains constant whilst the free spring contracts. At zero stress, there is a negative stress taken up by the friction block, equal and opposite to the stress in the hardening spring.

The slope of the elastic loading line is *E*. For the plastic hardening line,

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \frac{\sigma - Y}{H} \qquad \rightarrow \qquad K = \frac{d\sigma}{d\varepsilon} = \frac{EH}{E+H} \tag{8.1.12}$$

It can be seen that *H* is the plastic modulus.

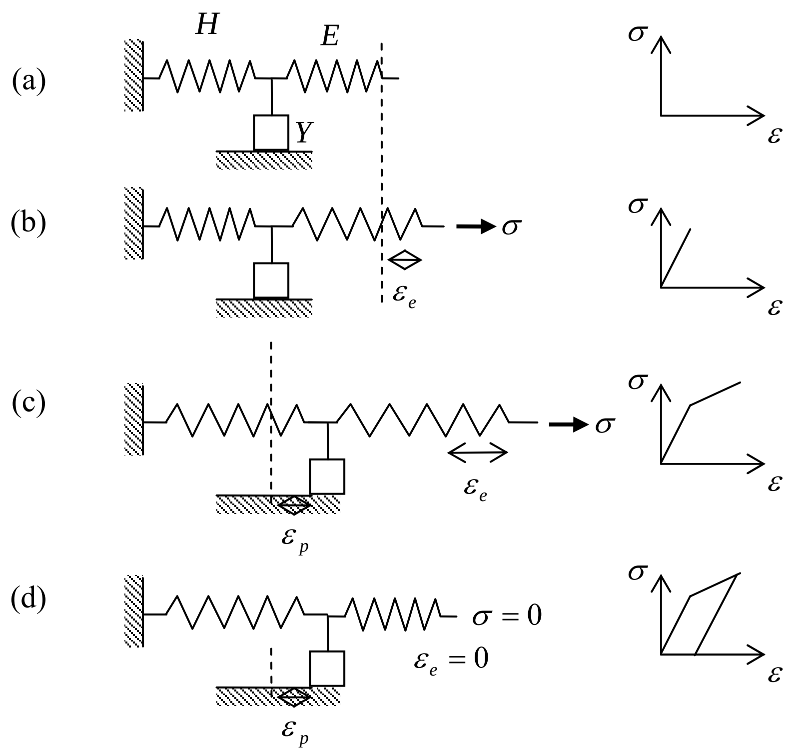


Figure 8.1.8: Friction block model for a linear elastic-plastic material with linear strain hardening; (a) stress-free, (b) elastic strain, (c) elastic and plastic strain, (d) unloading

8.1.6 Problems

1. Give two differences between plastic and viscoelastic materials.

2. A test specimen of initial length $0.01\,\mathrm{m}$ is extended to length $0.0101\,\mathrm{m}$. What is the percentage difference between the engineering and true strains (relative to the engineering strain)? What is this difference when the specimen is extended to length $0.015\,\mathrm{m}$?

3. Derive the relation 8.1.6, $\sigma/\sigma_n = l/l_0$.

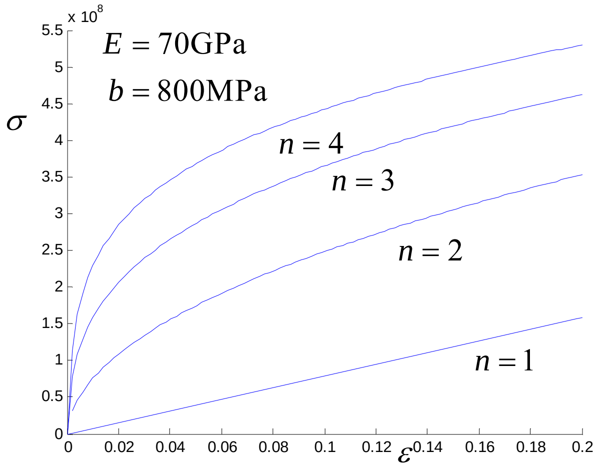
4. Derive Eqn. 8.1.10.

5. Which is larger, *H* or *K*? In the case of a perfectly-plastic material?

6. The **Ramberg-Osgood** model of plasticity is given by
$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^n$$
where *E* is the Young's modulus and *b* and *n* are model constants (material parameters) obtained from a curve-fitting of the uniaxial stress-strain curve.

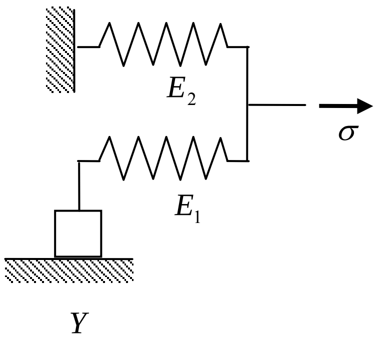
(i) Find the tangent and plastic moduli in terms of plastic strain ε^p (and the material constants).

(ii) A material with model parameters $n = 4$, $E = 70\,\text{GPa}$ and $b = 800\,\text{MPa}$ is strained in tension to $\varepsilon^p = 0.02$ and is subsequently unloaded and put into compression. Find the stress at the initiation of compressive yield assuming isotropic hardening.

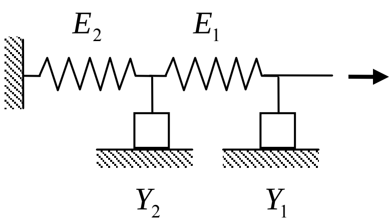


[Note that the yield stress is actually zero in this model, although the plastic strain at relatively low stress levels is small for larger values of *n*.]

7. Consider the plasticity model shown below.
   (i) What is the elastic modulus?
   (ii) What is the yield stress?
   (iii) What are the tangent and plastic moduli?
   Draw a typical loading and unloading curve.



8. Draw the stress-strain diagram for a cycle of loading and unloading to the rigid - plastic model shown here. Take the maximum load reached to be $\sigma_{\max} = 4Y_1$ and $Y_2 = 2Y_1$. What is the permanent deformation after complete removal of the load? [Hint: split the cycle into the following regions: (a) $0 \le \sigma \le Y_1$, (b) $Y_1 \le \sigma \le 3Y_1$, (c) $3Y_1 \le \sigma \le 4Y_1$, then unload, (d) $4Y_1 \le \sigma \le 3Y_1$, (e) $3Y_1 \le \sigma \le 2Y_1$, (f) $2Y_1 \le \sigma \le 0$.]



8.2 Stress Analysis for Plasticity

This section follows on from the analysis of three dimensional stress carried out in §7.2. The plastic behaviour of materials is often independent of a **hydrostatic stress** and this feature necessitates the study of the **deviatoric stress**.

8.2.1 Deviatoric Stress

Any state of stress can be decomposed into a **hydrostatic** (or **mean**) **stress** $\sigma_m \mathbf{I}$ and a **deviatoric stress s**, according to

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{8.2.1}$$

where

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \tag{8.2.2}$$

and

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix} \tag{8.2.3}$$

In index notation,

$$\sigma_{ij} = \sigma_m \delta_{ij} + s_{ij} \tag{8.2.4}$$

In a completely analogous manner to the derivation of the principal stresses and the principal scalar invariants of the stress matrix, §7.2.4, one can determine the principal stresses and principal scalar invariants of the deviatoric stress matrix. The former are denoted s_1, s_2, s_3 and the latter are denoted by J_1, J_2, J_3. The characteristic equation analogous to Eqn. 7.2.23 is

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \tag{8.2.5}$$

and the deviatoric invariants are (compare with 7.2.24, 7.2.26)[1]

[1] unfortunately, there is a convention (adhered to by most authors) to write the characteristic equation for stress with a $+I_2\sigma$ term and that for deviatoric stress with a $-J_2 s$ term; this means that the formulae for J_2 in Eqn. 8.2.5 are the negative of those for I_2 in Eqn. 7.2.24

$$
\begin{aligned}
J_1 &= s_{11} + s_{22} + s_{33} \\
&= s_1 + s_2 + s_3 \\
J_2 &= -\left(s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2\right) \\
&= -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right) \\
J_3 &= s_{11}s_{22}s_{33} - s_{11}s_{23}^2 - s_{22}s_{31}^2 - s_{33}s_{12}^2 + 2s_{12}s_{23}s_{31} \\
&= s_1 s_2 s_3
\end{aligned}
\tag{8.2.6}
$$

Since the hydrostatic stress remains unchanged with a change of coordinate system, the principal directions of stress coincide with the principal directions of the deviatoric stress, and the decomposition can be expressed with respect to the principal directions as

$$
\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \tag{8.2.7}
$$

Note that, from the definition Eqn. 8.2.3, the first invariant of the deviatoric stress, the sum of the normal stresses, is zero:

$$J_1 = 0 \tag{8.2.8}$$

The second invariant can also be expressed in the useful forms {▲Problem 3}

$$J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right), \tag{8.2.9}$$

and, in terms of the principal stresses, {▲Problem 4}

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \tag{8.2.10}$$

Further, the deviatoric invariants are related to the stress tensor invariants through {▲Problem 5}

$$J_2 = \tfrac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \tfrac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27I_3\right) \tag{8.2.11}$$

A State of Pure Shear

The stress state at a point is one of **pure shear** if for *any* one coordinate axes through the point one has only shear stress acting, i.e. the stress matrix is of the form

$$
\left[\sigma_{ij}\right] = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix} \tag{8.2.12}
$$

Applying the stress transformation rule 7.2.16 to this stress matrix and using the fact that the transformation matrix $\mathbf{Q}$ is orthogonal, i.e. $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, one finds that the first invariant is zero, $\sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$. Hence the deviatoric stress is one of pure shear.

8.2.2 The Octahedral Stresses

Examine now a material element subjected to principal stresses $\sigma_1, \sigma_2, \sigma_3$ as shown in Fig. 8.2.1. By definition, no shear stresses act on the planes shown.

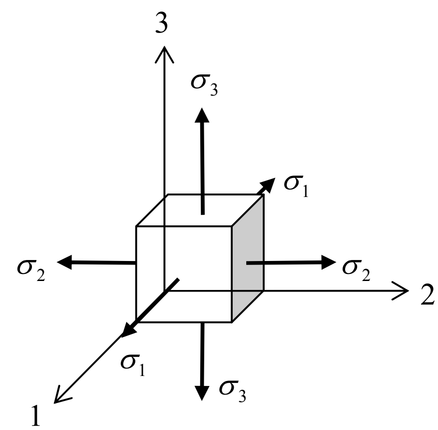


Figure 8.2.1: stresses acting on a material element

Consider next the **octahedral plane**; this is the plane shown shaded in Fig. 8.2.2, whose normal $\mathbf{n}_a$ makes equal angles with the principal directions. It is so-called because it cuts a cubic material element (with faces perpendicular to the principal directions) into a triangular plane and eight of these triangles around the origin form an octahedron.

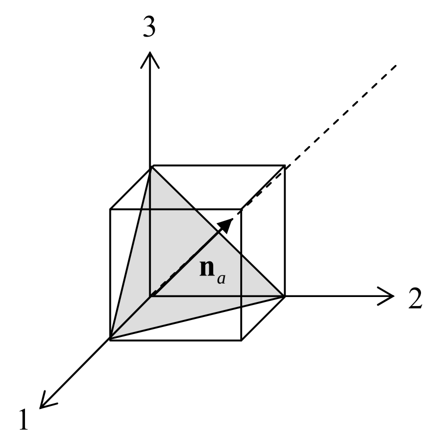


Figure 8.2.2: the octahedral plane

Next, a new Cartesian coordinate system is constructed with axes parallel and perpendicular to the octahedral plane, Fig. 8.2.3. One axis runs along the unit normal $\mathbf{n}_a$;

this normal has components $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ with respect to the principal axes. The angle θ_0 the normal direction makes with the 1 direction can be obtained from $\mathbf{n}_a \cdot \mathbf{e}_1 = \cos\theta_0$, where $\mathbf{e}_1 = (1,0,0)$ is a unit vector in the 1 direction, Fig. 8.2.3. To complete the new coordinate system, any two perpendicular unit vectors which lie in (parallel to) the octahedral plane can be chosen. Choose one which is along the projection of the 1 axis down onto the octahedral plane. The components of this vector are {▲Problem 6} $\mathbf{n}_c = (\sqrt{2/3}, -1/\sqrt{6}, -1/\sqrt{6})$. The final unit vector $\mathbf{n}_b$ is chosen so that it forms a right hand Cartesian coordinate system with $\mathbf{n}_a$ and $\mathbf{n}_c$, i.e. $\mathbf{n}_a \times \mathbf{n}_b = \mathbf{n}_c$. In summary,

$$\mathbf{n}_a = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \quad \mathbf{n}_c = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\-1\\-1\end{bmatrix} \tag{8.2.13}$$

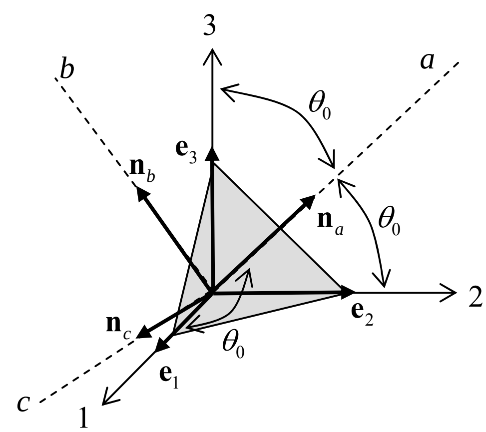


Figure 8.2.3: a new Cartesian coordinate system

To express the stress state in terms of components in the a,b,c directions, construct the stress transformation matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1\cdot\mathbf{n}_a & \mathbf{e}_1\cdot\mathbf{n}_b & \mathbf{e}_1\cdot\mathbf{n}_c \\ \mathbf{e}_2\cdot\mathbf{n}_a & \mathbf{e}_2\cdot\mathbf{n}_b & \mathbf{e}_2\cdot\mathbf{n}_c \\ \mathbf{e}_3\cdot\mathbf{n}_a & \mathbf{e}_3\cdot\mathbf{n}_b & \mathbf{e}_3\cdot\mathbf{n}_c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \tag{8.2.14}$$

and the new stress components are

$$\begin{aligned} \begin{bmatrix} \sigma_{aa} & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ba} & \sigma_{bb} & \sigma_{bc} \\ \sigma_{ca} & \sigma_{cb} & \sigma_{cc} \end{bmatrix} &= \mathbf{Q}^T \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \mathbf{Q} \\ &= \begin{bmatrix} \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3) & -\frac{1}{\sqrt{6}}(\sigma_2-\sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1-\sigma_2-\sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2-\sigma_3) & \frac{1}{2}(\sigma_2+\sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2-\sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1-\sigma_2-\sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2-\sigma_3) & \frac{1}{6}(4\sigma_1+\sigma_2+\sigma_3) \end{bmatrix} \end{aligned} \tag{8.2.15}$$

Now consider the stress components acting on the octahedral plane, $\sigma_{aa}, \sigma_{ab}, \sigma_{ac}$, Fig. 8.2.4. Recall from Cauchy's law, Eqn. 7.2.9, that these are the components of the traction vector $\mathbf{t}^{(\mathbf{n}_a)}$ acting on the octahedral plane, with respect to the (a,b,c) axes:

$$\mathbf{t}^{(\mathbf{n}_a)} = \sigma_{aa}\mathbf{n}_a + \sigma_{ab}\mathbf{n}_b + \sigma_{ac}\mathbf{n}_c \tag{8.2.16}$$

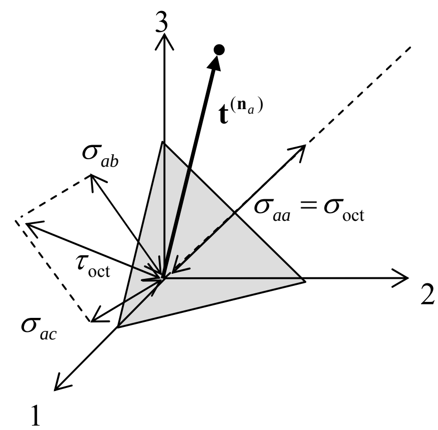


Figure 8.2.4: the stress vector σ and its components

The magnitudes of the normal and shear stresses acting on the octahedral plane are called the **octahedral normal stress** σ_{oct} and the **octahedral shear stress** τ_{oct}. Referring to Fig. 8.2.4, these can be expressed as {▲Problem 7}

$$\begin{aligned} \sigma_{oct} &= \sigma_{aa} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1 \\ \tau_{oct} &= \sqrt{\sigma_{ab}^2 + \sigma_{ac}^2} \\ &= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\frac{2J_2}{3}} \end{aligned} \tag{8.2.17}$$

The octahedral normal and shear stresses on all 8 octahedral planes around the origin are the same.

Note that the octahedral normal stress is simply the hydrostatic stress. This implies that the deviatoric stress has no normal component in the direction $\mathbf{n}_a$ and only contributes to shearing on the octahedral plane. Indeed, from Eqn. 8.2.15,

$$\begin{bmatrix} s_{aa} & s_{ab} & s_{ac} \\ s_{ba} & s_{bb} & s_{bc} \\ s_{ca} & s_{cb} & s_{cc} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & -\frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) & \frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) \end{bmatrix} \tag{8.2.18}$$

The σ 's on the right here can be replaced with s 's since $\sigma_i - \sigma_j = s_i - s_j$.

8.2.3 Problems

1. What are the hydrostatic and deviatoric stresses for the uniaxial stress $\sigma_{11} = \sigma_0$? What are the hydrostatic and deviatoric stresses for the state of pure shear $\sigma_{12} = \tau$? In both cases, verify that the first invariant of the deviatoric stress is zero: $J_1 = 0$.

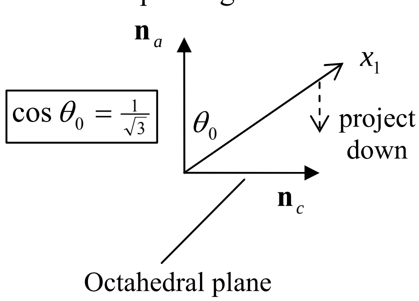
2. For the stress state $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$, calculate
   (a) the hydrostatic stress
   (b) the deviatoric stresses
   (c) the deviatoric invariants

3. The second invariant of the deviatoric stress is given by Eqn. 8.2.6,
$$J_2 = -(s_1 s_2 + s_2 s_3 + s_3 s_1)$$
By squaring the relation $J_1 = s_1 + s_2 + s_3 = 0$, derive Eqn. 8.2.9,
$$J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right)$$

4. Use Eqns. 8.2.9 (and your work from Problem 3) and the fact that $\sigma_1 - \sigma_2 = s_1 - s_2$, etc. to derive 8.2.10,
$$J_2 = \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

5. Use the fact that $J_1 = s_1 + s_2 + s_3 = 0$ to show that
$$\begin{aligned} I_1 &= 3\sigma_m \\ I_2 &= (s_1 s_2 + s_2 s_3 + s_3 s_1) + 3\sigma_m^2 \\ I_3 &= s_1 s_2 s_3 + \sigma_m (s_1 s_2 + s_2 s_3 + s_3 s_1) + \sigma_m^3 \end{aligned}$$
Hence derive Eqns. 8.2.11,
$$J_2 = \tfrac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \tfrac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27 I_3\right)$$

6. Show that a unit normal $\mathbf{n}_c$ in the octahedral plane in the direction of the projection of the 1 axis down onto the octahedral plane has coordinates $\left(\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$, Fig. 8.2.3. To do this, note the geometry shown below and the fact that when the 1 axis is projected down, it remains at equal angles to the 2 and 3 axes.



7. Use Eqns. 8.2.15 to derive Eqns. 8.2.17.

8. For the stress state of problem 2, calculate the octahedral normal stress and the octahedral shear stress

8.3 Yield Criteria in Three Dimensional Plasticity

The question now arises: a material yields at a stress level Y in a uniaxial tension test, but when does it yield when subjected to a complex three-dimensional stress state?

Let us begin with a very general case: an anisotropic material with different yield strengths in different directions. For example, consider the material shown in Fig. 8.3.1. This is a composite material with long fibres along the x_1 direction, giving it extra strength in that direction – it will yield at a higher tension when pulled in the x_1 direction than when pulled in other directions.

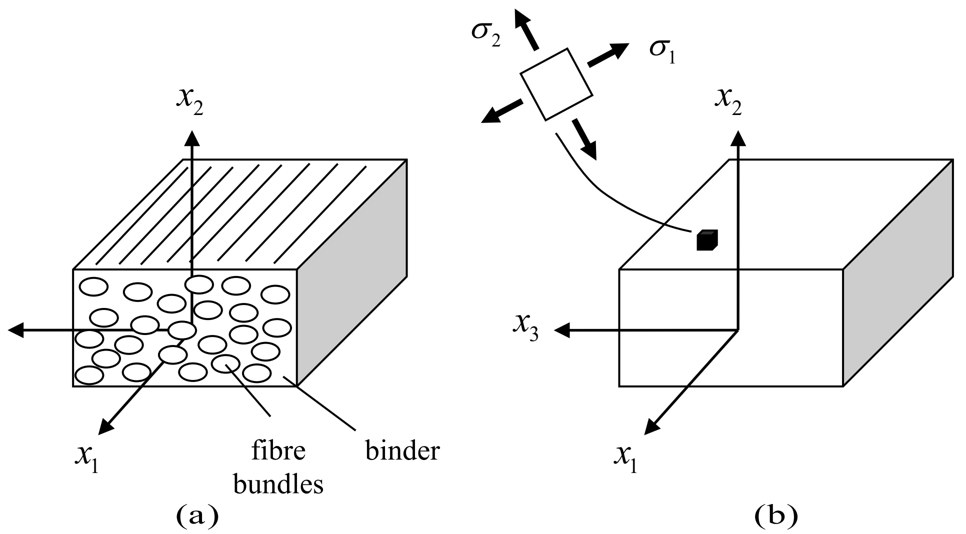


Figure 8.3.1: an anisotropic material; (a) microstructural detail, (b) continuum model

We can assume that yield will occur at a particle when some combination of the stress components reaches some critical value, say when

$$F(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}) = k \ . \tag{8.3.1}$$

Here, F is some function of the 6 independent components of the stress tensor and k is some material property which can be determined experimentally[1].

Alternatively, it is very convenient to express yield criteria in terms of principal stresses. Let us suppose that we know the principal stresses everywhere, $(\sigma_1, \sigma_2, \sigma_3)$, Fig. 8.3.1. Yield must depend somehow on the microstructure – on the orientation of the axes x_1, x_2, x_3, but this information is not contained in the three numbers $(\sigma_1, \sigma_2, \sigma_3)$. Thus we express the yield criterion in terms of principal stresses in the form

$$F(\sigma_1, \sigma_2, \sigma_3, \mathbf{n}_i) = k \tag{8.3.2}$$

[1] F will no doubt also contain other parameters which need to be determined experimentally

where $\mathbf{n}_i$ represent the principal directions – these give the orientation of the principal stresses relative to the material directions x_1, x_2, x_3.

If the material is isotropic, the response is independent of any material direction – independent of any "direction" the stress acts in, and so the yield criterion can be expressed in the simple form

$$F(\sigma_1, \sigma_2, \sigma_3) = k \tag{8.3.3}$$

Further, since it should not matter which direction is labelled '1', which '2' and which '3', *F* must be a symmetric function of the three principal stresses.

Alternatively, since the three principal invariants of stress are independent of material orientation, one can write

$$F(I_1, I_2, I_3) = k \tag{8.3.4}$$

or, more usually,

$$F(I_1, J_2, J_3) = k \tag{8.3.5}$$

where J_2, J_3 are the non-zero principal invariants of the deviatoric stress. With the further restriction that the yield stress is independent of the hydrostatic stress, one has

$$F(J_2, J_3) = k \tag{8.3.6}$$

8.3.1 The Tresca and Von Mises Yield Conditions

The two most commonly used and successful yield criteria for isotropic metallic materials are the **Tresca** and **Von Mises** criteria.

The Tresca Yield Condition

The Tresca yield criterion states that a material will yield if the maximum shear stress reaches some critical value, that is, Eqn. 8.3.3 takes the form

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} = k \tag{8.3.7}$$

The value of *k* can be obtained from a simple experiment. For example, in a tension test, $\sigma_1 = \sigma_0, \sigma_2 = \sigma_3 = 0$, and failure occurs when σ_0 reaches *Y*, the yield stress in tension. It follows that

$$k = \frac{Y}{2}. \tag{8.3.8}$$

In a shear test, $\sigma_1 = \tau, \sigma_2 = 0, \sigma_3 = -\tau$, and failure occurs when τ reaches τ_Y, the yield stress of a material in pure shear, so that $k = \tau_Y$.

The Von Mises Yield Condition

The Von Mises criterion states that yield occurs when the principal stresses satisfy the relation

$$\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} = k \tag{8.3.9}$$

Again, from a uniaxial tension test, one finds that the *k* in Eqn. 8.3.9 is

$$k = \frac{Y}{\sqrt{3}}. \tag{8.3.10}$$

Writing the Von Mises condition in terms of *Y*, one has

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = Y \tag{8.3.11}$$

The quantity on the left is called the **Von Mises Stress**, sometimes denoted by σ_{VM}. When it reaches the yield stress in pure tension, the material begins to deform plastically. In the shear test, one again finds that $k = \tau_Y$, the yield stress in pure shear.

Sometimes it is preferable to work with arbitrary stress components; for this purpose, the Von Mises condition can be expressed as {▲Problem 2}

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 6k^2 \tag{8.3.12}$$

The piecewise linear nature of the Tresca yield condition is sometimes a theoretical advantage over the quadratic Mises condition. However, the fact that in many problems one often does not know which principal stress is the maximum and which is the minimum causes difficulties when working with the Tresca criterion.

The Tresca and Von Mises Yield Criteria in terms of Invariants

From Eqn. 8.2.10 and 8.3.9, the Von Mises criterion can be expressed as

$$f(J_2) \equiv J_2 - k^2 = 0 \tag{8.3.13}$$

Note the relationship between $\sqrt{J_2}$ and the octahedral shear stress, Eqn. 8.2.17; the Von Mises criterion can be interpreted as predicting yield when the octahedral shear stress reaches a critical value.

With $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the Tresca condition can be expressed as

$$f(J_2, J_3) \equiv 4J_2^3 - 27J_3^2 - 36k^2 J_2^2 + 96k^4 J_2 - 64k^6 = 0 \qquad (8.3.14)$$

but this expression is too cumbersome to be of much use.

Experiments of Taylor and Quinney

In order to test whether the Von Mises or Tresca criteria best modelled the real behaviour of metals, G I Taylor & Quinney (1931), in a series of classic experiments, subjected a number of thin-walled cylinders made of copper and steel to combined tension and torsion, Fig. 8.3.2.

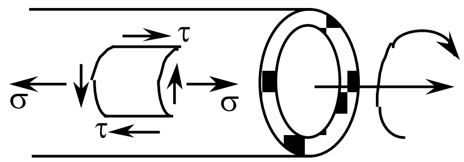


Figure 8.3.2: combined tension and torsion of a thin-walled tube

The cylinder wall is in a state of plane stress, with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and all other stress components zero. The principal stresses corresponding to such a stress-state are (zero and) {▲Problem 3}

$$\tfrac{1}{2}\sigma \pm \sqrt{\tfrac{1}{4}\sigma^2 + \tau^2} \qquad (8.3.15)$$

and so Tresca's condition reduces to

$$\sigma^2 + 4\tau^2 = 4k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/2}\right)^2 = 1 \qquad (8.3.16)$$

The Mises condition reduces to {▲Problem 4}

$$\sigma^2 + 3\tau^2 = 3k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1 \qquad (8.3.17)$$

Thus both models predict an elliptical **yield locus** in (σ, τ) **stress space**, but with different ratios of principal axes, Fig. 8.3.3. The origin in Fig. 8.3.3 corresponds to an unstressed state. The horizontal axes refer to uniaxial tension in the absence of shear, whereas the vertical axis refers to pure torsion in the absence of tension. When there is a combination of σ and τ, one is off-axes. If the combination remains "inside" the yield locus, the material remains elastic; if the combination is such that one reaches anywhere along the locus, then plasticity ensues.

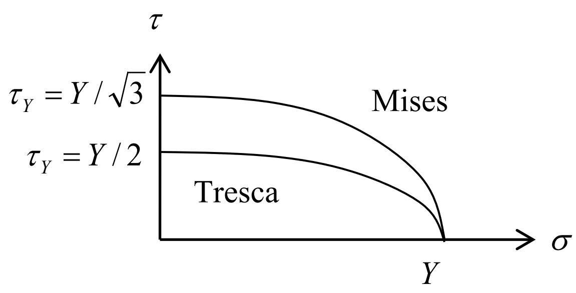


Figure 8.3.3: the yield locus for a thin-walled tube in combined tension and torsion

Taylor and Quinney, by varying the amount of tension and torsion, found that their measurements were closer to the Mises ellipse than the Tresca locus, a result which has been repeatedly confirmed by other workers[2].

2D Principal Stress Space

Fig. 8.3.3 gives a geometric interpretation of the Tresca and Von Mises yield criteria in (σ, τ) space. It is more usual to interpret yield criteria geometrically in a **principal stress space**. The Taylor and Quinney tests are an example of plane stress, where one principal stress is zero. Following the convention for plane stress, label now the two non-zero principal stresses σ_1 and σ_2, so that $\sigma_3 = 0$ (even if it is not the minimum principal stress). The criteria can then be displayed in (σ_1, σ_2) 2D principal stress space. With $\sigma_3 = 0$, one has

$$\text{Tresca:} \qquad \max\left\{ |\sigma_1 - \sigma_2|, |\sigma_2|, |\sigma_1| \right\} = Y$$

$$\text{Von Mises:} \qquad \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2 \qquad (8.3.18)$$

These are plotted in Fig. 8.3.4. The Tresca criterion is a hexagon. The Von Mises criterion is an ellipse with axes inclined at 45^0 to the principal axes, which can be seen by expressing Eqn. 8.3.18b in the canonical form for an ellipse:

[2] the maximum difference between the predicted stresses from the two criteria is about 15%. The two criteria can therefore be made to agree to within ± 7.5% by choosing *k* to be half-way between $Y/2$ and $Y/\sqrt{3}$

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1 & -1/2 \\ -1/2 & 1\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}$$

$$= \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/2 & 0 \\ 0 & 3/2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}$$

$$= \frac{1}{2}\left(\frac{\sigma_1 + \sigma_2}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{\sigma_2 - \sigma_1}{\sqrt{2}}\right)^2 = Y^2$$

$$\rightarrow \quad \left(\frac{\sigma_1'}{\sqrt{2}Y}\right)^2 + \left(\frac{\sigma_2'}{\sqrt{2/3}Y}\right)^2 = 1$$

where σ_1', σ_2' are coordinates along the new axes; the major axis is thus $\sqrt{2}Y$ and the minor axis is $\sqrt{2/3}Y$.

Some stress states are shown in the stress space: point *A* corresponds to a uniaxial tension, *B* to a equi-biaxial tension and *C* to a pure shear τ.

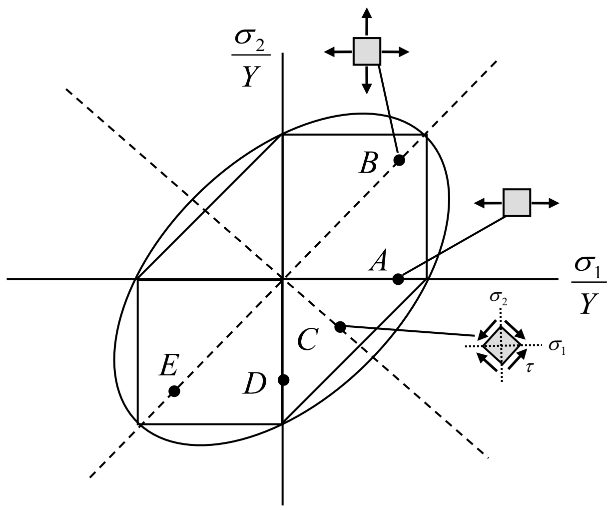


Figure 8.3.4: yield loci in 2D principal stress space

Again, points inside these loci represent an elastic stress state. Any combination of principal stresses which push the point out to the yield loci results in plastic deformation.

8.3.2 Three Dimensional Principal Stress Space

The 2D principal stress space has limited use. For example, a stress state that might start out two dimensional can develop into a fully three dimensional stress state as deformation proceeds.

In three dimensional principal stress space, one has a **yield surface** $f(\sigma_1, \sigma_2, \sigma_3) = 0$, Fig. 8.3.5 [3]. In this case, one can draw a line at equal angles to all three principal stress axes, the **space diagonal**. Along the space diagonal $\sigma_1 = \sigma_2 = \sigma_3$ and so points on it are in a state of hydrostatic stress.

Assume now, for the moment, that *hydrostatic stress does not affect yield* and consider some arbitrary point A, $(\sigma_1, \sigma_2, \sigma_3) = (a, b, c)$, on the yield surface, Fig. 8.3.5. A pure hydrostatic stress σ_h can be superimposed on this stress state without affecting yield, so any other point $(\sigma_1, \sigma_2, \sigma_3) = (a + \sigma_h, b + \sigma_h, c + \sigma_h)$ will also be on the yield surface. Examples of such points are shown at B, C and D, which are obtained from A by moving along a line parallel to the space diagonal. The yield behaviour of the material is therefore specified by a yield *locus* on a plane perpendicular to the space diagonal, and the yield surface is generated by sliding this locus up and down the space diagonal.

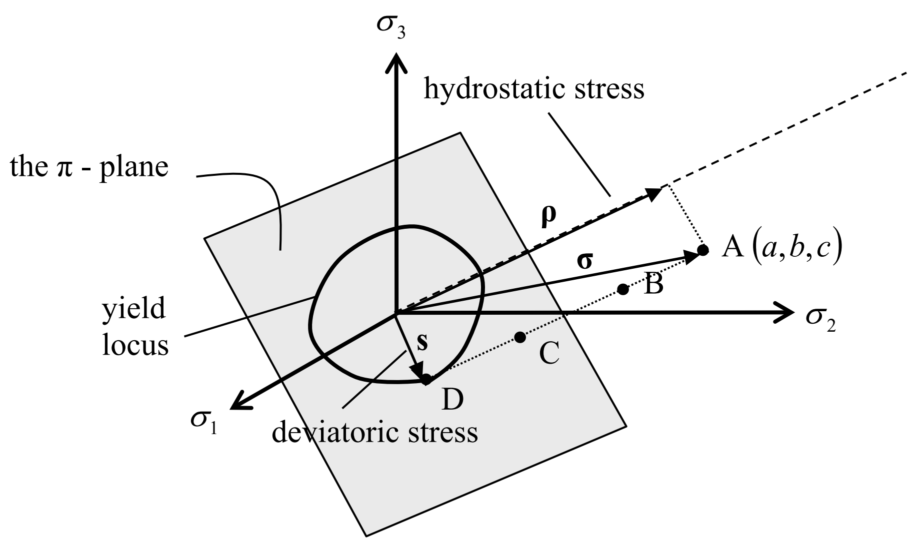


Figure 8.3.5: Yield locus/surface in three dimensional stress-space

The π-plane

Any surface in stress space can be described by an equation of the form

$$f(\sigma_1, \sigma_2, \sigma_3) = \text{const} \tag{8.3.19}$$

and a normal to this surface is the gradient vector

$$\frac{\partial f}{\partial \sigma_1}\mathbf{e}_1 + \frac{\partial f}{\partial \sigma_2}\mathbf{e}_2 + \frac{\partial f}{\partial \sigma_3}\mathbf{e}_3 \tag{8.3.20}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the stress space axes. In particular, any plane perpendicular to the space diagonal is described by the equation

[3] as mentioned, one has a six dimensional stress space for an anisotropic material and this cannot be visualised

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{const} \tag{8.3.21}$$

Without loss of generality, one can choose as a representative plane the **π – plane**, which is defined by $\sigma_1 + \sigma_2 + \sigma_3 = 0$. For example, the point $(\sigma_1, \sigma_2, \sigma_3) = (1, -1, 0)$ is on the π – plane and, with yielding independent of hydrostatic stress, is equivalent to points in principal stress space which differ by a hydrostatic stress, e.g. the points $(2,0,1)$, $(0,-2,-1)$, etc.

The stress state at any point A represented by the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ can be regarded as the sum of the stress state at the corresponding point on the π – plane, D, represented by the vector $\mathbf{s} = (s_1, s_2, s_3)$ together with a hydrostatic stress represented by the vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$:

$$(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_m, \sigma_2 - \sigma_m, \sigma_3 - \sigma_m) + (\sigma_m, \sigma_m, \sigma_m) \tag{8.3.22}$$

The components of the first term/vector on the right here sum to zero since it lies on the π – plane, and this is the deviatoric stress, whilst the hydrostatic stress is $\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$.

Projected view of the π-plane

Fig. 8.3.6a shows principal stress space and Fig. 8.3.6b shows the π – plane. The heavy lines $\sigma_1', \sigma_2', \sigma_3'$ in Fig. 8.3.6b represent the projections of the principal axes down onto the π – plane (so one is "looking down" the space diagonal). Some points, A, B, C in stress space and their projections onto the π – plane are also shown. Also shown is some point D *on* the π – plane. It should be kept in mind that the deviatoric stress vector **s** in the projected view of Fig. 8.3.6b is in reality a three dimensional vector (see the corresponding vector in Fig. 8.3.6a).

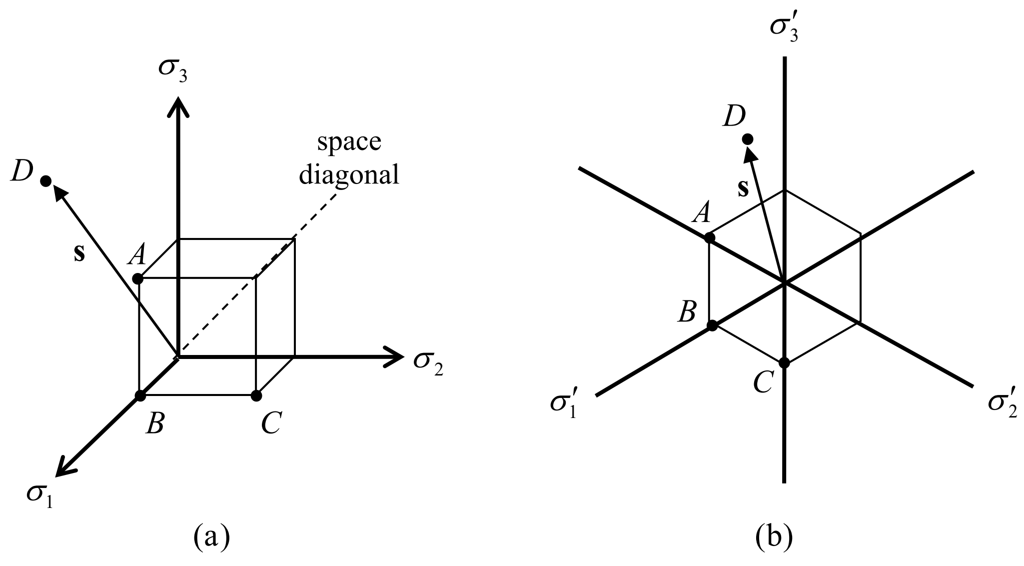


Figure 8.3.6: Stress space; (a) principal stress space, (b) the π – plane

Consider the more detailed Fig. 8.3.7 below. Point *A* here represents the stress state $(2,-1,0)$, as indicated by the arrows in the figure. It can also be "reached" in different ways, for example it represents $(3,0,1)$ and $(1,-2,-1)$. These three stress states of course differ by a hydrostatic stress. The actual $\pi-$plane value for *A* is the one for which $\sigma_1+\sigma_2+\sigma_3=0$, i.e. $(\sigma_1,\sigma_2,\sigma_3)=(s_1,s_2,s_3)=\left(\frac{5}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Points *B* and *C* also represent multiple stress states {▲Problem 7}.

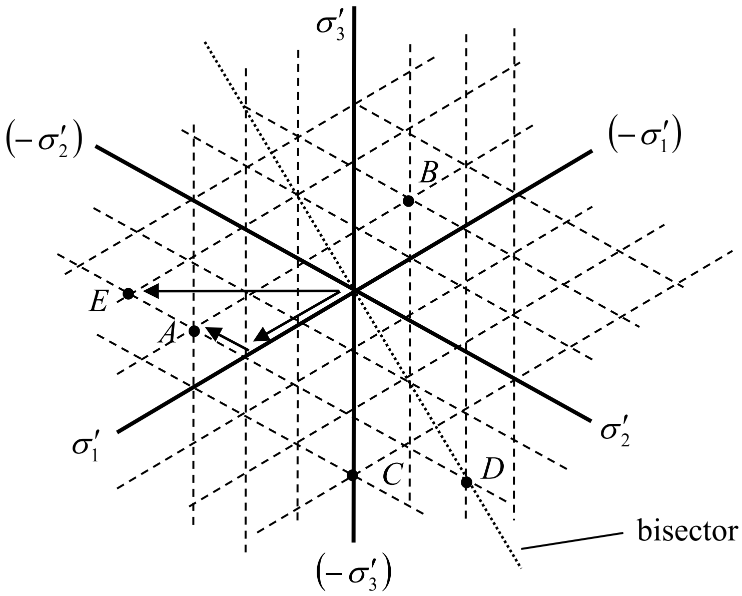


Figure 8.3.7: the π-plane

The bisectors of the principal plane projections, such as the dotted line in Fig. 8.3.7, represent states of pure shear. For example, the $\pi-$plane value for point *D* is $(0,2,-2)$, corresponding to a pure shear in the $\sigma_2-\sigma_3$ plane.

The dashed lines in Fig. 8.3.7 are helpful in that they allow us to plot and visualise stress states easily. The distance between each dashed line along the directions of the projected axes represents one unit of principal stress. Note, however, that these "units" are not consistent with the *actual* magnitudes of the deviatoric vectors in the $\pi-$plane. To create a more complete picture, note first that a unit vector along the space diagonal is $\mathbf{n}_\rho=\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, Fig. 8.3.8. The components of this normal are the direction cosines; for example, a unit normal along the '1' principal axis is $\mathbf{e}_1=[1,0,0]$ and so the angle θ_0 between the '1' axis and the space diagonal is given by $\mathbf{n}_\rho\cdot\mathbf{e}_1=\cos\theta_0=\frac{1}{\sqrt{3}}$. From Fig. 8.3.8, the angle θ between the '1' axis and the $\pi-$plane is given by $\cos\theta=\sqrt{\frac{2}{3}}$, and so a length of σ_1 units gets projected down to a length $s=|\mathbf{s}|=\sqrt{\frac{2}{3}}\sigma_1$.

For example, point *E* in Fig. 8.3.7 represents a pure shear $(\sigma_1,\sigma_2,\sigma_3)=(2,-2,0)$, which is on the $\pi-$plane. The length of the vector out to *E* in Fig. 8.3.7 is $2\sqrt{3}$ "units". To convert to actual magnitudes, multiply by $\sqrt{\frac{2}{3}}$ to get $s = 2\sqrt{2}$, which agrees with $s = |\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

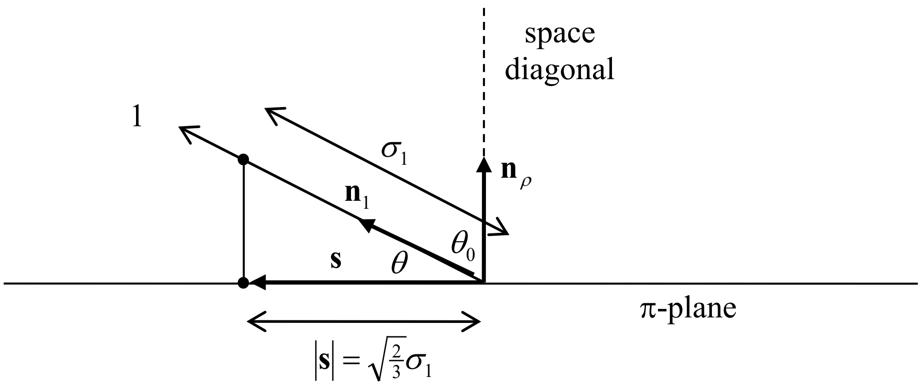


Figure 8.3.8: principal stress projected onto the π-plane

Typical π-plane Yield Loci

Consider next an arbitrary point (a,b,c) on the π −plane *yield locus*. If the material is isotropic, the points (a,c,b), (b,a,c), (b,c,a), (c,a,b) and (c,b,a) are also on the yield locus. If one assumes the same yield behaviour in tension as in compression, e.g. neglecting the Bauschinger effect, then so also are the points $(-a,-b,-c)$, $(-a,-c,-b)$, etc. Thus 1 point becomes 12 and one need only consider the yield locus in one 30º sector of the π −plane, the rest of the locus being generated through symmetry. One such sector is shown in Fig. 8.3.9, the axes of symmetry being the three projected principal axes and their (pure shear) bisectors.

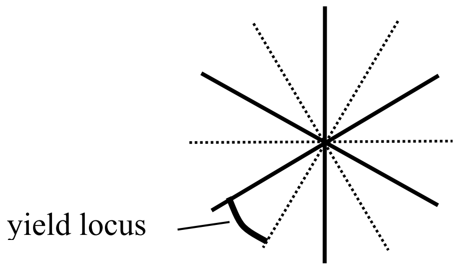


Figure 8.3.9: A typical sector of the yield locus

The Tresca and Von Mises Yield Loci in the π-plane

The Tresca criterion, Eqn. 8.3.7, is a regular hexagon in the π −plane as illustrated in Fig. 8.3.10. Which of the six sides of the locus is relevant depends on which of $\sigma_1, \sigma_2, \sigma_3$ is the maximum and which is the minimum, and whether they are tensile or compressive.

For example, yield at the pure shear $\sigma_1 = Y/2, \sigma_2 = 0, \sigma_3 = -Y/2$ is indicated by point A in the figure.

Point B represents yield under uniaxial tension, $\sigma_1 = Y$. The distance oB, the "magnitude" of the hexagon, is therefore $\sqrt{\frac{2}{3}}Y$; the corresponding point on the $\pi -$ plane is $(s_1, s_2 s_3) = \left(\frac{2}{3}Y, -\frac{1}{3}Y, -\frac{1}{3}Y\right)$.

A criticism of the Tresca criterion is that there is a sudden change in the planes upon which failure occurs upon a small change in stress at the sharp corners of the hexagon.

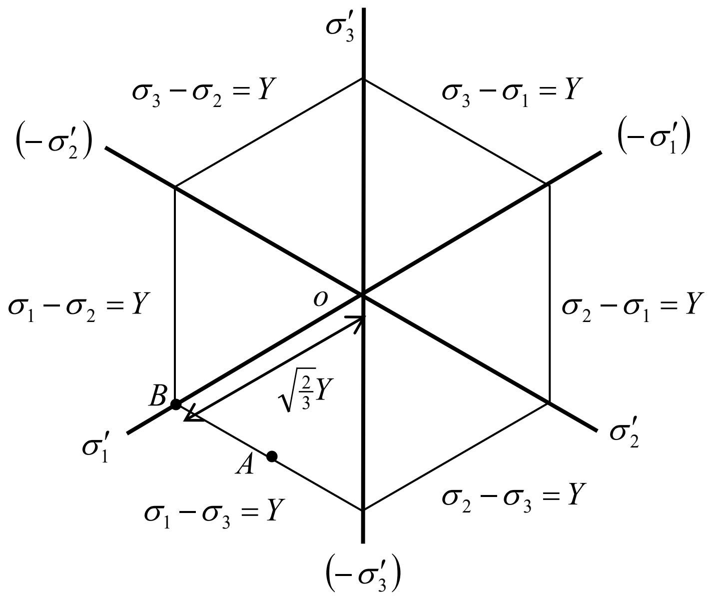


Figure 8.3.10: The Tresca criterion in the π-plane

Consider now the Von Mises criterion. From Eqns. 8.3.10, 8.3.13, the criterion is $\sqrt{J_2} = Y/\sqrt{3}$. From Eqn. 8.2.9, this can be re-written as

$$\sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{\frac{2}{3}}Y \tag{8.3.23}$$

Thus, the magnitude of the deviatoric stress vector is constant and one has a circular yield locus with radius $\sqrt{\frac{2}{3}}Y = \sqrt{2}k$, which circumscribes the Tresca hexagon, as illustrated in Fig. 8.3.11.

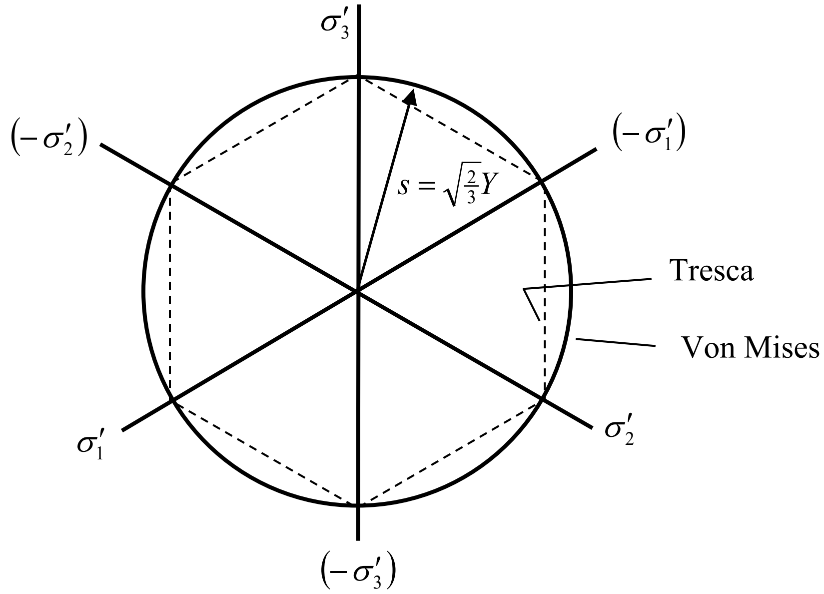


Figure 8.3.11: The Von Mises criterion in the π-plane

The yield surface is a circular cylinder with axis along the space diagonal, Fig. 8.3.12. The Tresca surface is a similar hexagonal cylinder.

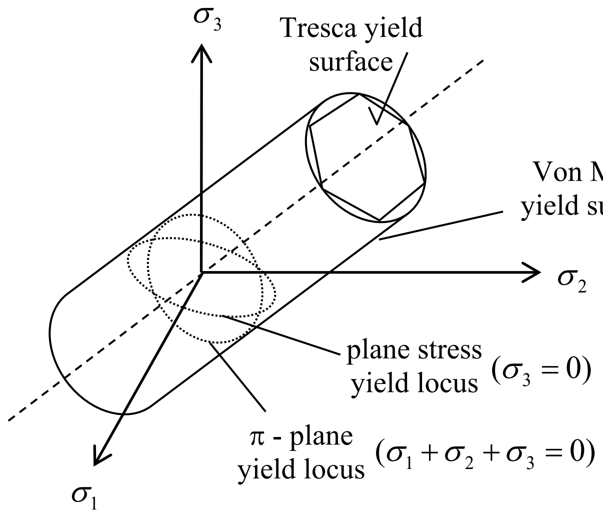


Figure 8.3.12: The Von Mises and Tresca yield surfaces

8.3.3 Haigh-Westergaard Stress Space

Thus far, yield criteria have been described in terms of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. It is often convenient to work with (ρ, s, θ) coordinates, Fig. 8.3.13; these cylindrical coordinates are called **Haigh-Westergaard coordinates**. They are particularly useful for describing and visualising geometrically pressure-dependent yield-criteria.

The coordinates (ρ, s) are simply the magnitudes of, respectively, the hydrostatic stress vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$ and the deviatoric stress vector $\mathbf{s} = (s_1, s_2, s_3)$. These are given by ($|\mathbf{s}|$ can be obtained from Eqn. 8.2.9)

$$\rho = |\boldsymbol{\rho}| = \sqrt{3}\sigma_m = I_1 / \sqrt{3}, \qquad s = |\mathbf{s}| = \sqrt{2J_2} \tag{8.3.24}$$

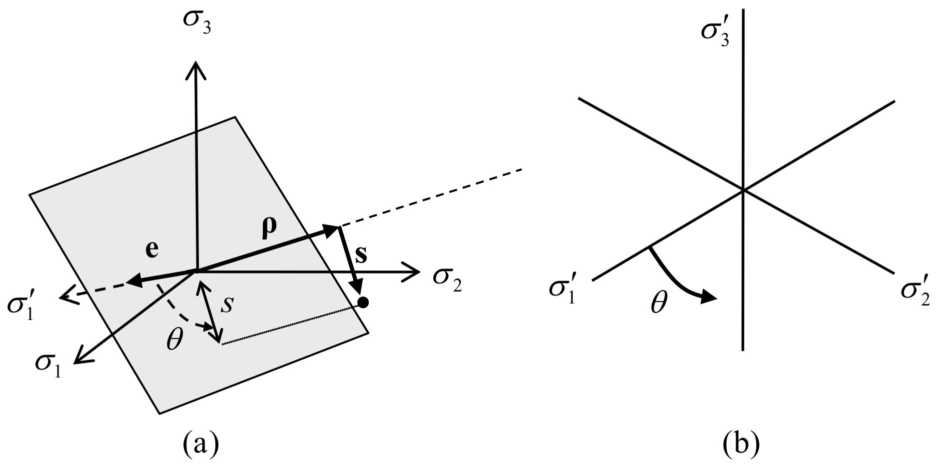


Figure 8.3.13: A point in stress space

θ is measured from the σ_1' (s_1) axis in the π-plane. To express θ in terms of invariants, consider a unit vector $\mathbf{e}$ in the π-plane in the direction of the σ_1' axis; this is the same vector $\mathbf{n}_c$ considered in Fig. 8.2.4 in connection with the octahedral shear stress, and it has coordinates $(\sigma_1, \sigma_2, \sigma_2) = \left(\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$, Fig. 8.3.13. The angle θ can now be obtained from $\mathbf{s} \cdot \mathbf{e} = s\cos\theta$ {▲Problem 9}:

$$\cos\theta = \frac{\sqrt{3}s_1}{2\sqrt{J_2}} \tag{8.3.25}$$

Further manipulation leads to the relation {▲Problem 10}

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \tag{8.3.26}$$

Since J_2 and J_3 are invariant, it follows that $\cos 3\theta$ is also. Note that J_3 enters through $\cos 3\theta$, and does not appear in ρ or s; it is J_3 which makes the yield locus in the π-plane non-circular.

From Eqn. 8.3.25 and Fig. 8.3.13b, the deviatoric stresses can be expressed in terms of the Haigh-Westergaard coordinates through

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \frac{2}{\sqrt{3}}\sqrt{J_2}\begin{bmatrix} \cos\theta \\ \cos(2\pi/3-\theta) \\ \cos(2\pi/3+\theta) \end{bmatrix} \tag{8.3.27}$$

The principal stresses and the Haigh-Westergaard coordinates can then be related through {▲Problem 12}

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} \rho \\ \rho \\ \rho \end{bmatrix} + \sqrt{\frac{2}{3}}s\begin{bmatrix} \cos\theta \\ \cos(\theta-2\pi/3) \\ \cos(\theta+2\pi/3) \end{bmatrix} \tag{8.3.28}$$

In terms of the Haigh-Westergaard coordinates, the yield criteria are

$$\begin{aligned} &\text{Von Mises:} && f(s) = \tfrac{1}{2}s^2 - k^2 = 0 \\ &\text{Tresca} && f(s,\theta) = \sqrt{2}s\sin\left(\theta + \tfrac{\pi}{3}\right) - Y = 0 \end{aligned} \tag{8.3.29}$$

8.3.4 Pressure Dependent Yield Criteria

The Tresca and Von Mises criteria are independent of hydrostatic pressure and are suitable for the modelling of plasticity in metals. For materials such as rock, soils and concrete, however, there is a strong dependence on the hydrostatic pressure.

The Drucker-Prager Criteria

The **Drucker-Prager** criterion is a simple modification of the Von Mises criterion, whereby the hydrostatic-dependent first invariant I_1 is introduced to the Von Mises Eqn. 8.3.13:

$$f(I_1, J_2) \equiv \alpha I_1 + \sqrt{J_2} - k = 0 \tag{8.3.30}$$

with α is a new material parameter. On the $\pi -$plane, $I_1 = 0$, and so the yield locus there is as for the Von Mises criterion, a circle of radius $\sqrt{2}k$, Fig. 8.3.14a. Off the $\pi -$ plane, the yield locus remains circular but the radius changes. When there is a state of pure hydrostatic stress, the magnitude of the hydrostatic stress vector is {▲Problem 13} $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$, with $s = |\mathbf{s}| = 0$. For large pressures, $\sigma_1 = \sigma_2 = \sigma_3 < 0$, the I_1 term in Eqn. 8.3.30 allows for large deviatoric stresses. This effect is shown in the **meridian plane** in Fig. 8.3.14b, that is, the (ρ, s) plane which includes the σ_1 axis.

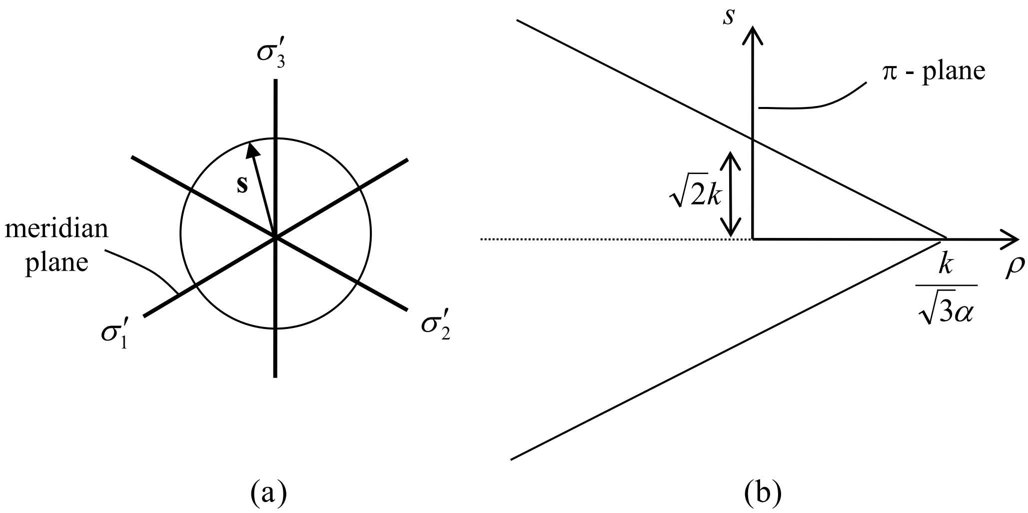


Figure 8.3.14: The Drucker-Prager criterion; (a) the π-plane, (b) the Meridian Plane

The Drucker-Prager surface is a right-circular cone with apex at $\rho = k/\sqrt{3}\alpha$, Fig. 8.3.15. Note that the plane stress locus, where the cone intersects the $\sigma_3 = 0$ plane, is an ellipse, but whose centre is off-axis, at some $(\sigma_1 < 0, \sigma_2 < 0)$.

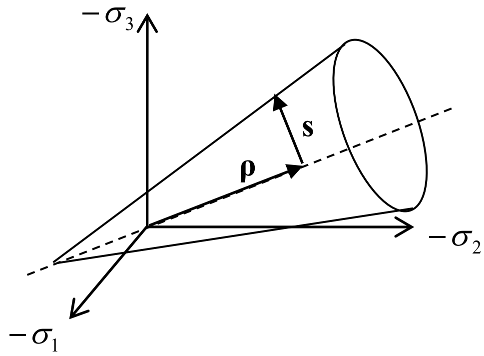


Figure 8.3.15: The Drucker-Prager yield surface

In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho, s) = \sqrt{6}\alpha\rho + s - \sqrt{2}k = 0 \tag{8.3.31}$$

The Mohr Coulomb Criteria

The Mohr-Coulomb criterion is based on Coulomb's 1773 friction equation, which can be expressed in the form

$$|\tau| = c - \sigma_n \tan\phi \tag{8.3.32}$$

where c, ϕ are material constants; c is called the **cohesion**[4] and ϕ is called the **angle of internal friction**. τ and σ_n are the shear and normal stresses acting on the plane where failure occurs (through a shearing effect), Fig. 8.3.16, with $\tan\phi$ playing the role of a coefficient of friction. The criterion states that the larger the pressure $-\sigma_n$, the more shear the material can sustain. Note that the Mohr-Coulomb criterion can be considered to be a generalised version of the Tresca criterion, since it reduces to Tresca's when $\phi = 0$ with $c = k$.

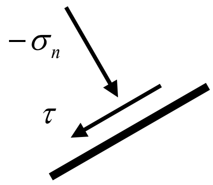


Figure 8.3.16: Coulomb friction over a plane

This criterion not only includes a hydrostatic pressure effect, but also allows for different yield behaviours in tension and in compression. Maintaining isotropy, there will now be three lines of symmetry in any deviatoric plane, and a typical sector of the yield locus is as shown in Fig. 8.3.17 (compare with Fig. 8.3.9)

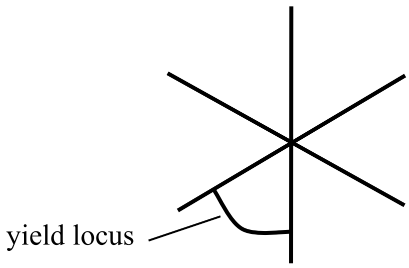


Figure 8.3.17: A typical sector of the yield locus for an isotropic material with different yield behaviour in tension and compression

Given values of c and ϕ, one can draw the failure locus (lines) of the Mohr-Coulomb criterion in (σ_n, τ) stress space, with intercepts $\tau = \pm c$ and slopes $\mp \tan\phi$, Fig. 8.3.18. Given some stress state $\sigma_1 \geq \sigma_2 \geq \sigma_3$, a Mohr stress circle can be drawn also in (σ_n, τ) space (see §7.2.6). When the stress state is such that this circle reaches out and touches the failure lines, yield occurs.

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[4] $c = 0$ corresponds to a cohesionless material such as sand or gravel, which has no strength in tension

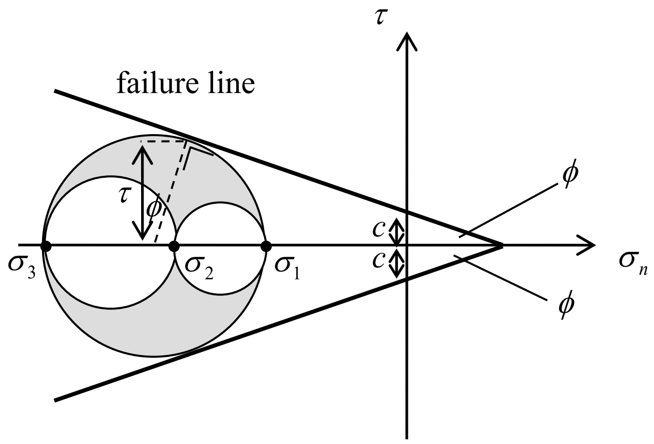


Figure 8.3.18: Mohr-Coulomb failure criterion

From Fig. 8.3.18, and noting that the large Mohr circle has centre $\left(\frac{1}{2}(\sigma_1+\sigma_3),0\right)$ and radius $\frac{1}{2}(\sigma_1-\sigma_3)$, one has

$$\begin{aligned} \tau &= \frac{\sigma_1-\sigma_3}{2}\cos\phi \\ \sigma_n &= \frac{\sigma_1+\sigma_3}{2}+\frac{\sigma_1-\sigma_3}{2}\sin\phi \end{aligned} \tag{8.3.33}$$

Thus the Mohr-Coulomb criterion in terms of principal stresses is

$$(\sigma_1-\sigma_3) = 2c\cos\phi-(\sigma_1+\sigma_3)\sin\phi \tag{8.3.34}$$

The strength of the Mohr-Coulomb material in uniaxial tension, f_{Yt}, and in uniaxial compression, f_{Yc}, are thus

$$f_{Yt} = \frac{2c\cos\phi}{1+\sin\phi}, \quad f_{Yc} = \frac{2c\cos\phi}{1-\sin\phi} \tag{8.3.35}$$

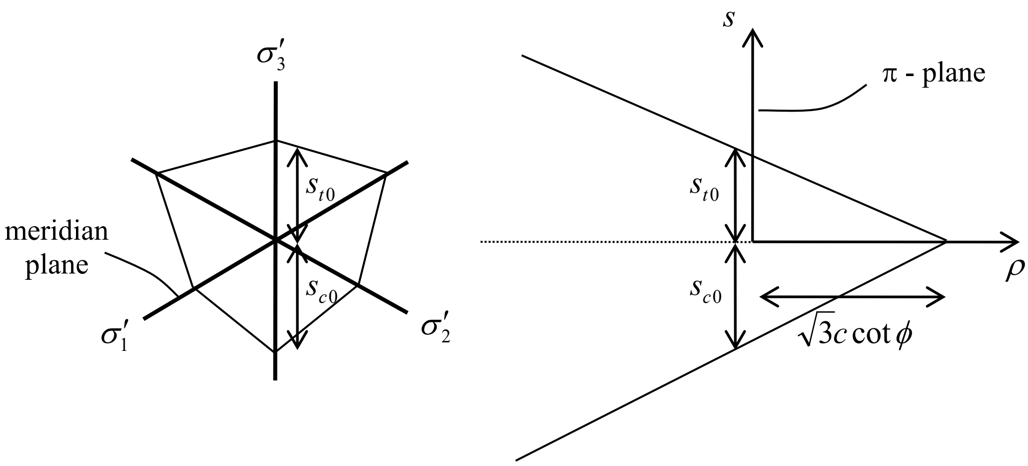
In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho,s,\theta) = \sqrt{2}\rho\sin\phi+\sqrt{3}s\sin\left(\theta+\tfrac{\pi}{3}\right)+s\cos\left(\theta+\tfrac{\pi}{3}\right)\sin\phi-\sqrt{6}c\cos\phi = 0 \tag{8.3.36}$$

The Mohr-Coulomb yield surface in the π – plane and meridian plane are displayed in Fig. 8.3.19. In the π – plane one has an irregular hexagon which can be constructed from two lengths: the magnitude of the deviatoric stress in uniaxial tension at yield, s_{t0}, and the corresponding (larger) value in compression, s_{c0}; these are given by:

$$s_{t0} = \frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3+\sin\phi}, \quad s_{c0} = \frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3-\sin\phi} \tag{8.3.37}$$

In the meridian plane, the failure surface cuts the $s = 0$ axis at $\rho = \sqrt{3}c \cot\phi$ {▲Problem 14}.



(a) (b)

Figure 8.3.19: The Mohr-Coulomb criterion; (a) the π-plane, (b) the Meridian Plane

The Mohr-Coulomb surface is thus an irregular hexagonal pyramid, Fig. 8.3.20.

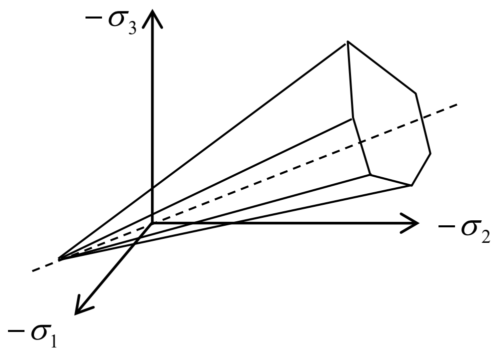


Figure 8.3.20: The Mohr-Coulomb yield surface

By adjusting the material parameters α, k, c, ϕ, the Drucker-Prager cone can be made to match the Mohr-Coulomb hexagon, either inscribing it at the minor vertices, or circumscribing it at the major vertices, Fig. 8.3.21.

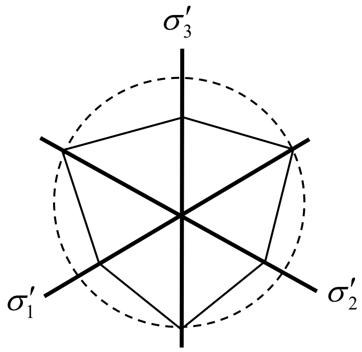


Figure 8.3.21: The Mohr-Coulomb and Drucker-Prager criteria matched in the π-plane

Capped Yield Surfaces

The Mohr-Coulomb and Drucker-Prager surfaces are open in that a pure hydrostatic pressure can be applied without affecting yield. For many geomaterials, however, for example soils, a large enough hydrostatic pressure will induce permanent deformation. In these cases, a **closed (capped) yield surface** is more appropriate, for example the one illustrated in Fig. 8.3.22.

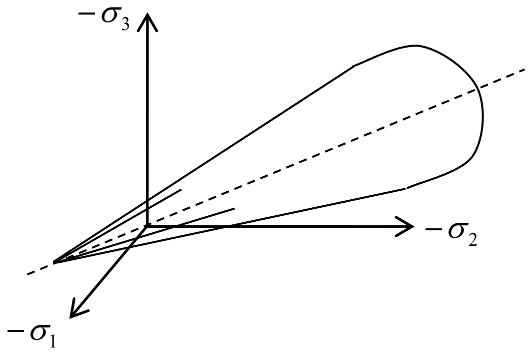


Figure 8.3.22: a capped yield surface

An example is the **modified Cam-Clay** criterion:

$$3J_2 = -\tfrac{1}{3}I_1 M^2\left(2p_c + \tfrac{1}{3}I_1\right) \text{ or } s = \sqrt{-\tfrac{2}{3\sqrt{3}}\rho M^2\left(2p_c + \tfrac{1}{\sqrt{3}}\rho\right)}, \quad \rho < 0 \tag{8.3.38}$$

with M and p_c material constants. In terms of the standard geomechanics notation, it reads

$$q^2 = M^2 p\left(2p_c - p\right) \tag{8.3.39}$$

where

$$p = -\frac{1}{3} I_1 = -\frac{1}{\sqrt{3}} \rho, \qquad q = \sqrt{3J_2} = \sqrt{\frac{3}{2}} s \tag{8.3.40}$$

The modified Cam-Clay locus in the meridian plane is shown in Fig. 8.3.23. Since s is constant for any given ρ, the locus in planes parallel to the π - plane are circles. The material parameter p_c is called the **critical state pressure**, and is the pressure which carries the maximum deviatoric stress. M is the slope of the dotted line shown in Fig. 8.3.23, known as the **critical state line**.

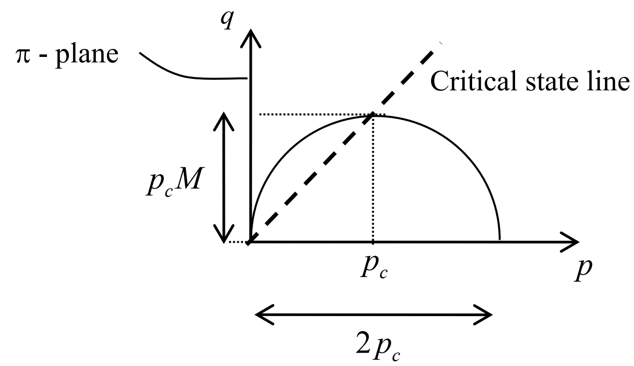


Figure 8.3.23: The modified Cam-Clay criterion in the Meridian Plane

8.3.5 Anisotropy

Many materials will display anisotropy. For example metals which have been processed by rolling will have characteristic material directions, the tensile yield stress in the direction of rolling being typically 15% greater than that in the transverse direction. The form of anisotropy exhibited by rolled sheets is such that the material properties are symmetric about three mutually orthogonal planes. The lines of intersection of these planes form an orthogonal set of axes known as the **principal axes of anisotropy**. The axes are (a) in the rolling direction, (b) normal to the sheet, (c) in the plane of the sheet but normal to rolling direction. This form of anisotropy is called **orthotropy** (see Part I, §6.2.2). Hill (1948) proposed a yield condition for such a material which is a natural generalisation of the Mises condition:

$$\begin{aligned} f(\sigma_{ij}) = F\left(\sigma_{22} - \sigma_{33}\right)^2 + G\left(\sigma_{33} - \sigma_{11}\right)^2 + H\left(\sigma_{11} - \sigma_{22}\right)^2 \\ + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 - 1 = 0 \end{aligned} \tag{8.3.41}$$

where F, G, H, L, M, N are material constants. One needs to carry out 6 tests: uniaxial tests in the three coordinate directions to find the uniaxial yield strengths $(\sigma_Y)_x, (\sigma_Y)_y, (\sigma_Y)_z$, and shear tests to find the shear strengths $(\tau_Y)_{xy}, (\tau_Y)_{yz}, (\tau_Y)_{zx}$. For a uniaxial test in the x direction, Eqn. 8.3.41 reduces to $G + H = 1/(\sigma_Y)_x^2$. By considering the other simple uniaxial and shear tests, one can solve for the material parameters:

$$F = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_y^2} + \frac{1}{(\sigma_Y)_z^2} - \frac{1}{(\sigma_Y)_x^2}\right] \qquad L = \frac{1}{2}\frac{1}{(\tau_Y)_{yz}^2}$$
$$G = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_z^2} + \frac{1}{(\sigma_Y)_x^2} - \frac{1}{(\sigma_Y)_y^2}\right] \qquad M = \frac{1}{2}\frac{1}{(\tau_Y)_{zx}^2} \tag{8.3.42}$$
$$H = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_x^2} + \frac{1}{(\sigma_Y)_y^2} - \frac{1}{(\sigma_Y)_z^2}\right] \qquad N = \frac{1}{2}\frac{1}{(\tau_Y)_{xy}^2}$$

The criterion reduces to the Mises condition 8.3.12 when

$$F = G = H = \frac{L}{3} = \frac{M}{3} = \frac{N}{3} = \frac{1}{6k^2} \tag{8.3.43}$$

The 1, 2, 3 axes of reference in Eqn. 8.3.41 are the principal axes of anisotropy. The form appropriate for a general choice of axes can be derived by using the usual stress transformation formulae. It is complicated and involves cross-terms such as $\sigma_{11}\sigma_{23}$, etc.

8.3.6 Problems

1. A material is to be loaded to a stress state
$$[\sigma_{ij}] = \begin{bmatrix} 50 & -30 & 0 \\ -30 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$
What should be the minimum uniaxial yield stress of the material so that it does not fail, according to the
(a) Tresca criterian
(b) Von Mises criterion
What do the theories predict when the yield stress of the material is 80MPa?

2. Use Eqn. 8.2.6, $J_2 = s_{12}^2 + s_{23}^2 + s_{31}^2 - (s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11})$ to derive Eqn. 8.3.12, $(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 6k^2$, for the Von Mises criterion.

3. Use the plane stress principal stress formula $\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$ to derive Eqn. 8.3.15 for the Taylor-Quinney tests.

4. Derive Eqn. 8.3.17 for the Taylor-Quinney tests.

5. Describe the states of stress represented by the points D and E in Fig. 8.3.4. (The complete stress states can be visualised with the help of Mohr's circles of stress, Fig. 7.2.17.)

6. Suppose that, in the Taylor and Quinney tension-torsion tests, one has $\sigma = Y/2$ and $\tau = \sqrt{3}Y/4$. Plot this stress state in the 2D principal stress state, Fig. 8.3.4. (Use Eqn. 8.3.15 to evaluate the principal stresses.) Keeping now the normal stress at $\sigma = Y/2$, what value can the shear stress be increased to before the material yields, according to the von Mises criterion?

7. What are the π −plane principal stress values for the points *B* and *C* in Fig. 8.3.7?

8. Sketch on the π −plane Fig. 8.3.7 a line corresponding to $\sigma_1 = \sigma_2$ and also a region corresponding to $\sigma_1 > \sigma_2 > 0 > \sigma_3$

9. Using the relation $\mathbf{s} \cdot \mathbf{e} = s\cos\theta$ and $s_2 + s_3 = -s_1$, derive Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$.

10. Using the trigonometric relation $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$, show that $\cos 3\theta = \dfrac{3\sqrt{3}s_1}{2J_2^{3/2}}\left(s_1^2 - J_2\right)$. Then using the relations 8.2.6, $J_2 = -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right)$, with $J_1 = 0$, derive Eqn. 8.3.26, $\cos 3\theta = \dfrac{3\sqrt{3}}{2}\dfrac{J_3}{J_2^{3/2}}$

11. Consider the following stress states. For each one, evaluate the space coordinates (ρ, s, θ) and plot in the π −plane (see Fig. 8.3.13b):
    (a) triaxial tension: $\sigma_1 = T_1 > \sigma_2 = \sigma_3 = T_2 > 0$
    (b) triaxial compression: $\sigma_3 = -p_1 < \sigma_1 = \sigma_2 = -p_2 < 0$ (this is an important test for geomaterials, which are dependent on the hydrostatic pressure)
    (c) a pure shear $\sigma_{xy} = \tau$: $\sigma_1 = +\tau,\ \sigma_2 = 0,\ \sigma_3 = -\tau$
    (d) a pure shear $\sigma_{xy} = \tau$ in the presence of hydrostatic pressure *p*:
    $\sigma_1 = -p + \tau,\ \sigma_2 = -p,\ \sigma_3 = -p - \tau$, i.e. $\sigma_1 - \sigma_2 = \sigma_2 - \sigma_3$

12. Use relations 8.3.24, $\rho = I_1/\sqrt{3},\ s = \sqrt{2J_2}$ and Eqns. 8.3.27 to derive Eqns. 8.3.28.

13. Show that the magnitude of the hydrostatic stress vector is $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$ for the Drucker-Prager yield criterion when the deviatoric stress is zero

14. Show that the magnitude of the hydrostatic stress vector is $\rho = \sqrt{3}c\cot\phi$ for the Mohr-Coulomb yield criterion when the deviatoric stress is zero

15. Show that, for a Mohr-Coulomb material, $\sin\phi = (r-1)/(r+1)$, where $r = f_{Yc}/f_{Yt}$ is the compressive to tensile strength ratio

16. A sample of concrete is subjected to a stress $\sigma_{11} = \sigma_{22} = -p,\ \sigma_{33} = -Ap$ where the constant $A > 1$. Using the Mohr-Coulomb criterion and the result of Problem 15, show that the material will not fail provided $A < f_{Yc}/p + r$

8.4 Elastic Perfectly Plastic Materials

Once yield occurs, a material will deform plastically. Predicting and modelling this plastic deformation is the topic of this section. For the most part, in this section, the material will be assumed to be perfectly plastic, that is, there is no work hardening.

8.4.1 Plastic Strain Increments

When examining the strains in a plastic material, it should be emphasised that one works with *increments in strain* rather than a total accumulated strain. One reason for this is that when a material is subjected to a certain stress state, the corresponding strain state could be one of many. Similarly, the strain state could correspond to many different stress states. Examples of this state of affairs are shown in Fig. 8.4.1.

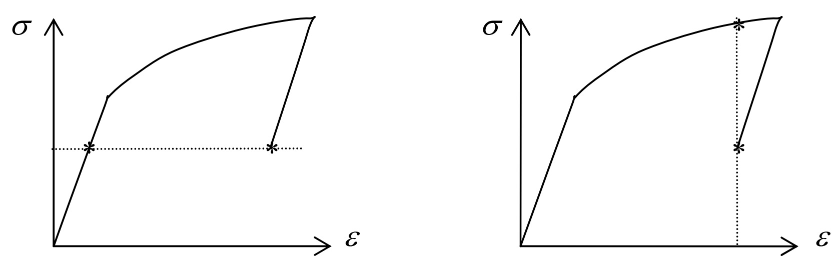


Figure 8.4.1: stress-strain curve; (a) different strains at a certain stress, (b) different stress at a certain strain

One cannot therefore make use of stress-strain relations in plastic regions (except in some special cases), since there is no unique relationship between the current stress and the current strain. However, one can relate the current stress to the current **increment in strain**, and these are the "stress-strain" laws which are used in plasticity theory. The total strain can be obtained by summing up, or integrating, the strain increments.

8.4.2 The Prandtl-Reuss Equations

An increment in strain $d\varepsilon$ can be decomposed into an elastic part $d\varepsilon^e$ and a plastic part $d\varepsilon^p$. If the material is isotropic, it is reasonable to suppose that the principal plastic strain increments $d\varepsilon_i^p$ are proportional to the principal deviatoric stresses s_i:

$$\frac{d\varepsilon_1^p}{s_1} = \frac{d\varepsilon_2^p}{s_2} = \frac{d\varepsilon_3^p}{s_3} = d\lambda \geq 0 \qquad (8.4.1)$$

This relation only gives the ratios of the plastic strain increments to the deviatoric stresses. To determine the precise relationship, one must specify the positive scalar $d\lambda$ (see later). Note that the plastic volume constancy is inherent in this relation: $d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = 0$.

Eqns. 8.4.1 are in terms of the principal deviatoric stresses and principal plastic strain increments. In terms of Cartesian coordinates, one has

$$\frac{d\varepsilon_{xx}^p}{s_{xx}} = \frac{d\varepsilon_{yy}^p}{s_{yy}} = \frac{d\varepsilon_{zz}^p}{s_{zz}} = \frac{d\varepsilon_{xy}^p}{s_{xy}} = \frac{d\varepsilon_{xz}^p}{s_{xz}} = \frac{d\varepsilon_{yz}^p}{s_{yz}} = d\lambda \tag{8.4.2}$$

or, succinctly,

$$d\varepsilon_{ij}^p = s_{ij}\, d\lambda \tag{8.4.3}$$

These equations are often expressed in the alternative forms

$$\frac{d\varepsilon_{xx}^p - d\varepsilon_{yy}^p}{s_{xx} - s_{yy}} = \frac{d\varepsilon_{yy}^p - d\varepsilon_{zz}^p}{s_{yy} - s_{zz}} = \cdots \quad = \frac{d\varepsilon_{xx}^p - d\varepsilon_{yy}^p}{\sigma_{xx} - \sigma_{yy}} = \frac{d\varepsilon_{yy}^p - d\varepsilon_{zz}^p}{\sigma_{yy} - \sigma_{zz}} = \cdots \quad = d\lambda \tag{8.4.4}$$

or, dividing by dt to get the **rate** equations,

$$\frac{\dot\varepsilon_{xx}^p - \dot\varepsilon_{yy}^p}{s_{xx} - s_{yy}} = \cdots = \frac{\dot\varepsilon_{xx}^p - \dot\varepsilon_{yy}^p}{\sigma_{xx} - \sigma_{yy}} = \cdots = \dot\lambda \tag{8.4.5}$$

In terms of actual stresses, one has, from 8.2.3,

$$\begin{aligned}
d\varepsilon_{xx}^p &= \tfrac{2}{3} d\lambda\left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy}^p &= \tfrac{2}{3} d\lambda\left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz}^p &= \tfrac{2}{3} d\lambda\left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy}^p &= d\lambda \sigma_{xy} \\
d\varepsilon_{yz}^p &= d\lambda \sigma_{yz} \\
d\varepsilon_{zx}^p &= d\lambda \sigma_{zx}
\end{aligned} \tag{8.4.6}$$

This *plastic* stress-strain law is known as a **flow rule**. Other flow rules will be considered later on. Note that one cannot propose a flow rule which gives the plastic strain increments as explicit functions of the stress, otherwise the yield criterion might not be met (in particular, when there is strain hardening); one must include the to-be-determined scalar plastic multiplier λ. The plastic multiplier is determined by ensuring the stress-state lies on the yield surface during plastic flow.

The full elastic-plastic stress-strain relations are now, using Hooke's law,

$$
\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu\left(d\sigma_{yy} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy} &= \frac{1}{E}\left[d\sigma_{yy} - \nu\left(d\sigma_{xx} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{yy} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu\left(d\sigma_{xx} + d\sigma_{yy}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{zz} - \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \\
d\varepsilon_{yz} &= \frac{1+\nu}{E}d\sigma_{yz} + d\lambda\sigma_{yz} \\
d\varepsilon_{zx} &= \frac{1+\nu}{E}d\sigma_{zx} + d\lambda\sigma_{zx}
\end{aligned}
\qquad (8.4.7)
$$

or

$$
d\varepsilon_{ij} = \frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}\delta_{ij}d\sigma_{kk} + d\lambda s_{ij}
$$

These expressions are called the **Prandtl-Reuss equations**. If the first, elastic, terms are neglected, they are known as the **Lévy-Mises** equations.

8.4.3 Application: Plane Strain Compression of a Block

Consider the plane strain compression of a thick block, Fig. 8.4.2. The block is subjected to an increasing pressure $\sigma_{xx} = -p$, is constrained in the *z* direction, so $\varepsilon_{zz} = 0$, and is free to move in the *y* direction, so $\sigma_{yy} = 0$.

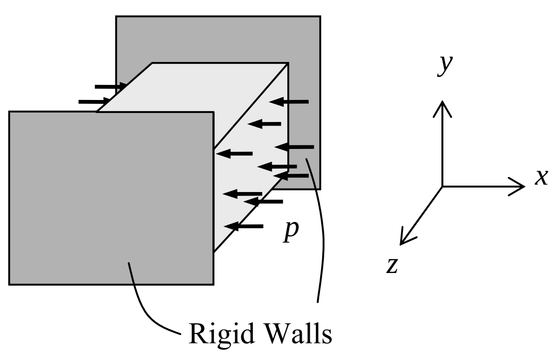


Figure 8.4.2: Plane strain compression of a thick block

The solution to the *elastic* problem is obtained from 8.4.7 (disregarding the plastic terms). One finds that {▲Problem 1}

$$
\sigma_{xx} = -p, \quad \sigma_{zz} = -\nu p, \quad \varepsilon_{xx} = -\frac{p}{E}\left(1-\nu^2\right), \quad \varepsilon_{yy} = +\frac{p}{E}\nu\left(1+\nu\right) \qquad (8.4.8)
$$

and all other stress and strain components are zero. In this elastic phase, the principal stresses are clearly

$$\sigma_1 = \sigma_{yy} = 0 > \sigma_2 = \sigma_{zz} > \sigma_3 = \sigma_{xx} \tag{8.4.9}$$

The Prandtl-Reuss equations are

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu d\sigma_{zz}\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}\left(d\sigma_{xx} + d\sigma_{zz}\right) - \frac{1}{3}d\lambda\left(\sigma_{xx} + \sigma_{zz}\right) \\ d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu d\sigma_{xx}\right] + \frac{2}{3}d\lambda\left[-\frac{1}{2}\sigma_{xx} + \sigma_{zz}\right] \end{aligned} \tag{8.4.10}$$

The magnitude of the plastic straining is determined by the multiplier $d\lambda$. This can be evaluated by noting that plastic deformation proceeds so long as the stress state remains on the yield surface, the so-called **consistency condition**. By definition, a perfectly plastic material is one whose yield surface remains unchanged during deformation.

A Tresca Material

Take now the Tresca yield criterion, which states that yield occurs when $\sigma_{xx} = -Y$, where Y is the uniaxial yield stress (in compression). Assume further perfect plasticity, so that $\sigma_{xx} = -Y$ holds during all subsequent plastic flow. Thus, with $d\sigma_{xx} = 0$, and since $d\varepsilon_{zz} = 0$, 8.4.10 reduce to

$$\begin{aligned} d\varepsilon_{xx} &= -\frac{\nu}{E}d\sigma_{zz} - \frac{2}{3}d\lambda\left[Y + \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}d\sigma_{zz} + \frac{1}{3}d\lambda\left(Y - \sigma_{zz}\right) \\ 0 &= \frac{1}{E}d\sigma_{zz} + \frac{2}{3}d\lambda\left[\frac{1}{2}Y + \sigma_{zz}\right] \end{aligned} \tag{8.4.11}$$

Thus

$$d\lambda = -\frac{3}{E}\frac{d\sigma_{zz}}{2\sigma_{zz} + Y} \tag{8.4.12}$$

and, eliminating $d\lambda$ from Eqns. 8.4.11 {▲Problem 2},

$$\begin{aligned} Ed\varepsilon_{xx} &= -\nu d\sigma_{zz} + Y\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \\ Ed\varepsilon_{yy} &= -\nu d\sigma_{zz} - \frac{Y}{2}\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \end{aligned} \tag{8.4.13}$$

Using the relation

$$\int \frac{x}{x+a} dx = x - a\ln(x+a) \tag{8.4.14}$$

and the initial (yield point) conditions, i.e. Eqns. 8.4.8 with $p = Y$, one can integrate 8.4.13 to get {▲Problem 2}

$$\begin{aligned} \frac{E}{Y}\varepsilon_{xx} &= -\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} - \frac{1}{2}(2-\nu) \\ \frac{E}{Y}\varepsilon_{yy} &= +\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} + \frac{3}{2}\nu \end{aligned}, \quad \frac{\sigma_{zz}}{Y} < -\nu \tag{8.4.15}$$

The stress-strain curves are shown in Fig. 8.4.3 below for $\nu = 0.3$. Note that, for a typical metal, $E/Y \sim 10^3$, and so the strains are very small right through the plastic compression; the plastic strains are of comparable size to the elastic strains. There is a rapid change of stress and then little change once σ_{zz} has approached close to its limiting value of $-Y/2$.

The above plastic analysis was based on σ_{xx} remaining the minimum principal stress. This assumption has proved to be valid, since σ_{zz} remains between 0 and $-Y$ in the plastic region.

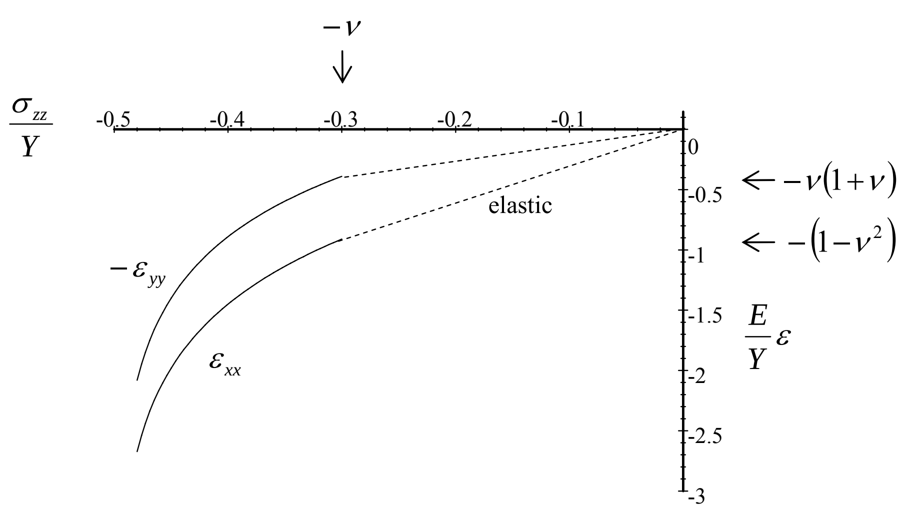


Figure 8.4.3: Stress-strain results for plane strain compression of a thick block for $\nu = 0.3$

A Von Mises Material

Slightly different results are obtained with the Von Mises yield criterion, Eqn. 8.4.11, which for this problem reads

$$\sigma_{xx}^2 - \sigma_{xx}\sigma_{zz} + \sigma_{zz}^2 = Y^2 \tag{8.4.16}$$

The Prandtl-Reuss equations can be solved by making the substitution

$$\sigma_{xx} = -\frac{2Y}{\sqrt{3}}\cos\theta \tag{8.4.17}$$

in the plastic region. In what follows, use is made of the trigonometric relations

$$\begin{aligned}\sin\left(\frac{\pi}{6}-\theta\right) &= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \\ \cos\left(\frac{\pi}{6}-\theta\right) &= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\end{aligned} \tag{8.4.18}$$

From Eqn. 8.4.16,

$$\sigma_{zz} = -\frac{2Y}{\sqrt{3}}\sin\left(\frac{\pi}{6}-\theta\right) \tag{8.4.19}$$

Substituting into 8.4.10 then leads to

$$\begin{aligned}\frac{E}{Y}d\varepsilon_{xx} &= \frac{2}{\sqrt{3}}\left\{\left[\sin\theta - \nu\cos\left(\frac{\pi}{6}-\theta\right)\right]d\theta - \frac{1}{\sqrt{3}}Ed\lambda\cos\left(\frac{\pi}{6}-\theta\right)\right\} \\ \frac{E}{Y}d\varepsilon_{yy} &= \frac{2}{\sqrt{3}}\left\{-\nu\left(\sin\theta + \cos\left(\frac{\pi}{6}-\theta\right)\right)d\theta - \frac{1}{3}Ed\lambda\left(-\cos\theta - \sin\left(\frac{\pi}{6}-\theta\right)\right)\right\} \\ \frac{E}{Y}d\varepsilon_{zz} &= \frac{2}{\sqrt{3}}\left\{\left[\cos\left(\frac{\pi}{6}-\theta\right) - \nu\sin\theta\right]d\theta + \frac{1}{\sqrt{3}}Ed\lambda\sin\theta\right\}\end{aligned} \tag{8.4.20}$$

Using $d\varepsilon_{zz} = 0$ leads to

$$d\lambda = -\frac{\sqrt{3}}{E}\frac{\cos\left(\frac{\pi}{6}-\theta\right) - \nu\sin\theta}{\sin\theta}d\theta \tag{8.4.21}$$

and

$$\frac{E}{Y}d\varepsilon_{xx} = \frac{2}{\sqrt{3}}\left\{(1-2\nu)\cos\left(\frac{\pi}{6}-\theta\right) + \frac{3}{4}\operatorname{cosec}\theta\right\}d\theta \tag{8.4.22}$$

An integration gives

$$\frac{E}{Y}\varepsilon_{xx} = -\frac{2}{\sqrt{3}}(1-2\nu)\sin\left(\frac{\pi}{6}-\theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta}{2}\right| + C \tag{8.4.23}$$

To determine the constant of integration, consider again the conditions at first yield. Suppose the block first yields when σ_{xx} reaches σ_{xx}^Y. Then $\sigma_{zz}^Y = \nu\sigma_{xx}^Y$ and

$$\sigma_{xx}^Y = -\frac{Y}{\sqrt{1-\nu+\nu^2}} \tag{8.4.24}$$

Note that in this case it is predicted that first yield occurs when $\sigma_{xx} < -Y$. From Eqn. 8.4.17, the value of θ at first yield is

$$\cos\theta^Y = \frac{\sqrt{3}}{2\sqrt{1-\nu+\nu^2}} \quad \text{or} \quad \tan\theta^Y = \frac{1-2\nu}{\sqrt{3}} \tag{8.4.25}$$

Thus, with $\varepsilon_{xx} = \sigma_{xx}^Y\left(1-\nu^2\right)/E$ at yield,

$$C = -\sqrt{1-\nu+\nu^2} - \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^Y}{2}\right| \tag{8.4.26}$$

and so

$$-\frac{E}{Y}\varepsilon_{xx} = \frac{2}{\sqrt{3}}(1-2\nu)\sin\left(\frac{\pi}{6}-\theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^Y}{2}\cot\frac{\theta}{2}\right| + \sqrt{1-\nu+\nu^2} \tag{8.4.27}$$

This leads to a similar stress-strain curve as for the Tresca criterion, only now the limiting value of σ_{zz} is $-Y/\sqrt{3} \approx -0.58Y$.

8.4.4 Application: Combined Tension/Torsion of a thin walled tube

Consider now the combined tension/torsion of a thin-walled tube as in the Taylor/Quinney tests. The only stresses in the tube are $\sigma_{xx} = \sigma$ due to the tension along the axial direction and $\sigma_{xy} = \tau$ due to the torsion. The Prandtl-Reuss equations reduce to

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \end{aligned} \tag{8.4.28}$$

Consider the case where the tube is twisted up to the yield point. Torsion is then halted and tension is applied, holding the angle of twist constant. In that case, during the tension, $d\varepsilon_{xy} = 0$ and so {▲Problem 3}

$$d\varepsilon_{xx} = \frac{1}{E} d\sigma - \frac{2}{3}\frac{1+\nu}{E}\frac{d\tau}{\tau}\sigma \tag{8.4.29}$$

If one takes the Von Mises criterion, then $\sigma^2 + 3\tau^2 = Y^2$ (see Eqn. 8.3.17). Assuming perfect plasticity, one has {▲Problem 4},

$$d\varepsilon_{xx} = \frac{1}{E} d\sigma + \frac{2}{3}\frac{1+\nu}{E}\frac{\sigma^2 d\sigma}{Y^2 - \sigma^2} \tag{8.4.30}$$

Using the relation

$$\int \frac{x^2}{a^2 - x^2} dx = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right), \tag{8.4.31}$$

an integration leads to {▲Problem 5}

$$\frac{E}{Y}\varepsilon_{xx} = \frac{1}{3}\left\{(1-2\nu)\frac{\sigma}{Y} + (1+\nu)\ln\left(\frac{1+\sigma/Y}{1-\sigma/Y}\right)\right\} \tag{8.4.32}$$

This result is plotted in Fig. 8.4.4. Note that, with $\sigma^2 + 3\tau^2 = Y^2$, as σ increases (rapidly) to its limiting value Y, τ decreases from its yield value of $Y/\sqrt{3}$ to zero.

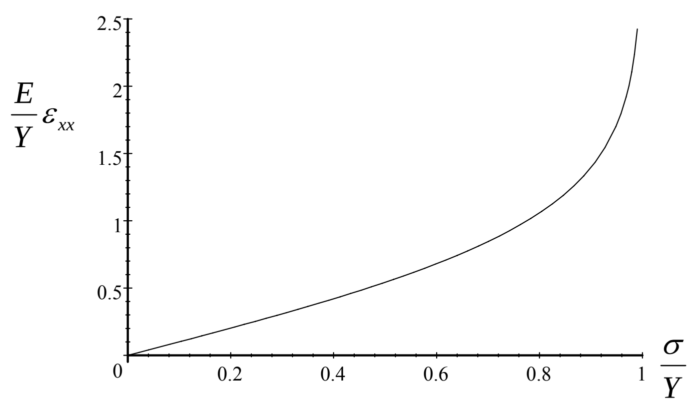


Figure 8.4.4: Stress-strain results for combined tension/torsion of a thin walled tube for $\nu = 0.3$

8.4.5 The Tresca Flow Rule

The flow rule used in the preceding applications was the Prandtl-Reuss rule 8.4.7. Many other flow rules have been proposed. For example, the Tresca flow rule is simply (for $\sigma_1 > \sigma_2 > \sigma_3$)

$$\begin{aligned} d\varepsilon_1^p &= +d\lambda \\ d\varepsilon_2^p &= 0 \\ d\varepsilon_3^p &= -d\lambda \end{aligned} \qquad (8.4.33)$$

This flow rule will be used in the next section, which details the classic solution for the plastic deformation and failure of a thick cylinder under internal pressure.

A unifying theory of flow rules will be presented in a later section, in which the reason for the name "Treca flow rule" will become clear.

8.4.6 Problems

1. Derive the elastic strains for the plane strain compression of a thick block, Eqns. 8.4.8.

2. Derive Eqns. 8.4.13 and 8.4.15

3. Derive Eqn. 8.4.29

4. Use Eqn. 8.3.17 to show that $\sigma d\tau / \tau = -\sigma^2 d\sigma / \left(Y^2 - \sigma^2\right)$ and hence derive Eqn. 8.4.30

5. Derive Eqns. 8.4.32

6. Does the axial stress-stress curve of Fig. 8.4.4 differ when the Tresca criterion is used?

7. Consider the uniaxial straining of a perfectly plastic isotropic Von Mises metallic block. There is only one non-zero strain, ε_{xx}. One only need consider two stresses, σ_{xx}, σ_{yy} since $\sigma_{zz} = \sigma_{yy}$ by isotropy.
   (i) Write down the two relevant Prandtl-Reuss equations
   (ii) Evaluate the stresses and strains at first yield
   (iii) For plastic flow, show that $d\sigma_{xx} = d\sigma_{yy}$ and that the plastic modulus is

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}} = \frac{E}{3(1-2\nu)}$$

8. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of a perfectly plastic Von Mises metal and Y is the uniaxial yield strength in tension. The only stresses are $\sigma_{xx} = \sigma$ and $\sigma_{xy} = \tau$ and the Prandtl-Reuss equations reduce to

$$
\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx} \\
d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx} \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy}
\end{aligned}
$$

The axial strain is increased from zero until yielding occurs (with $\varepsilon_{xy} = 0$). From first yield, the axial strain is held constant and the shear strain is increased up to its final value of $(1+\nu)Y/\sqrt{3}E$

(i) Write down the yield criterion in terms of σ and τ only and sketch the yield locus in $\sigma - \tau$ space
(ii) Evaluate the stresses and strains at first yield
(iii) Evaluate $d\lambda$ in terms of $\sigma, d\sigma$
(iv) Relate $\sigma, d\sigma$ to $\tau, d\tau$ and hence derive a differential equation for shear strain in terms of τ only
(v) Solve the differential equation and evaluate any constant of integration
(vi) Evaluate the shear stress when ε_{xy} reaches its final value of $(1+\nu)Y/\sqrt{3}E$. Taking $\nu = 1/2$, put in the form $\tau = \alpha Y$ with α to 3 d.p.

8.5 The Internally Pressurised Cylinder

8.5.1 Elastic Solution

Consider the problem of a long thick hollow cylinder, with internal and external radii *a* and *b*, subjected to an internal pressure *p*. This can be regarded as a plane problem, with stress and strain independent of the axial direction *z*. The solution to the axisymmetric elastic problem is (see §4.3.5)

$$\begin{aligned}
\sigma_{rr} &= -p\frac{b^2/r^2-1}{b^2/a^2-1} \\
\sigma_{\theta\theta} &= +p\frac{b^2/r^2+1}{b^2/a^2-1} \\
\sigma_{zz} &= +p\frac{1}{b^2/a^2-1}\times\begin{cases} 2\nu, & \text{plane strain} \\ 0, & \text{open end} \\ 1, & \text{closed end}\end{cases}
\end{aligned} \tag{8.5.1}$$

There are no shear stresses and these are the principal stresses. The strains are (with constant axial strain $\bar{\varepsilon}_{zz}$),

$$\begin{aligned}
\bar{\varepsilon}_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] \quad \to \quad \sigma_{zz} = E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right) \\
\varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr}-\nu\left(\sigma_{\theta\theta}+\sigma_{zz}\right)\right] = -\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[\left(1-\nu\right)\sigma_{rr}-\nu\sigma_{\theta\theta}\right] \\
\varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta}-\nu\left(\sigma_{rr}+\sigma_{zz}\right)\right] = -\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+\left(1-\nu\right)\sigma_{\theta\theta}\right]
\end{aligned} \tag{8.5.2}$$

with

$$\bar{\varepsilon}_{zz} = \frac{p}{E}\frac{1}{b^2/a^2-1}\times\begin{cases} 0, & \text{plane strain} \\ -2\nu, & \text{open end} \\ 1-2\nu, & \text{closed end}\end{cases} \tag{8.5.3}$$

Axial Force

The axial force in the elastic tube can be calculated through

$$\begin{aligned}
P &= \int_0^{2\pi}\int_a^b \sigma_{zz} r dr d\theta = \int_0^{2\pi}\int_a^b \left[E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] r dr d\theta \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+\nu\int_0^{2\pi}\int_a^b \left(\sigma_{rr}+\sigma_{\theta\theta}\right) r dr d\theta \qquad , \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+2\nu\pi p a^2
\end{aligned} \tag{8.5.4}$$

which is the result expected:

$$P = \pi a^2 p \times \begin{cases} 2\nu, & \text{plane strain} \\ 0, & \text{open end} \\ 1, & \text{closed end} \end{cases} \tag{8.5.5}$$

i.e., consistent with the result obtained from a simple consideration of the axial stress in 8.5.1c.

8.5.2 Plastic Solution

The pressure is now increased so that the cylinder begins to deform plastically. It will be assumed that the material is isotropic and elastic perfectly-plastic and that it satisfies the Tresca criterion.

First Yield

It can be seen from 8.5.1 that $\sigma_{\theta\theta} > \sigma_{zz} \geq 0 > \sigma_{rr}$ (σ_{rr} does equal zero on the outer surface, $r = b$, but that is not relevant here, as plastic flow will begin on the inner wall), so the intermediate stress is σ_{zz}, and so the Tresca criterion reads

$$\left|\sigma_{rr} - \sigma_{\theta\theta}\right| = \sigma_{\theta\theta} - \sigma_{rr} = 2p\frac{b^2/r^2}{b^2/a^2 - 1} \equiv 2k \tag{8.5.6}$$

This expression has its maximum value at the inner surface, $r = a$, and hence it is here that plastic flow first begins. From this, plastic deformation begins when

$$p_{flow} = k\left(1 - \frac{a^2}{b^2}\right), \tag{8.5.7}$$

irrespective of the end conditions.

Confined Plastic Flow and Collapse

As the pressure increases above p_{flow}, the plastic region spreads out from the inner face; suppose that it reaches out to $r = c$. With the material perfectly plastic, the material in the annulus $a < r < c$ satisfies the yield condition 8.5.6 at all times. Consider now the equilibrium of this *plastic* material. Since this is an axi-symmetric problem, there is only one equilibrium equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0\,. \tag{8.5.8}$$

It follows that

$$\frac{d\sigma_{rr}}{dr} - \frac{2k}{r} = 0 \quad \to \quad \sigma_{rr} = 2k\ln r + C_1 . \qquad (8.5.9)$$

The constant of integration can be obtained from the pressure boundary condition at $r = a$, leading to

$$\sigma_{rr} = -p + 2k\ln(r/a) \qquad (a \le r \le c) \quad \textbf{Plastic} \qquad (8.5.10)$$

The stresses in the elastic region are again given by the elastic stress solution 8.5.1, only with *a* replaced by *c* and the pressure *p* is now replaced by the pressure exerted by the plastic region at $r = c$, i.e. $p - 2k\ln(c/a)$.

The precise location of the boundary *c* can be obtained by noting that the elastic stresses must satisfy the yield criterion at $r = c$. Since in the elastic region,

$$\sigma_{\theta\theta} - \sigma_{rr} = 2(p - 2k\ln(c/a))\frac{b^2/r^2}{b^2/c^2 - 1} \qquad (c \le r \le b) \quad \textbf{Elastic} \qquad (8.5.11)$$

one has from $(\sigma_{\theta\theta} - \sigma_{rr})_{r=c} = 2k$ that

$$(1 - c^2/b^2) + 2\ln(c/a) = \frac{p}{k} \qquad (8.5.12)$$

Fig. 8.5.1 shows a plot of Eqn. 8.5.12.

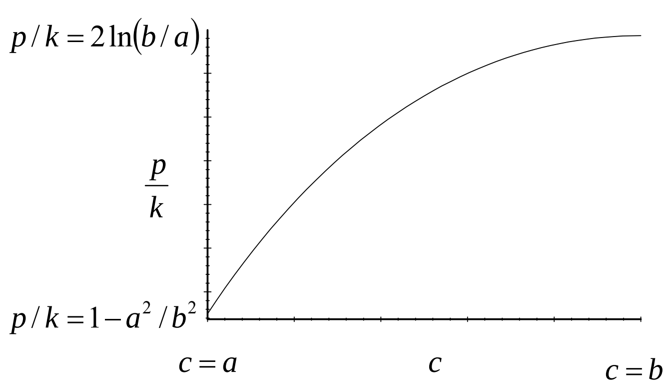


Figure 8.5.1: Extent of the plastic region $r = c$ during confined plastic flow

The complete cylinder will become plastic when *c* reaches *b*, or when the pressure reaches the **collapse pressure** (or **ultimate pressure**)

$$p_U = 2k\ln(b/a) . \qquad (8.5.13)$$

This problem illustrates a number of features of elastic-plastic problems in general. First, **confined plastic flow** occurs. This is where the plastic region is surrounded by an elastic region, and so the plastic strains are of the same order as the elastic strains. It is only when the pressure reaches the collapse pressure does catastrophic failure occur.

Stress Field

As discussed after Eqn. 8.5.10, the stresses in the elastic region are derived from Eqns. 8.5.1 with the pressure given by $p - 2k\ln(c/a)$ and a replaced with c:

$$\begin{aligned}
\sigma_{rr} &= -k\frac{c^2}{b^2}\left(\frac{b^2}{r^2}-1\right) \\
\sigma_{\theta\theta} &= +k\frac{c^2}{b^2}\left(\frac{b^2}{r^2}+1\right), \qquad c \le r \le b \quad \textbf{Elastic} \\
\sigma_{zz} &= +2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz}
\end{aligned} \tag{8.5.14}$$

For the plastic region, the radial and hoop stresses can be obtained from 8.5.10 and 8.5.6. The Tresca flow rule, 8.4.33, implies that $d\varepsilon_{zz}^p = 0$ and so the axial strain $\bar{\varepsilon}_{zz}$ is purely elastic. Thus the elastic Hooke's Law relation $\bar{\varepsilon}_{zz} = \left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right]/E$ holds also in the plastic region, and

$$\begin{aligned}
\sigma_{rr} &= -k\left(1-\frac{c^2}{b^2}+2\ln\frac{c}{r}\right) \\
\sigma_{\theta\theta} &= +k\left(1+\frac{c^2}{b^2}-2\ln\frac{c}{r}\right) \qquad , \qquad a \le r \le c \quad \textbf{Plastic} \\
\sigma_{zz} &= +2k\nu\left(\frac{c^2}{b^2}-2\ln\frac{c}{r}\right) + E\bar{\varepsilon}_{zz}
\end{aligned} \tag{8.5.15}$$

Note that, as in the fully elastic solution, the radial and tangential stresses are independent of the end conditions, i.e. they are independent of $\bar{\varepsilon}_{zz}$. Unlike the elastic case, the axial stress is not constant in the plastic zone, and in fact it is tensile and compressive in different regions of the cylinder, depending on the end-conditions.

Axial Force and Strain

The total axial force can be obtained by integrating the axial stresses over the elastic zone and the plastic zone:

$$\begin{aligned}
P &= \int_0^{2\pi}\int_a^c \sigma_{zz}^p r\,dr\,d\theta + \int_0^{2\pi}\int_c^b \sigma_{zz}^e r\,dr\,d\theta \\
&= 2\pi\left\{-4k\nu\ln\frac{c}{r}\int_a^c \sigma_{zz}^p r\,dr + \left(2k\nu\frac{c^2}{b^2}+E\bar{\varepsilon}_{zz}\right)\int_a^c \sigma_{zz}^p r\,dr + \left(2k\nu\frac{c^2}{b^2}+E\bar{\varepsilon}_{zz}\right)\int_c^b r\,dr\right\} \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right) + 2k\pi\nu a^2\left[\left(1-c^2/b^2\right)+\ln(c/a)\right] \\
&= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right) + 2\nu\pi p a^2
\end{aligned} \tag{8.5.16}$$

the last line coming from Eqn. 8.5.12.

A neat alternative way of deriving this result for this problem is as follows: first, the equation of equilibrium 8.5.8, which of course applies in both elastic and plastic zones, can be used to write

$$(\sigma_{rr}+\sigma_{\theta\theta})r = r(\sigma_{\theta\theta}-\sigma_{rr})+2r\sigma_{rr} = r^2\frac{d\sigma_{rr}}{dr}+2r\sigma_{rr} = \frac{d}{dr}(r^2\sigma_{rr}) \tag{8.5.17}$$

Using the same method as used in Eqn. 8.5.4, but this time using 8.5.17 and the boundary conditions on the radial stresses at the inner and outer walls:

$$\begin{aligned} P &= \int_0^{2\pi}\int_a^b \left[E\bar{\varepsilon}_{zz}+\nu(\sigma_{rr}+\sigma_{\theta\theta})\right] r\,dr\,d\theta \\ &= E\bar{\varepsilon}_{zz}\pi(b^2-a^2)+\nu 2\pi\left[r^2\sigma_{rr}\right]_a^b \\ &= E\bar{\varepsilon}_{zz}\pi(b^2-a^2)+2\nu\pi pa^2 \end{aligned} \tag{8.5.18}$$

which is the same as 8.5.16.

This axial force is the same as Eqn. 8.5.4. In other words, although σ_{zz} in general varies in the plastic zone, the axial force is independent of the plastic zone size *c*.

For the open cylinder, $P=0$, for plane strain, $\bar{\varepsilon}_{zz}=0$, and for the closed cylinder, $P=pa^2$. Unsurprisingly, since 8.5.16 (or 8.5.18) is the same as the elastic version, 8.5.4, one sees that the axial strains are as for the elastic solution, Eqn. 8.5.3. In terms of *c* and *k*, and using Eqn. 8.5.12, the axial strain is

$$\bar{\varepsilon}_{zz} = \frac{k}{E}\frac{1}{b^2/a^2-1}\left[1-\frac{c^2}{b^2}+2\ln(c/a)\right]\times\begin{cases} 0, & \text{plane strain} \\ -2\nu, & \text{open end} \\ 1-2\nu, & \text{closed end}\end{cases} \tag{8.5.19}$$

This can now be substituted into Eqns. 8.5.14 and 8.5.15 to get explicit expressions for the axial stresses. As an example, the stresses are plotted in Fig. 8.5.2 for the case of $b/a=2$, and $c/b=0.6, 0.8$.

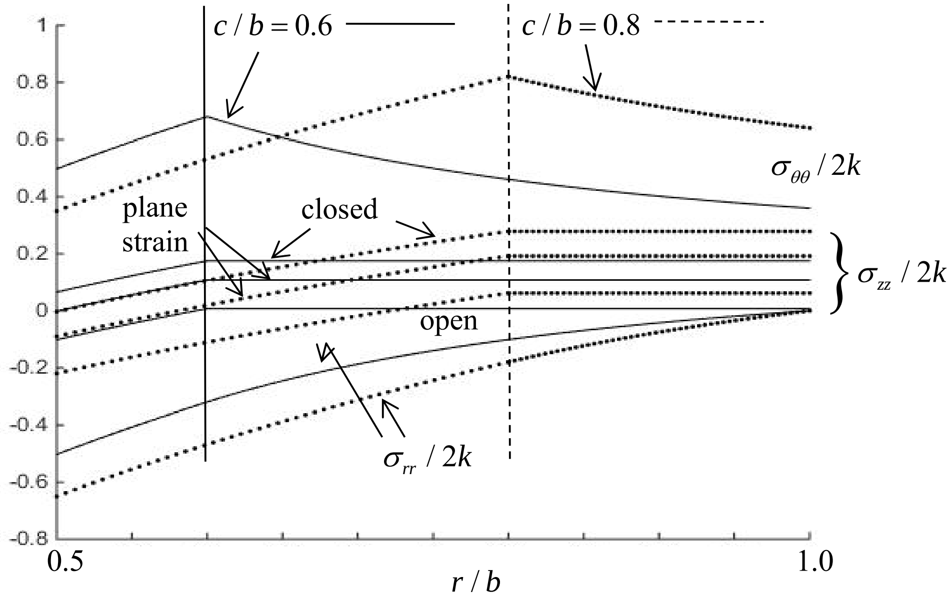


Figure 8.5.2: Stress field in the cylinder for the case of $b/a = 2$ and $c/b = 0.6, 0.8$

Strains and Displacement

In the elastic region, the strains are given by Hooke's law

$$\begin{aligned} \varepsilon_{rr} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right] - \nu\bar{\varepsilon}_{zz} \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right] - \nu\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.20}$$

From Eqns. 8.5.14,

$$\left.\begin{aligned} \varepsilon_{rr} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu) - \frac{b^2}{r^2}\right] - \nu\bar{\varepsilon}_{zz} \\ \varepsilon_{\theta\theta} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu) + \frac{b^2}{r^2}\right] - \nu\bar{\varepsilon}_{zz} \end{aligned}\right., \quad c \le r \le b \quad \textbf{Elastic} \tag{8.5.21}$$

From the definition of strain

$$\begin{aligned} \varepsilon_{rr} &= \frac{du_r}{dr} \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r} \end{aligned} \tag{8.5.22}$$

Substituting Eqn. 8.5.21a into Eqn. 8.5.22a (or Eqn. 8.5.21b into Eqn. 8.5.22b) gives the displacement in the elastic region:

$$u_r = k\frac{1+\nu}{E}\frac{c^2}{b^2}\left((1-2\nu)r + \frac{b^2}{r}\right) - \nu r\bar{\varepsilon}_{zz}, \quad c \le r \le b \quad \textbf{Elastic} \tag{8.5.23}$$

Now for a Tresca material, from 8.4.33, the increments in plastic strain are $d\varepsilon_{zz}^p = 0$ and $d\varepsilon_{rr}^p = -d\varepsilon_{\theta\theta}^p$, so, from the elastic Hooke's Law,

$$\begin{aligned}\varepsilon_{rr} + \varepsilon_{\theta\theta} &= \varepsilon_{rr}^e + \varepsilon_{\theta\theta}^e \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\sigma_{zz}\right] \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\left(E\bar{\varepsilon}_{zz} + \nu\right)(\sigma_{rr} + \sigma_{\theta\theta})\right] \\ &= \frac{(1+\nu)(1-2\nu)}{E}(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\bar{\varepsilon}_{zz}\end{aligned} \tag{8.5.24}$$

Using 8.5.22 and 8.5.17, one has

$$\frac{d}{dr}(ru_r) = \frac{(1+\nu)(1-2\nu)}{E}\frac{d}{dr}\left(r^2\sigma_{rr}\right) - 2\nu r\bar{\varepsilon}_{zz}, \tag{8.5.25}$$

which integrates to

$$u_r = \frac{(1+\nu)(1-2\nu)}{E}r\sigma_{rr} - \nu r\bar{\varepsilon}_{zz} + \frac{C}{r} \tag{8.5.26}$$

Equations 8.5.24-26 are valid in both the elastic and plastic regions. The constant of integration can be obtained from the condition $\sigma_{rr} = 0$ at $r = b$, where u_r equals the elastic displacement 8.5.23, and so $C = 2k(1-\nu^2)c^2 / E$ and

$$u_r = \frac{(1+\nu)(1-2\nu)}{E}r\sigma_{rr} + 2k\frac{(1-\nu^2)c^2}{Er} - \nu r\bar{\varepsilon}_{zz}, \quad a \le r \le c \quad \textbf{Plastic} \tag{8.5.27}$$

8.5.3 Unloading

Residual Stress

Suppose that the cylinder is loaded beyond p_{flow} but not up to the collapse pressure, to a pressure p_0 say. It is then unloaded completely. After unloading the cylinder is still subjected to a stress field – these stresses which are locked into the cylinder are called **residual stresses**. If the unloading process is fully elastic, the new stresses are obtained by subtracting 8.5.1 from 8.5.14-15. Using Eqn. 8.5.7, 8.5.12 {▲Problem 1},

$$\sigma_{rr} = -k\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2} - \frac{a^2}{b^2}\right)$$

$$\sigma_{\theta\theta} = +k\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2} + \frac{a^2}{b^2}\right), \qquad c \le r \le b \qquad \textbf{Elastic} \qquad (8.5.28)$$

$$\sigma_{zz} = +2k\nu\left(\frac{c^2}{a^2} - \frac{p_0}{p_{flow}}\right)\frac{a^2}{b^2}$$

$$\sigma_{rr} = -k\left[\frac{p_0}{p_{flow}}\left(1 - \frac{a^2}{r^2}\right) - 2\ln\frac{r}{a}\right]$$

$$\sigma_{\theta\theta} = -k\left[\frac{p_0}{p_{flow}}\left(1 + \frac{a^2}{r^2}\right) - 2 - 2\ln\frac{r}{a}\right], \qquad a \le r \le c \qquad \textbf{Plastic} \qquad (8.5.29)$$

$$\sigma_{zz} = -2k\nu\left[\frac{p_0}{p_{flow}} - 1 - 2\ln\frac{r}{a}\right]$$

Consider as an example the case $a = 1$, $b = 2$, with $c = 1.5$, for which

$$\frac{p_{flow}}{2k} = 0.375, \qquad \frac{p_0}{2k} = 0.624 \qquad (8.5.30)$$

The residual stresses are as shown in Fig. 8.5.3.

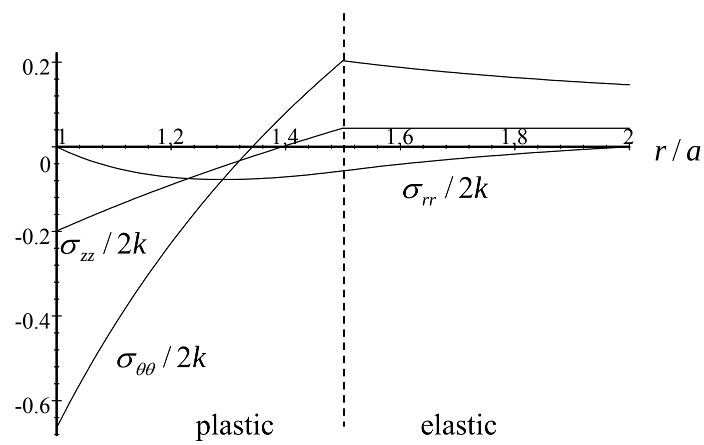


Figure 8.5.3: Residual stresses in the unloaded cylinder for the case of $a = 1, b = 2, c = 1.5$

Note that the axial strain, being purely elastic, is completely removed, and the axial stresses, as given by Eqns. 8.5.28c, 8.5.29c, are independent of the end condition.

There is the possibility that if the original pressure p_0 is very large, the unloading will lead to compressive yield. The maximum value of $|\sigma_{\theta\theta} - \sigma_{rr}|$ occurs at $r = a$ {▲Problem 2}:

$$\left|\sigma_{\theta\theta} - \sigma_{rr}\right|_{r=a} = 2k\left(\frac{p_0}{p_{flow}} - 1\right) \tag{8.5.31}$$

and so, neglecting any Bauschinger effect, yield will occur if $p_0 \geq 2p_{flow}$. Yielding in compression will not occur right up to the collapse pressure p_U if the wall ratio b/a is such that $p_0|_{c=b} = p_U < 2p_{flow}$, or when {▲Problem 3}

$$\ln\frac{b}{a} < 1 - \frac{a^2}{b^2} \tag{8.5.32}$$

The largest wall ratio for which the unloading is completely elastic is $b/a \approx 2.22$. For larger wall ratios, a new plastic zone will develop at the inner wall, with $\sigma_{\theta\theta} - \sigma_{rr} = -2k$.

Shakedown

When the cylinder is initially loaded, plasticity begins at a pressure $p = p_{flow}$. If it is loaded to some pressure p_0, with $p_{flow} < p_0 < 2p_{flow}$, then unloading will be completely elastic. When the cylinder is reloaded again it will remain elastic up to pressure p_0. In this way, it is possible to strengthen the cylinder by an initial loading; theoretically it is possible to increase the flow pressure by a factor of 2. This maximum possible new flow pressure is called the **shakedown pressure** $p_s = \min(2p_{flow}, p_U)$. **Shakedown** is said to have occurred when any subsequent loading/unloading cycles are purely elastic. The strengthening of the cylinder is due to the compressive residual hoop stresses at the inner wall – similar to the way a barrel can be strengthened with hoops. This method of strengthening is termed **autofrettage**, a French term meaning "self-hooping".

8.5.4 Validity of the Solution

One needs to check whether the assumption of the ordering of the principal stresses, $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$, holds through the deformation in the plastic region. It can be confirmed that the inequality $\sigma_{zz} \geq \sigma_{rr}$ always holds. For the inequality $\sigma_{\theta\theta} \geq \sigma_{zz}$, consider the inequality $\sigma_{\theta\theta} - \sigma_{zz} \geq 0$. The quantity on the left is a minimum when $r = a$, where it equals {▲Problem 4}

$$k\left[(1-2\nu)\left(1+\frac{c^2}{b^2} - 2\ln\frac{c}{a}\right) + 2\nu - \frac{1}{b^2/a^2 - 1}\left(1 - \frac{c^2}{b^2} + 2\ln\frac{c}{a}\right)\begin{bmatrix} 0 \\ -2\nu \\ 1-2\nu \end{bmatrix}\right] \tag{8.5.33}$$

This quantity must be positive for all values of c up to the maximum value b, where it takes its minimum value, and so one must have

$$2(1-2\nu)\left(1-\ln\frac{b}{a}\right)+2\nu-\frac{1}{b^2/a^2-1}\left(2\ln\frac{b}{a}\right)\begin{bmatrix}0\\-2\nu\\1-2\nu\end{bmatrix}\geq 0 \qquad (8.5.34)$$

The solution is thus valid only for limited values of b/a. For $\nu = 0.3$, one must have $b/a < 5.42$ (closed ends), $b/a < 5.76$ (plane strain), $b/a < 6.19$ (open ends). For higher wall ratios, the axial stress becomes equal to the hoop stress. In this case, a solution based on large changes in geometry is necessary for higher pressures.

8.5.5 Problems

1. Derive Eqns. 8.5.28-29.
2. Use Eqns. 8.5.29a,b to show that the maximum value of $|\sigma_{\theta\theta}-\sigma_{rr}|$ occurs at $r = a$, where it equals $2k(p_0 / p_{flow} - 1)$, Eqn. 8.5.31.
3. Derive Eqn. 8.5.32.
4. Use Eqns. 8.5.15, 8.5.12 to derive Eqn. 8.5.33.

Hardening

In the applications discussed in the preceding two sections, the material was assumed to be perfectly plastic. The issue of hardening (softening) materials is addressed in this section.

8.6.1 Hardening

In the one-dimensional (uniaxial test) case, a specimen will deform up to yield and then generally harden, Fig. 8.6.1. Also shown in the figure is the perfectly-plastic idealisation. In the perfectly plastic case, once the stress reaches the yield point (A), plastic deformation ensues, so long as the stress is maintained at Y. If the stress is reduced, elastic unloading occurs. In the hardening case, once yield occurs, the stress needs to be continually increased in order to drive the plastic deformation. If the stress is held constant, for example at B, no further plastic deformation will occur; at the same time, no elastic unloading will occur. Note that this condition cannot occur in the perfectly-plastic case, where there is one of plastic deformation or elastic unloading.

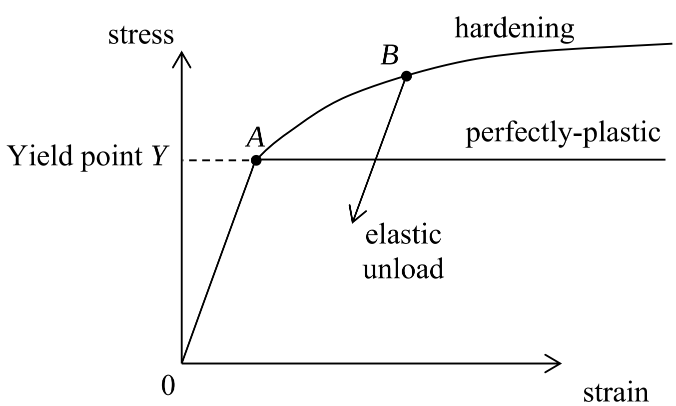


Figure 8.6.1: uniaxial stress-strain curve (for a typical metal)

These ideas can be extended to the multiaxial case, where the initial yield surface will be of the form

$$f_0(\sigma_{ij}) = 0 \tag{8.6.1}$$

In the perfectly plastic case, the yield surface remains unchanged.. In the more general case, the yield surface may change size, shape and position, and can be described by

$$f(\sigma_{ij}, \mathrm{K}_i) = 0 \tag{8.6.2}$$

Here, K_i represents one or more **hardening parameters**, which change during plastic deformation and determine the evolution of the yield surface. They may be scalars or higher-order tensors. At first yield, the hardening parameters are zero, and $f(\sigma_{ij}, 0) = f_0(\sigma_{ij})$.

The description of how the yield surface changes with plastic deformation, Eqn. 8.6.2, is called the **hardening rule**.

Strain Softening

Materials can also **strain soften**, for example soils. In this case, the stress-strain curve "turns down", as in Fig. 8.6.2. The yield surface for such a material will in general decrease in size with further straining.

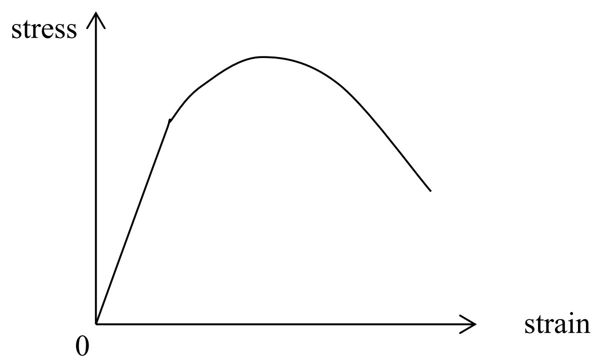


Figure 8.6.2: uniaxial stress-strain curve for a strain-softening material

8.6.2 Hardening Rules

A number of different hardening rules are discussed in this section.

Isotropic Hardening

Isotropic hardening is where the yield surface remains the same shape but expands with increasing stress, Fig. 8.6.3.

In particular, the yield function takes the form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij}) - \mathrm{K} = 0 \tag{8.6.3}$$

The shape of the yield function is specified by the initial yield function and its size changes as the hardening parameter K changes.

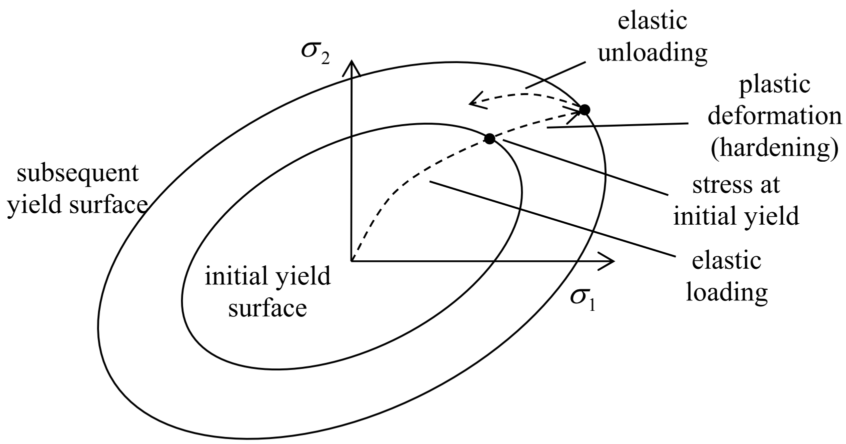


Figure 8.6.3: isotropic hardening

For example, consider the Von Mises yield surface. At initial yield, one has

$$\begin{aligned} f_0(\sigma_{ij}) &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}-Y \\ &= \sqrt{3J_2}-Y \\ &= \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}-Y \end{aligned} \tag{8.6.4}$$

where Y is the yield stress in uniaxial tension. Subsequently, one has

$$f(\sigma_{ij},\mathrm{K}_i) = \sqrt{3J_2}-Y-\mathrm{K}=0 \tag{8.6.5}$$

The initial cylindrical yield surface in stress-space with radius $\sqrt{\tfrac{2}{3}}Y$ (see Fig. 8.3.11) develops with radius $\sqrt{\tfrac{2}{3}}(Y+\mathrm{K})$. The details of how the hardening parameter K actually changes with plastic deformation have not yet been specified.

As another example, consider the Drucker-Prager criterion, Eqn. 8.3.30, $f_0(\sigma_{ij}) = \alpha I_1+\sqrt{J_2}-k=0$. In uniaxial tension, $I_1 = Y$, $\sqrt{J_2}=Y/\sqrt{3}$, so $k=(\alpha+1/\sqrt{3})Y$. Isotropic hardening can then be expressed as

$$f(\sigma_{ij},\mathrm{K}_i) = \frac{1}{\alpha+1/\sqrt{3}}\left(\alpha I_1+\sqrt{J_2}\right)-Y-\mathrm{K}=0 \tag{8.6.6}$$

Kinematic Hardening

The isotropic model implies that, if the yield strength in tension and compression are initially the same, i.e. the yield surface is symmetric about the stress axes, they remain equal as the yield surface develops with plastic strain. In order to model the Bauschinger effect, and similar responses, where a hardening in tension will lead to a softening in a subsequent compression, one can use the **kinematic hardening** rule. This is where the yield surface remains the same shape and size but merely translates in stress space, Fig. 8.6.4.

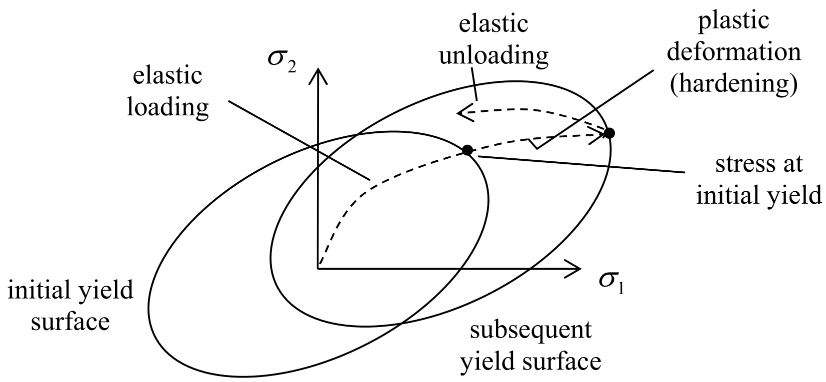


Figure 8.6.4: kinematic hardening

The yield function now takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) = 0 \qquad (8.6.7)$$

The hardening parameter here is the stress α_{ij}, known as the **back-stress** or **shift-stress**; the yield surface is shifted relative to the stress-space axes by α_{ij}, Fig. 8.6.5.

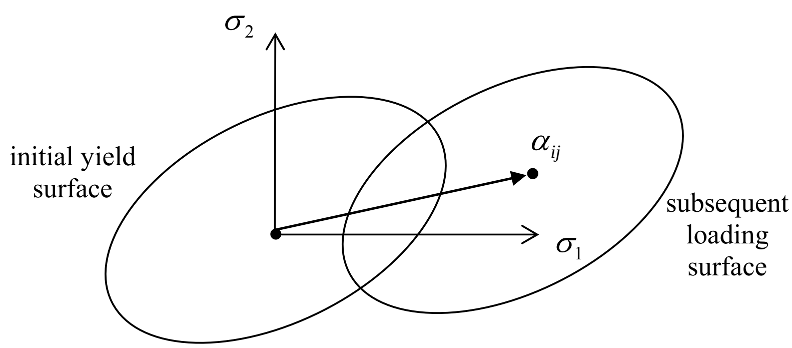


Figure 8.6.5: kinematic hardening; a shift by the back-stress

For example, again considering the Von Mises material, one has, from 8.6.4, and using the deviatoric part of $\boldsymbol{\sigma} - \boldsymbol{\alpha}$ rather than the deviatoric part of $\boldsymbol{\sigma}$,

$$f(\sigma_{ij}, \mathrm{K}_i) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0 \qquad (8.6.8)$$

where $\boldsymbol{\alpha}^d$ is the deviatoric part of $\boldsymbol{\alpha}$. Again, the details of how the hardening parameter α_{ij} might change with deformation will be discussed later.

Other Hardening Rules

More complex hardening rules can be used. For example, the **mixed hardening** rule combines features of both the isotropic and kinematic hardening models, and the loading function takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) - \mathrm{K} = 0 \qquad (8.6.9)$$

The hardening parameters are now the scalar K and the tensor α_{ij}.

8.6.3 The Flow Curve

In order to model plastic deformation and hardening in a complex three-dimensional geometry, one will generally have to use but the data from a simple test. For example, in the uniaxial tension test, one will have the data shown in Fig. 8.6.6a, with stress plotted against plastic strain. The idea now is to define a scalar **effective stress** $\hat{\sigma}$ and a scalar **effective plastic strain** $\hat{\varepsilon}^p$, functions respectively of the stresses and plastic strains in the loaded body. The following hypothesis is then introduced: a plot of effective stress against effective plastic strain follows the same **universal plastic stress-strain curve** as in the uniaxial case. This assumed universal curve is known as the **flow curve**.

The question now is: how should one define the effective stress and the effective plastic strain?

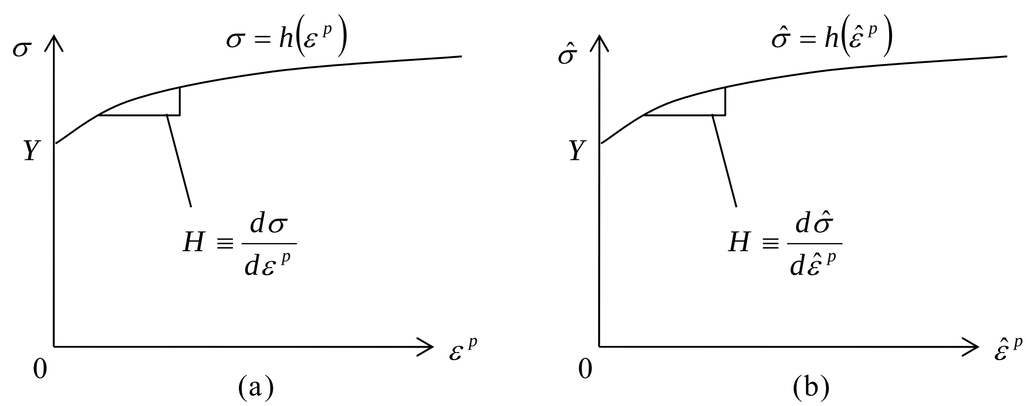


Figure 8.6.6: the flow curve; (a) uniaxial stress – plastic strain curve, (b) effective stress – effective plastic strain curve

8.6.4 A Von Mises Material with Isotropic Hardening

Consider a Von Mises material. Here, it is appropriate to define the effective stress to be

$$\hat{\sigma}(\sigma_{ij}) = \sqrt{3J_2} \qquad (8.6.10)$$

This has the essential property that, in the uniaxial case, $\hat{\sigma}(\sigma_{ij}) = Y$. (In the same way, for example, the effective stress for the Drucker-Prager material, Eqn. 8.6.6, would be $\hat{\sigma}(\sigma_{ij}) = \left(\alpha I_1 + \sqrt{J_2}\right)/(\alpha + 1/\sqrt{3})$.)

For the effective plastic strain, one possibility is to define it in the following rather intuitive, non-rigorous, way. The deviatoric stress **s** and plastic strain (increment) tensor $d\boldsymbol{\varepsilon}^p$ are of a similar character. In particular, their traces are zero, albeit for different physical reasons; $J_1 = 0$ because of independence of hydrostatic pressure, $d\varepsilon_{ii}^p = 0$ because of material incompressibility in the plastic range. For this reason, one chooses the effective plastic strain (increment) $d\hat{\varepsilon}^p$ to be a similar function of $d\varepsilon_{ij}^p$ as $\hat{\sigma}$ is of the s_{ij}. Thus, in lieu of $\hat{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$, one chooses $d\hat{\varepsilon}^p = C\sqrt{d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$. One can determine the constant *C* by ensuring that the expression reduces to $d\hat{\varepsilon}^p = d\varepsilon_1^p$ in the uniaxial case. Considering this uniaxial case, $d\varepsilon_{11}^p = d\varepsilon_1^p$, $d\varepsilon_{22}^p = d\varepsilon_{33}^p = -\frac{1}{2} d\varepsilon_1^p$, one finds that

$$\begin{aligned} d\hat{\varepsilon}^p &= \sqrt{\tfrac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p} \\ &= \frac{\sqrt{2}}{3}\sqrt{\left(d\varepsilon_1^p - d\varepsilon_2^p\right)^2 + \left(d\varepsilon_2^p - d\varepsilon_3^p\right)^2 + \left(d\varepsilon_3^p - d\varepsilon_1^p\right)^2} \end{aligned} \tag{8.6.11}$$

Let the hardening in the uniaxial tension case be described using a relationship of the form (see Fig. 8.6.6)

$$\sigma = h\left(\varepsilon^p\right) \tag{8.6.12}$$

The slope of this flow curve is the plastic modulus, Eqn. 8.1.9,

$$H = \frac{d\sigma}{d\varepsilon^p} \tag{8.6.13}$$

The effective stress and effective plastic strain for any conditions are now assumed to be related through

$$\hat{\sigma} = h\left(\hat{\varepsilon}^p\right) \tag{8.6.14}$$

and the effective plastic modulus is given by

$$H = \frac{d\hat{\sigma}}{d\hat{\varepsilon}^p} \tag{8.6.15}$$

Isotropic Hardening

Assuming isotropic hardening, the yield surface is given by Eqn. 8.6.5, and with the definition of the effective stress, Eqn. 8.6.10,

$$f(\sigma_{ij}, \mathrm{K}_i) = \hat{\sigma} - Y - \mathrm{K} = 0 \tag{8.6.16}$$

Differentiating with respect to the effective plastic strain,

$$H = \frac{\partial \hat{\sigma}}{\partial \hat{\varepsilon}^p} = \frac{\partial \mathrm{K}}{\partial \hat{\varepsilon}^p} \tag{8.6.17}$$

One can now see how the hardening parameter evolves with deformation: K here is a function of the effective plastic strain, and its functional dependence on the effective plastic strain is given by the plastic modulus H of the universal flow curve.

Loading Histories

Each material particle undergoes a plastic strain history. One such path is shown in Fig. 8.6.7. At point q, the plastic strain is $\varepsilon_i^p(q)$. The effective plastic strain at q must be evaluated through an integration over the complete history of deformation:

$$\hat{\varepsilon}^p(q) = \int_0^q d\hat{\varepsilon}^p = \int_0^q \sqrt{\tfrac{2}{3} d\varepsilon_i^p d\varepsilon_i^p} \tag{8.6.18}$$

Note that the effective plastic strain at q is not simply $\sqrt{\tfrac{2}{3}\varepsilon_i^p(q)\varepsilon_i^p(q)}$, hence the definition of an effective plastic strain *increment* in Eqn. 8.6.11.

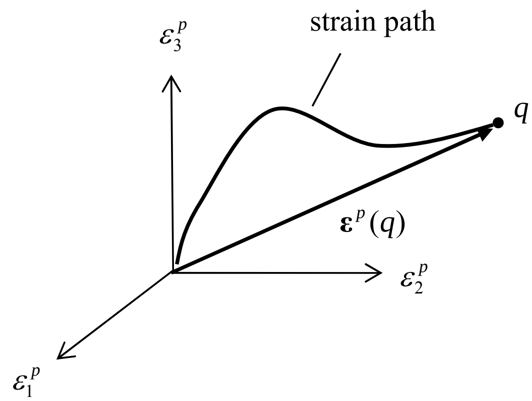


Figure 8.6.7: plastic strain space

Prandtl-Reuss Relations in terms of Effective Parameters

Using the Prandtl-Reuss (Levy-Mises) flow rule 8.4.1, and the definitions 8.6.10-11 for effective stress and effective plastic strain, one can now express the plastic multiplier as{▲Problem 1}

$$d\lambda = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} \tag{8.6.19}$$

and the plastic strain increments, Eqn. 8.4.6, now read

$$\begin{aligned}
d\varepsilon_{xx}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{xy} \\
d\varepsilon_{yz}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{yz} \\
d\varepsilon_{zx}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{zx}
\end{aligned} \tag{8.6.20}$$

or

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} s_{ij}\,. \tag{8.6.21}$$

Knowledge of the plastic modulus, Eqn. 8.6.15, now makes equations 8.6.21 complete.

Note here that the plastic modulus in the Prandtl-Reuss equations is conveniently expressible in a simple way in terms of the effective stress and plastic strain increment, Eqn. 8.6.19. It will be shown in the next section that this is no coincidence, and that the Prandtl-Reuss flow-rule is indeed naturally associated with the Von-Mises criterion.

8.6.5 Application: Combined Tension/Torsion of a thin walled tube with Isotropic Hardening

Consider again the thin-walled tube under combined tension and torsion. The Von Mises yield function in terms of the axial stress σ and the shear stress τ is, as in §8.3.1, $f_0(\sigma_{ij}) = \sqrt{\sigma^2 + 3\tau^2} - Y = 0$. This defines the ellipse of Fig. 8.3.2. Subsequent yield surfaces are defined by

$$\begin{aligned}
f(\sigma_{ij}, \mathrm{K}_i) &= \sqrt{\sigma^2 + 3\tau^2} - Y - K \\
&= f_0(\sigma_{ij}) - K \\
&= \hat{\sigma} - (Y + K) \\
&= 0
\end{aligned} \tag{8.6.22}$$

Whereas the initial yield surface is the ellipse with major and minor axes Y and $Y/\sqrt{3}$, subsequent yield ellipses have axes $Y + K$ and $(Y + K)/\sqrt{3}$, Fig. 8.6.8.

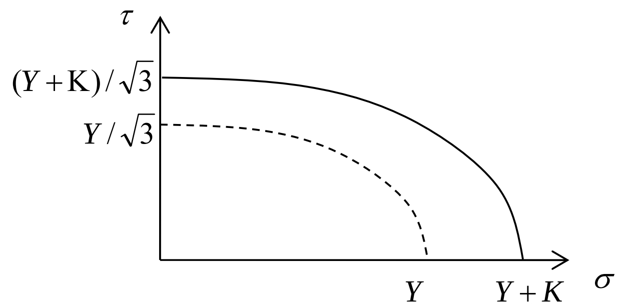


Figure 8.6.8: expansion of the yield locus (ellipse) for a thin-walled tube under isotropic hardening

The Prandtl-Reuss equations in terms of effective stress and effective plastic strain, 8.6.20-21, reduce to

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}d\sigma + \frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\sigma \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{\nu}{E}d\sigma - \frac{1}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\sigma \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}}\tau \end{aligned} \tag{8.6.23}$$

Consider the case where the material is brought to first yield through tension only, in which case the Von Mises condition reduces to $\sigma = Y$. Let the material then be subjected to a twist whilst maintaining the axial stress constant. The expansion of the yield surface is then as shown in Fig. 8.6.9.

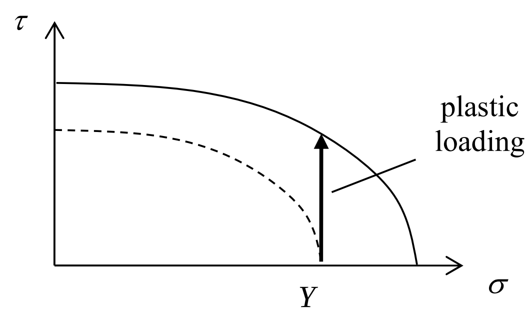


Figure 8.6.9: expansion of the yield locus for a thin-walled tube under constant axial loading

Introducing the plastic modulus, then, one has

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{1}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}\tau \end{aligned} \tag{8.6.24}$$

Using $\hat{\sigma} = \sqrt{Y^2 + 3\tau^2}$,

$$\begin{aligned} d\varepsilon_{xx} &= \frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{1}{2}\frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{\tau^2 d\tau}{\tau^2 + Y^2/3} \end{aligned} \tag{8.6.25}$$

These equations can now be integrated. If the material is **linear hardening**, so H is constant, then they can be integrated exactly using

$$\int \frac{x}{x^2 + a^2}dx = \frac{1}{2}\ln\left(x^2 + a^2\right), \qquad \int \frac{x^2}{x^2 + a^2}dx = x - a\arctan\left(\frac{x}{a}\right) \tag{8.6.26}$$

leading to {▲Problem 2}

$$\begin{aligned} \frac{E}{Y}\varepsilon_{xx} &= 1 + \frac{1}{2}\frac{E}{H}\ln\left(1 + 3\frac{\tau^2}{Y^2}\right) \\ \frac{E}{Y}\varepsilon_{yy} &= \frac{E}{Y_0}\varepsilon_{zz} = -\frac{E}{4H}\ln\left(1 + 3\frac{\tau^2}{Y^2}\right) \\ \frac{E}{Y}\varepsilon_{xy} &= (1+\nu)\left(\frac{\tau}{Y}\right) + \frac{3}{2}\frac{E}{H}\left[\frac{\tau}{Y} - \frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\frac{\tau}{Y}\right)\right] \end{aligned} \tag{8.6.27}$$

Results are presented in Fig. 8.6.10 for the case of $\nu = 0.3$, $E/H = 10$. The axial strain grows logarithmically and is eventually dominated by the faster-growing shear strain.

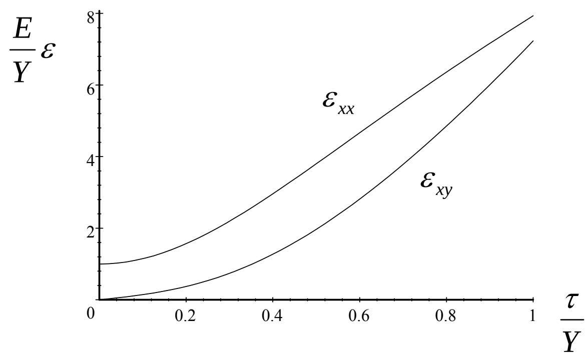


Figure 8.6.10: Stress-strain curves for thin-walled tube with isotropic linear strain hardening

8.6.6 Kinematic Hardening Rules

A typical uniaxial kinematic hardening curve is shown in Fig. 8.6.11a (see Fig. 8.1.3). During cyclic loading, the elastic zone always remains at $2Y$. Depending on the stress history, one can even have the situation shown in Fig. 8.6.11b, where yielding occurs upon unloading, even though the stress is still tensile.

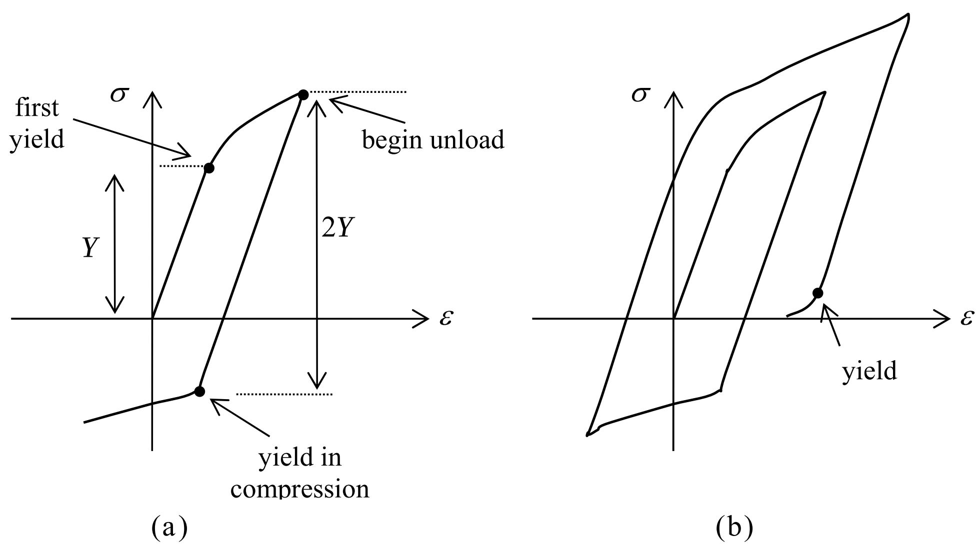


Figure 8.6.11: Kinematic Hardening; (a) load-unload, (b) cyclic loading

The multiaxial yield function for a kinematic hardening Von Mises is given by Eqn. 8.6.8,

$$f\left(\sigma_{ij}, \mathrm{K}_i\right) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0$$

The deviatoric shift stress α_{ij}^d describes the shift in the centre of the Von Mises cylinder, as viewed in the π-plane, Fig. 8.6.12. This is a generalisation of the uniaxial case, in that the radius of the Von Mises cylinder remains constant, just as the elastic zone in the uniaxial case remains constant (at $2Y$).

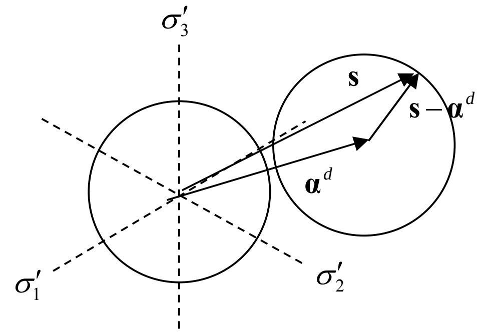


Figure 8.6.12: The Von Mises cylinder shifted in the π-plane

One needs to specify, by specifying the evolution of the hardening paremter $\boldsymbol{\alpha}$, how the yield surface shifts with deformation. In the multiaxial case, one has the added complication that the direction in which the yield surface shifts in stress space needs to be specified. The simplest model is the **linear kinematic** (or **Prager's**) **hardening rule**. Here, the back stress is assumed to depend on the plastic strain according to

$$\alpha_{ij} = c\varepsilon_{ij}^p \quad \text{or} \quad d\alpha_{ij} = cd\varepsilon_{ij}^p \tag{8.6.28}$$

where c is a material parameter, which might change with deformation. Thus the yield surface is translated in the same direction as the plastic strain increment. This is illustrated in Fig. 8.6.13, where the principal directions of stress and plastic strain are superimposed.

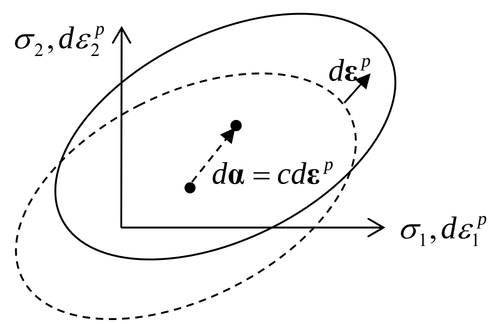


Figure 8.6.13: Linear kinematic hardening rule

One can use the uniaxial (possibly cyclic) curve to again define a universal plastic modulus H. Using the effective plastic strain, one can relate the constant c to H. This will be discussed in §8.8, where a more general formulation will be used.

Ziegler's hardening rule is

$$d\alpha_{ij} = da\left(\varepsilon_{ij}^p\right)\left(\sigma_{ij} - \alpha_{ij}\right) \tag{8.6.29}$$

where a is some scalar function of the plastic strain. Here, then, the loading function translates in the direction of $\sigma_{ij} - \alpha_{ij}$, Fig. 8.6.14.

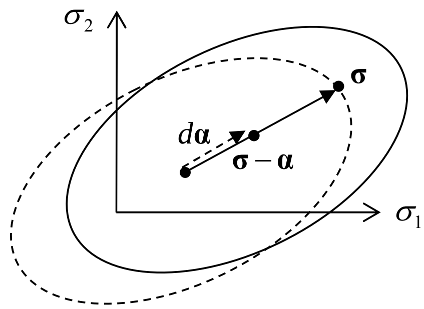


Figure 8.6.14: Ziegler's kinematic hardening rule

8.6.7 Strain Hardening and Work Hardening

In the models considered above, the hardening parameters have been functions of the plastic strains. For example, in the Von Mises isotropic hardening model, the hardening parameter K is a function of the effective plastic strain, $\hat{\varepsilon}^p$. Hardening expressed in this way is called **strain hardening**.

Another means of generalising the uniaxial results to multiaxial conditions is to use the **plastic work** (per unit volume), also known as the **plastic dissipation**,

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.30}$$

The total plastic work is the area under the stress – plastic strain curve of Fig. 8.6.6a,

$$W^p = \int \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.31}$$

A plot of stress against the plastic work can therefore easily be generated, as in Fig. 8.6.15.

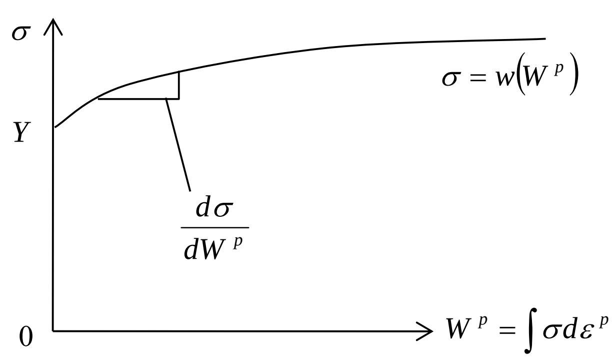


Figure 8.6.15: uniaxial stress – plastic work curve (for a typical metal)

The stress is now expressed in the form (compare with Eqn. 8.6.12)

$$\sigma = w(W^p) = w\left(\int \sigma d\varepsilon^p\right) \tag{8.6.32}$$

Again defining an effective stress $\hat{\sigma}$, the universal flow curve to be used for arbitrary loading conditions is then (compare with Eqn. 8.6.14)

$$\hat{\sigma} = w(W^p) \tag{8.6.33}$$

where now W^p is the plastic work during the multiaxial deformation. This is known as a **work hardening** formulation.

Equivalence of Strain and Work Hardening for the Isotropic Hardening Von Mises Material

Consider the Prandtl-Reuss flow rule, Eqn. 8.4.1, $d\varepsilon_i^p = s_i d\lambda$ (other flow rules will be examined more generally in §8.7). In this case, working with principal stresses, the plastic work increment is (see Eqns. 8.2.7-10)

$$\begin{aligned} dW^p &= \sigma_i d\varepsilon_i^p \\ &= \sigma_i s_i d\lambda \\ &= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] d\lambda \end{aligned} \tag{8.6.34}$$

Using the Von Mises effective stress 8.6.10, and Eqn. 8.6.19,

$$\begin{aligned} dW^p &= \tfrac{2}{3}\hat{\sigma}^2 d\lambda \\ &= \hat{\sigma}\, d\hat{\varepsilon}^p \end{aligned} \tag{8.6.35}$$

where $\hat{\varepsilon}^p$ is the very same effective plastic strain as used in the strain hardening isotropic model, Eqn. 8.6.11. Although true for the Von Mises yield condition, this will not be so in general.

8.6.8 Problems

1. Staring with the definition of the effective plastic strain, Eqn. 8.6.11, and using Eqn. 8.4.1, $d\varepsilon_i^p = d\lambda s_i$, derive Eqns. 8.6.19, $d\lambda = \dfrac{3}{2}\dfrac{d\hat{\varepsilon}^p}{\hat{\sigma}}$

2. Integrate Eqns. 8.6.25 and use the initial (first yield) conditions to get Eqns. 8.6.27.

3. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of an isotropic hardening Von Mises metal with uniaxial yield stress Y. The strain-hardening is linear with plastic modulus *H*. The tube is loaded, keeping the ratio $\sigma / \tau = \sqrt{3}$ at all times throughout the elasto-plastic deformation, until $\sigma = Y$.
   (i) Show that the stresses and strains at first yield are given by
   $$\sigma^Y = \frac{1}{\sqrt{2}} Y, \quad \tau^Y = \frac{1}{\sqrt{6}} Y, \quad \varepsilon_{xx}^Y = \frac{1}{\sqrt{2}} \frac{Y}{E}, \quad \varepsilon_{xy}^Y = \frac{1+\nu}{\sqrt{6}} \frac{Y}{E}$$
   (ii) The Prandtl-Reuss equations in terms of the effective stress and effective plastic strain are given by Eqns. 8.6.23. Eliminate τ from these equations (using $\sigma / \tau = \sqrt{3}$).
   (iii) Eliminate the effective plastioc strain using the plastic modulus.
   (iv) The effective stress is defined as $\hat{\sigma} = \sqrt{\sigma^2 + 3\tau^2}$ (see Eqn. 8.6.22). Eliminate the effective stress to obtain
   $$d\varepsilon_{xx} = \frac{1}{E} d\sigma + \frac{1}{H} d\sigma$$
   $$d\varepsilon_{xy} = \frac{1}{\sqrt{3}} \frac{1+\nu}{E} d\sigma + \frac{\sqrt{3}}{2} \frac{1}{H} d\sigma$$
   (v) Integrate the differential equations and evaluate any constants of integration
   (vi) Hence, show that the strains at the final stress values $\sigma = Y$, $\tau = Y / \sqrt{3}$ are given by
   $$\frac{E}{Y} \varepsilon_{xx} = 1 + \frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$
   $$\frac{E}{Y} \varepsilon_{xy} = \frac{1+\nu}{\sqrt{3}} + \frac{\sqrt{3}}{2} \frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$
   (vii) Sketch the initial yield (elliptical) locus and the final yield locus in (σ, τ) space and the loading path.
   (viii) Plot σ against ε_{xx}.

8.7 Associated and Non-associated Flow Rules

Recall the Levy-Mises flow rule, Eqn. 8.4.3,

$$d\varepsilon_{ij}^{p} = d\lambda s_{ij} \tag{8.7.1}$$

The plastic multiplier can be determined from the hardening rule. Given the hardening rule one can more generally, instead of the particular flow rule 8.7.1, write

$$d\varepsilon_{ij}^{p} = d\lambda G_{ij}, \tag{8.7.2}$$

where G_{ij} is some function of the stresses and perhaps other quantities, for example the hardening parameters. It is symmetric because the strains are symmetric.

A wide class of material behaviour (perhaps all that one would realistically be interested in) can be modelled using the general form

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}. \tag{8.7.3}$$

Here, g is a scalar function which, when differentiated with respect to the stresses, gives the plastic strains. It is called the **plastic potential**. The flow rule 8.7.3 is called a **non-associated flow rule**.

Consider now the sub-class of materials whose plastic potential *is* the yield function, $g = f$:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}. \tag{8.7.4}$$

This flow rule is called an **associated flow-rule**, because the flow rule is associated with a particular yield criterion.

8.7.1 Associated Flow Rules

The yield surface $f(\sigma_{ij}) = 0$ is displayed in Fig 8.7.1. The axes of principal stress and principal plastic strain are also shown; the material being isotropic, these are taken to be coincident. The normal to the yield surface is in the direction $\partial f / \sigma_{ij}$ and so the associated flow rule 8.7.4 can be interpreted as saying that *the plastic strain increment vector is normal to the yield surface*, as indicated in the figure. This is called the **normality rule**.

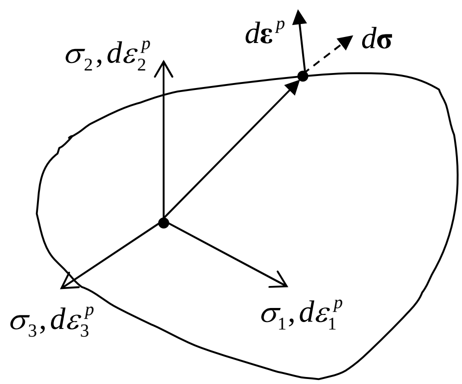


Figure 8.7.1: Yield surface

The normality rule has been confirmed by many experiments on metals. However, it is found to be seriously in error for soils and rocks, where, for example, it overestimates plastic volume expansion. For these materials, one must use a non-associative flow-rule.

Next, the Tresca and Von Mises yield criteria will be discussed. First note that, to make the differentiation easier, the associated flow-rule 8.7.4 can be expressed in terms of principal stresses as

$$d\varepsilon_i^p = d\lambda \frac{\partial f}{\partial \sigma_i}. \tag{8.7.5}$$

Tresca

Taking $\sigma_1 > \sigma_2 > \sigma_3$, the Tresca yield criterion is

$$f = \frac{\sigma_1 - \sigma_3}{2} - k \tag{8.7.6}$$

One has

$$\frac{\partial f}{\partial \sigma_1} = +\frac{1}{2}, \quad \frac{\partial f}{\partial \sigma_2} = 0, \quad \frac{\partial f}{\partial \sigma_3} = -\frac{1}{2} \tag{8.7.7}$$

so, from 8.7.5, the flow-rule associated with the Tresca criterion is

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} +\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}. \tag{8.7.8}$$

This is the flow-rule of Eqns. 8.4.33. The plastic strain increment is illustrated in Fig. 8.7.2 (see Fig. 8.3.9). All plastic deformation occurs in the $1-3$ plane. Note that 8.7.8 is independent of stress.

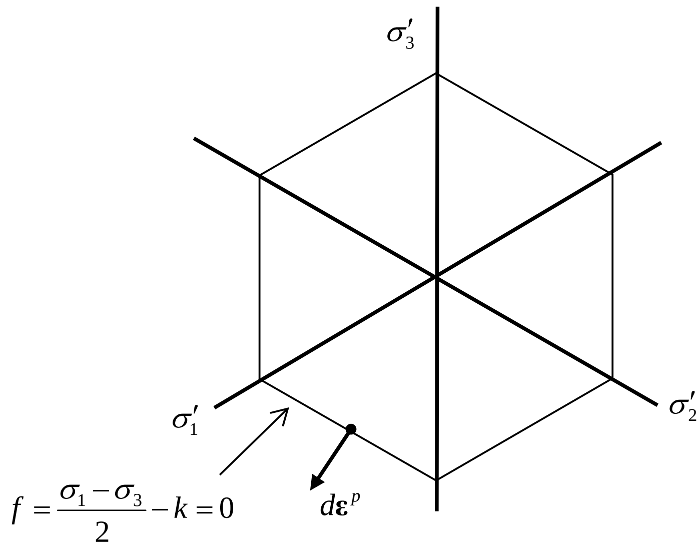


Figure 8.7.2: The plastic strain increment vector and the Tresca criterion in the π-plane (for the associated flow-rule)

Von Mises

The Von Mises yield criterion is $f = J_2 - k^2 = 0$. With

$$\frac{\partial J_2}{\partial \sigma_1} = \frac{\partial}{\partial \sigma_1}\frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] = \frac{2}{3}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] \quad (8.7.9)$$

one has

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} \frac{2}{3}\left(\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right) \\ \frac{2}{3}\left(\sigma_2 - \frac{1}{2}(\sigma_1+\sigma_3)\right) \\ \frac{2}{3}\left(\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)\right) \end{bmatrix}. \quad (8.7.10)$$

This is none other than the Levy-Mises flow rule 8.4.6[1].

The associative flow-rule is very appealing, connecting as it does the yield surface to the flow-rule. Many attempts have been made over the years to justify this rule, both

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[1] note that if one were to use the alternative but equivalent expression $f = \sqrt{J_2} - k = 0$, one would have a $1/2\sqrt{J_2}$ term common to all three principal strain increments, which could be "absorbed" into the $d\lambda$ giving the same flow-rule 8.7.10

mathematically and physically. However, it should be noted that the associative flow-rule is not a law of nature by any means. It is simply very convenient. That said, it does agree with experimental observations of many plastically deforming materials, particularly metals.

In order to put the notion of associative flow-rules on a sounder footing, one can define more clearly the type of material for which the associative flow-rule applies; this is tied closely to the notion of stable and unstable materials.

8.7.2 Drucker's Postulate

Stress Cycles

First, consider the one-dimensional loading of a hardening material. The material may have undergone any type of deformation (e.g. elastic or plastic) and is now subjected to the stress σ^*, point A in Fig. 8.7.3. An *additional* load is now applied to the material, bringing it to the current yield stress σ at point B (if σ^* is below the yield stress) and then plastically (greatly exaggerated in the figure) through the infinitesimal increment $d\sigma$ to point C. It is conventional to call these additional loads the **external agency**. The external agency is then removed, bringing the stress back to σ^* and point D. The material is said to have undergone a **stress cycle**.

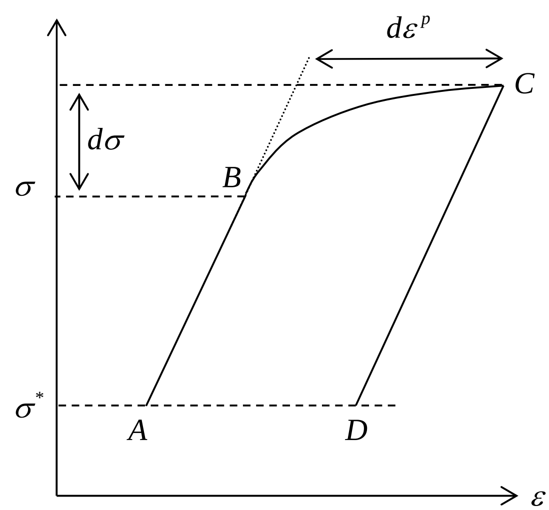

Figure 8.7.3: A stress cycle for a hardening material

Consider now a softening material, Fig. 8.7.4. The external agency first brings the material to the current yield stress σ at point B. To reach point C, the loads must be reduced. This cannot be achieved with a stress (force) control experiment, since a reduction in stress at B will induce elastic unloading towards A. A strain (displacement) control must be used, in which case the stress required to induce the (plastic) strain will be seen to drop to $\sigma + d\sigma$ ($d\sigma < 0$) at C. The stress cycle is completed by unloading from C to D.

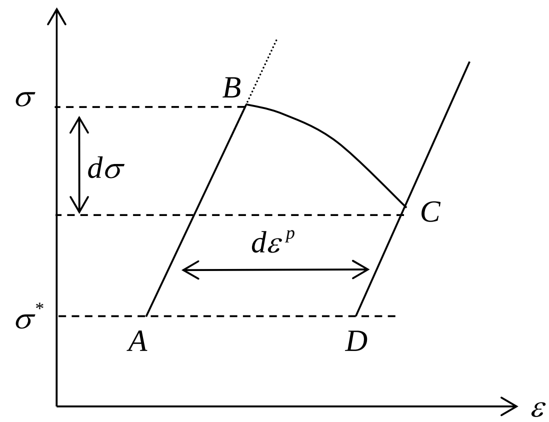

Figure 8.7.4: A stress cycle for a softening material

Suppose now that $\sigma = \sigma^*$, so the material is at point B, on the yield surface, before action by the external agency. It is now not possible for the material to undergo a stress cycle, since the stress cannot be increased. This provides a means of distinguishing between strain hardening and softening materials:

Strain-hardening … Material can always undergo a stress-cycle
Strain-softening … Material cannot always undergo a stress-cycle

Drucker's Postulate

The following statements define a **stable material**: (these statements are also known as **Drucker's postulate**):

(1) Positive work is done by the external agency during the application of the loads
(2) The net work performed by the external agency over a stress cycle is nonnegative

By this definition, it is clear that a strain hardening material is stable (and satisfies Drucker's postulates). For example, considering plastic deformation ($\sigma = \sigma^*$ in the above), the work done during an increment in stress is $d\sigma d\varepsilon$. The work done by the external agency is the area shaded in Fig. 8.7.5a and is clearly positive (note that the work referred to here is not the *total* work, $\int_{\varepsilon}^{\varepsilon+d\varepsilon} \sigma d\varepsilon$, but only that part which is done by the external agency[2]). Similarly, the net work over a stress cycle will be positive.

On the other hand, note that plastic loading of a softening (or perfectly plastic) material results in a non-positive work, Fig. 8.7.5b.

[2] the laws of thermodynamics insist that the total work is positive (or zero) in a complete cycle.

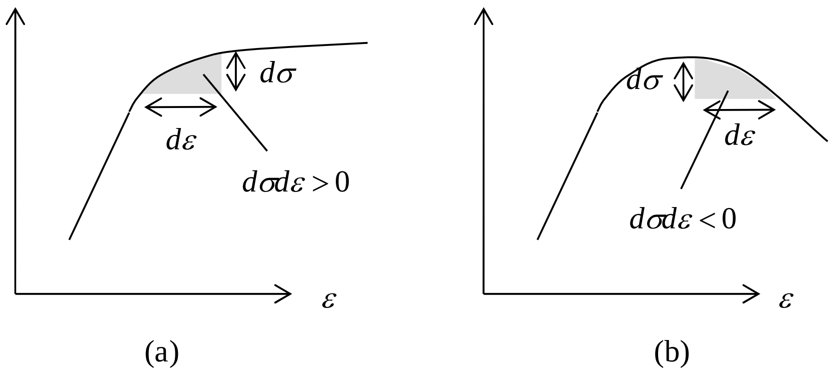

Figure 8.7.5: Stable (a) and unstable (b) stress-strain curves

The work done (per unit volume) by the additional loads during a stress cycle A-B-C-D is given by:

$$W = \int_{A-B-C-D} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon \qquad (8.7.11)$$

This is the shaded work in Fig. 8.7.6. Writing $d\varepsilon = d\varepsilon^e + d\varepsilon^p$ and noting that the elastic work is recovered, i.e. the net work due to the elastic strains is zero, this work is due to the plastic strains,

$$W = \int_{B-C} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon^p \qquad (8.7.12)$$

With $d\sigma$ infinitesimal, this equals

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \qquad (8.7.13)$$

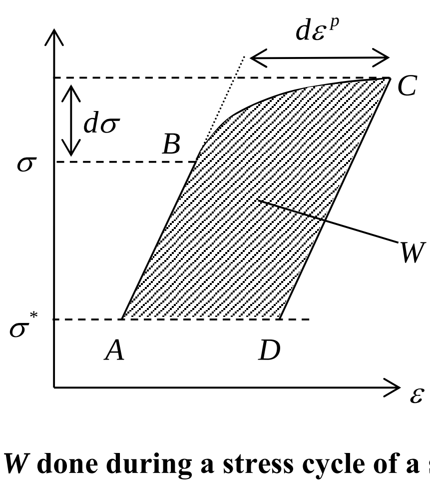

Figure 8.7.6: Work *W* done during a stress cycle of a strain-hardening material

The requirement (2) of a stable material is that this work be non-negative,

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \geq 0 \qquad (8.7.14)$$

Making $\sigma - \sigma^* >> d\sigma$, this reads

$$\left(\sigma - \sigma^*\right) d\varepsilon^p \geq 0 \tag{8.7.15}$$

On the other hand, making $\sigma = \sigma^*$, it reads

$$d\sigma d\varepsilon^p \geq 0 \tag{8.7.16}$$

The three dimensional case is illustrated in Fig. 8.7.7, for which one has

$$\left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p \geq 0, \quad d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \tag{8.7.17}$$

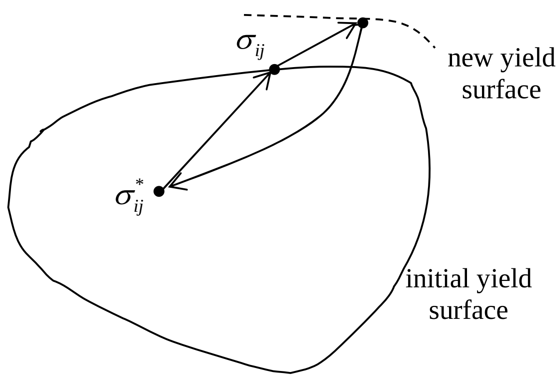

Figure 8.7.7: Stresses during a loading/unloading cycle

8.7.3 Consequences of the Drucker's Postulate

The criteria that a material be stable have very interesting consequences.

Normality

In terms of vectors in principal stress (plastic strain increment) space, Fig. 8.7.8, Eqn. 8.7.17 reads

$$\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\right) \cdot d\boldsymbol{\varepsilon}^p \geq 0 \tag{8.7.18}$$

These vectors are shown with the solid lines in Fig. 8.7.8. Since the dot product is non-negative, the angle between the vectors $\boldsymbol{\sigma} - \boldsymbol{\sigma}^*$ and $d\boldsymbol{\varepsilon}^p$ (with their starting points coincident) must be less than 90°. This implies that the plastic strain increment vector must be normal to the yield surface since, if it were not, an initial stress state $\boldsymbol{\sigma}^*$ could be found for which the angle was greater than 90° (as with the dotted vectors in Fig. 8.7.8). Thus a consequence of a material satisfying the stability requirements is that the **normality rule** holds, i.e. the flow rule is associative, Eqn. 8.7.4.

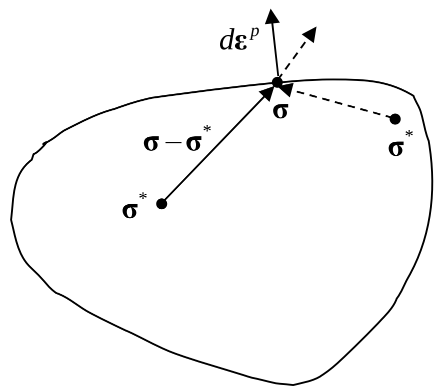

Figure 8.7.8: Normality of the plastic strain increment vector

When the yield surface has sharp corners, as with the Tresca criterion, it can be shown that the plastic strain increment vector must lie within the cone bounded by the normals on either side of the corner, as illustrated in Fig. 8.7.9.

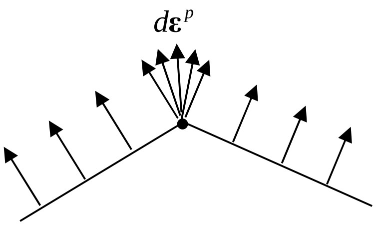

Figure 8.7.9: The plastic strain increment vector for sharp corners

Convexity

Using the same arguments, one cannot have a yield surface like the one shown in Fig. 8.7.10. In other words, the yield surface is **convex**: the entire elastic region lies to one side of the tangent plane to the yield surface[3].

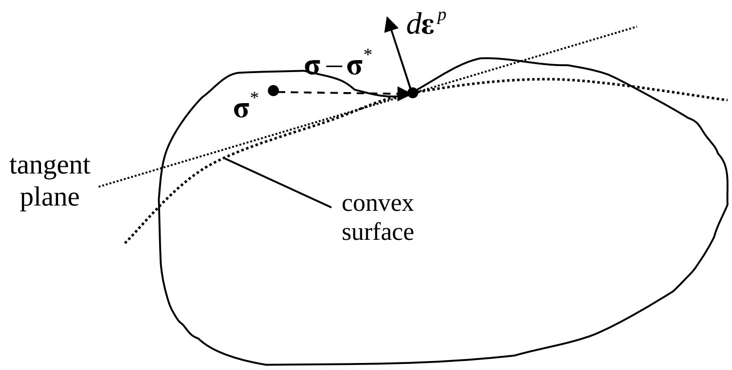

Figure 8.7.10: A non-convex surface

[3] note that when the plastic deformation affects the elastic response of the material, it can be shown that the stability postulate again ensures normality, but that the convexity does not necessarily hold

In summary then, Drucker's Postulate, which is satisfied by a stable, strain-hardening material, implies normality (associative flow rule) and convexity[4].

8.7.4 The Principle of Maximum Plastic Dissipation

The rate form of Eqn. 8.7.18 is

$$\left(\sigma_{ij} - \sigma_{ij}^{*}\right)\dot{\varepsilon}_{ij}^{p} \geq 0 \tag{8.7.19}$$

The quantity $\sigma_{ij}\dot{\varepsilon}_{ij}^{p}$ is called the **plastic dissipation**, and is a measure of the *rate* at which energy is being dissipated as deformation proceeds.

Eqn. 8.7.19 can be written as

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} \geq \sigma_{ij}^{*}\dot{\varepsilon}_{ij}^{p} \quad \text{or} \quad \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}^{p} \geq \boldsymbol{\sigma}^{*}\cdot\dot{\boldsymbol{\varepsilon}}^{p} \tag{8.7.20}$$

and in this form is known as the **principle of maximum plastic dissipation**: of all possible stress states σ_{ij}^{*} (within or on the yield surface), the one which arises is that which requires the maximum plastic work.

Although the principle of maximum plastic dissipation was "derived" from Drucker's postulate in the above, it is more general, holding also for the case of perfectly plastic and softening materials. To see this, disregard stress cycles and consider a stress state σ^{*} which is at or below the current (yield) stress σ, and apply a strain $d\varepsilon > 0$. For a perfectly plastic material, $\sigma - \sigma^{*} \geq 0$ and $d\varepsilon = d\varepsilon^{p} > 0$. For a softening material, again $\sigma - \sigma^{*} \geq 0$ and $d\varepsilon^{e} < 0$, $d\varepsilon^{p} > d\varepsilon > 0$.

It follows that the normality rule and convexity hold also for the perfectly plastic and softening materials which satisfy the principle of maximum plastic dissipation.

In summary:

Drucker's postulate leads to the Principle of maximum plastic dissipation
For hardening materials
    Principle of maximum plastic dissipation leads to Drucker's postulate
For softening materials
    Principle of maximum plastic dissipation does not lead to Drucker's postulate

Finally, note that, for many materials, hardening and softening, a non-associative flow rule is required, as in Eqn. 8.7.3. Here, the plastic strain increment is no longer normal to the yield surface and the principle of maximum plastic dissipation does not hold in general. In this case, when there is hardening, i.e. the stress increment is directed out from the yield surface, it is easy to see that one can have $d\sigma_{ij}d\varepsilon_{ij}^{p} < 0$, Fig. 8.7.11, contradicting the stability postulate (1),. With hardening, there is no

[4] it also ensures the uniqueness of solution to the boundary value elastoplastic problem

obvious instability, and so it could be argued that the use of the term “stability” in Drucker’s postulate is inappropriate.

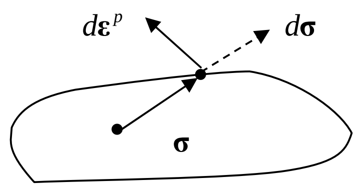

Figure 8.7.11: plastic strain increment vector not normal to the yield surface; non-associated flow-rule

8.7.5 Problems

1. Derive the flow-rule associated with the Drucker-Prager yield criterion $f = \alpha I_1 + \sqrt{J_2} - k$

2. Derive the flow-rule associated with the Mohr-Coulomb yield criterion, i.e. with $\sigma_1 > \sigma_2 > \sigma_3$,
$$\frac{\alpha\sigma_1 - \sigma_3}{2} = k$$
Here,
$$\alpha = \frac{1+\sin\phi}{1-\sin\phi} > 1, \qquad 0 < \phi < \frac{\pi}{2}$$
Evaluate the volumetric plastic strain increment, that is
$$\frac{\Delta V^p}{\Delta V} = d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p,$$
and hence show that the model predicts dilatancy (expansion).

3. Consider the plastic potential
$$g = \frac{\beta\sigma_1 - \sigma_3}{2} - k$$
Derive the non-associative flow-rule corresponding to this potential. Hence show that compaction of material can be modelled by choosing an appropriate value of β.