8.4 Elastic Perfectly Plastic Materials

Once yield occurs, a material will deform plastically. Predicting and modelling this plastic deformation is the topic of this section. For the most part, in this section, the material will be assumed to be perfectly plastic, that is, there is no work hardening.

8.4.1 Plastic Strain Increments

When examining the strains in a plastic material, it should be emphasised that one works with *increments in strain* rather than a total accumulated strain. One reason for this is that when a material is subjected to a certain stress state, the corresponding strain state could be one of many. Similarly, the strain state could correspond to many different stress states. Examples of this state of affairs are shown in Fig. 8.4.1.

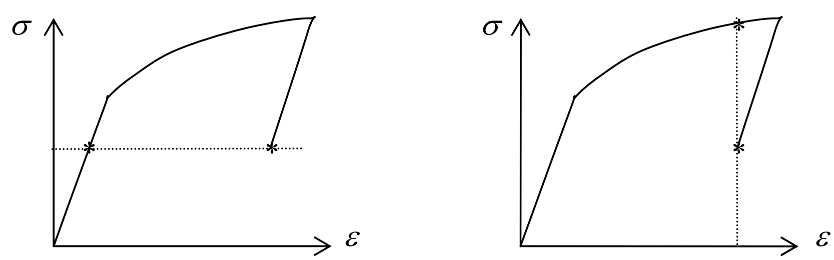

Figure 8.4.1: stress-strain curve; (a) different strains at a certain stress, (b) different stress at a certain strain

One cannot therefore make use of stress-strain relations in plastic regions (except in some special cases), since there is no unique relationship between the current stress and the current strain. However, one can relate the current stress to the current **increment in strain**, and these are the "stress-strain" laws which are used in plasticity theory. The total strain can be obtained by summing up, or integrating, the strain increments.

8.4.2 The Prandtl-Reuss Equations

An increment in strain $d\varepsilon$ can be decomposed into an elastic part $d\varepsilon^e$ and a plastic part $d\varepsilon^p$. If the material is isotropic, it is reasonable to suppose that the principal plastic strain increments $d\varepsilon_i^p$ are proportional to the principal deviatoric stresses s_i:

$$\frac{d\varepsilon_1^p}{s_1} = \frac{d\varepsilon_2^p}{s_2} = \frac{d\varepsilon_3^p}{s_3} = d\lambda \geq 0 \tag{8.4.1}$$

This relation only gives the ratios of the plastic strain increments to the deviatoric stresses. To determine the precise relationship, one must specify the positive scalar $d\lambda$ (see later). Note that the plastic volume constancy is inherent in this relation: $d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = 0$.

Eqns. 8.4.1 are in terms of the principal deviatoric stresses and principal plastic strain increments. In terms of Cartesian coordinates, one has

$$\frac{d\varepsilon_{xx}^{p}}{s_{xx}} = \frac{d\varepsilon_{yy}^{p}}{s_{yy}} = \frac{d\varepsilon_{zz}^{p}}{s_{zz}} = \frac{d\varepsilon_{xy}^{p}}{s_{xy}} = \frac{d\varepsilon_{xz}^{p}}{s_{xz}} = \frac{d\varepsilon_{yz}^{p}}{s_{yz}} = d\lambda \tag{8.4.2}$$

or, succinctly,

$$d\varepsilon_{ij}^{p} = s_{ij}\, d\lambda \tag{8.4.3}$$

These equations are often expressed in the alternative forms

$$\frac{d\varepsilon_{xx}^{p} - d\varepsilon_{yy}^{p}}{s_{xx} - s_{yy}} = \frac{d\varepsilon_{yy}^{p} - d\varepsilon_{zz}^{p}}{s_{yy} - s_{zz}} = \cdots \quad = \frac{d\varepsilon_{xx}^{p} - d\varepsilon_{yy}^{p}}{\sigma_{xx} - \sigma_{yy}} = \frac{d\varepsilon_{yy}^{p} - d\varepsilon_{zz}^{p}}{\sigma_{yy} - \sigma_{zz}} = \cdots \quad = d\lambda \tag{8.4.4}$$

or, dividing by dt to get the **rate** equations,

$$\frac{\dot{\varepsilon}_{xx}^{p} - \dot{\varepsilon}_{yy}^{p}}{s_{xx} - s_{yy}} = \cdots = \frac{\dot{\varepsilon}_{xx}^{p} - \dot{\varepsilon}_{yy}^{p}}{\sigma_{xx} - \sigma_{yy}} = \cdots = \dot{\lambda} \tag{8.4.5}$$

In terms of actual stresses, one has, from 8.2.3,

$$\begin{aligned} d\varepsilon_{xx}^{p} &= \tfrac{2}{3} d\lambda \left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\ d\varepsilon_{yy}^{p} &= \tfrac{2}{3} d\lambda \left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\ d\varepsilon_{zz}^{p} &= \tfrac{2}{3} d\lambda \left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\ d\varepsilon_{xy}^{p} &= d\lambda \sigma_{xy} \\ d\varepsilon_{yz}^{p} &= d\lambda \sigma_{yz} \\ d\varepsilon_{zx}^{p} &= d\lambda \sigma_{zx} \end{aligned} \tag{8.4.6}$$

This *plastic* stress-strain law is known as a **flow rule**. Other flow rules will be considered later on. Note that one cannot propose a flow rule which gives the plastic strain increments as explicit functions of the stress, otherwise the yield criterion might not be met (in particular, when there is strain hardening); one must include the to-be-determined scalar plastic multiplier λ. The plastic multiplier is determined by ensuring the stress-state lies on the yield surface during plastic flow.

The full elastic-plastic stress-strain relations are now, using Hooke's law,

$$
\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu\left(d\sigma_{yy} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy} &= \frac{1}{E}\left[d\sigma_{yy} - \nu\left(d\sigma_{xx} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{yy} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu\left(d\sigma_{xx} + d\sigma_{yy}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{zz} - \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \\
d\varepsilon_{yz} &= \frac{1+\nu}{E}d\sigma_{yz} + d\lambda\sigma_{yz} \\
d\varepsilon_{zx} &= \frac{1+\nu}{E}d\sigma_{zx} + d\lambda\sigma_{zx}
\end{aligned}
\qquad (8.4.7)
$$

or

$$
d\varepsilon_{ij} = \frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}\delta_{ij}d\sigma_{kk} + d\lambda s_{ij}
$$

These expressions are called the **Prandtl-Reuss equations**. If the first, elastic, terms are neglected, they are known as the **Lévy-Mises** equations.

8.4.3 Application: Plane Strain Compression of a Block

Consider the plane strain compression of a thick block, Fig. 8.4.2. The block is subjected to an increasing pressure $\sigma_{xx} = -p$, is constrained in the *z* direction, so $\varepsilon_{zz} = 0$, and is free to move in the *y* direction, so $\sigma_{yy} = 0$.

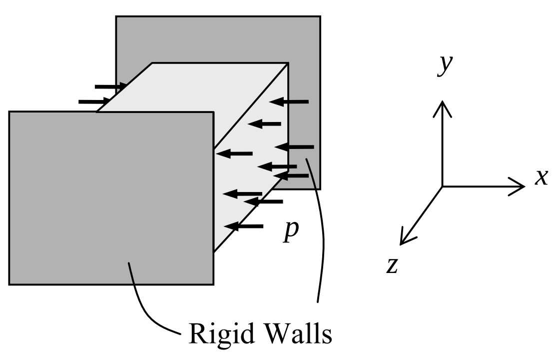

Figure 8.4.2: Plane strain compression of a thick block

The solution to the *elastic* problem is obtained from 8.4.7 (disregarding the plastic terms). One finds that {▲Problem 1}

$$
\sigma_{xx} = -p, \quad \sigma_{zz} = -\nu p, \quad \varepsilon_{xx} = -\frac{p}{E}\left(1-\nu^2\right), \quad \varepsilon_{yy} = +\frac{p}{E}\nu\left(1+\nu\right) \qquad (8.4.8)
$$

and all other stress and strain components are zero. In this elastic phase, the principal stresses are clearly

$$\sigma_1 = \sigma_{yy} = 0 > \sigma_2 = \sigma_{zz} > \sigma_3 = \sigma_{xx} \tag{8.4.9}$$

The Prandtl-Reuss equations are

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu d\sigma_{zz}\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}\left(d\sigma_{xx} + d\sigma_{zz}\right) - \frac{1}{3}d\lambda\left(\sigma_{xx} + \sigma_{zz}\right) \\ d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu d\sigma_{xx}\right] + \frac{2}{3}d\lambda\left[-\frac{1}{2}\sigma_{xx} + \sigma_{zz}\right] \end{aligned} \tag{8.4.10}$$

The magnitude of the plastic straining is determined by the multiplier $d\lambda$. This can be evaluated by noting that plastic deformation proceeds so long as the stress state remains on the yield surface, the so-called **consistency condition**. By definition, a perfectly plastic material is one whose yield surface remains unchanged during deformation.

A Tresca Material

Take now the Tresca yield criterion, which states that yield occurs when $\sigma_{xx} = -Y$, where Y is the uniaxial yield stress (in compression). Assume further perfect plasticity, so that $\sigma_{xx} = -Y$ holds during all subsequent plastic flow. Thus, with $d\sigma_{xx} = 0$, and since $d\varepsilon_{zz} = 0$, 8.4.10 reduce to

$$\begin{aligned} d\varepsilon_{xx} &= -\frac{\nu}{E}d\sigma_{zz} - \frac{2}{3}d\lambda\left[Y + \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}d\sigma_{zz} + \frac{1}{3}d\lambda\left(Y - \sigma_{zz}\right) \\ 0 &= \frac{1}{E}d\sigma_{zz} + \frac{2}{3}d\lambda\left[\frac{1}{2}Y + \sigma_{zz}\right] \end{aligned} \tag{8.4.11}$$

Thus

$$d\lambda = -\frac{3}{E}\frac{d\sigma_{zz}}{2\sigma_{zz} + Y} \tag{8.4.12}$$

and, eliminating $d\lambda$ from Eqns. 8.4.11 {▲Problem 2},

$$\begin{aligned} Ed\varepsilon_{xx} &= -\nu d\sigma_{zz} + Y\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \\ Ed\varepsilon_{yy} &= -\nu d\sigma_{zz} - \frac{Y}{2}\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \end{aligned} \tag{8.4.13}$$

Using the relation

$$\int \frac{x}{x+a}dx = x - a\ln(x+a) \qquad (8.4.14)$$

and the initial (yield point) conditions, i.e. Eqns. 8.4.8 with $p = Y$, one can integrate 8.4.13 to get {▲Problem 2}

$$\begin{aligned} \frac{E}{Y}\varepsilon_{xx} &= -\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} - \frac{1}{2}(2-\nu) \\ \frac{E}{Y}\varepsilon_{yy} &= +\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} + \frac{3}{2}\nu \end{aligned}, \qquad \frac{\sigma_{zz}}{Y} < -\nu \qquad (8.4.15)$$

The stress-strain curves are shown in Fig. 8.4.3 below for $\nu = 0.3$. Note that, for a typical metal, $E/Y \sim 10^3$, and so the strains are very small right through the plastic compression; the plastic strains are of comparable size to the elastic strains. There is a rapid change of stress and then little change once σ_{zz} has approached close to its limiting value of $-Y/2$.

The above plastic analysis was based on σ_{xx} remaining the minimum principal stress. This assumption has proved to be valid, since σ_{zz} remains between 0 and $-Y$ in the plastic region.

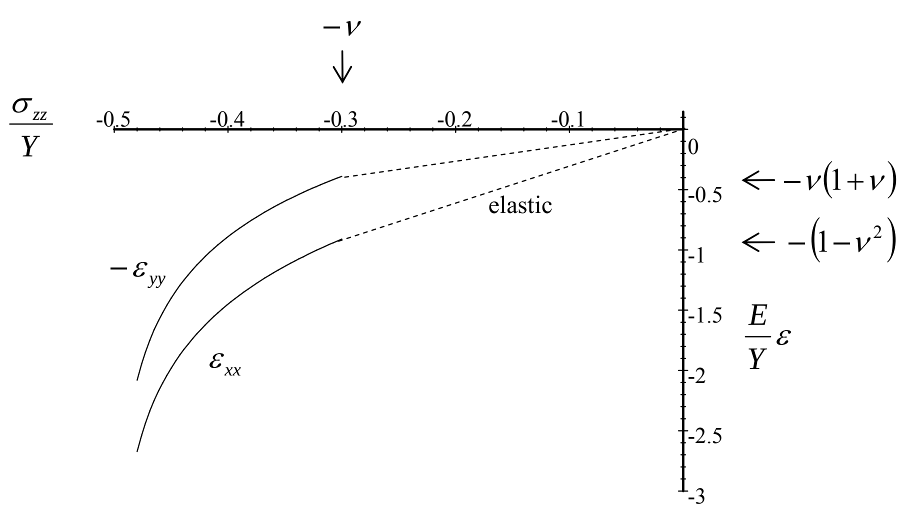

Figure 8.4.3: Stress-strain results for plane strain compression of a thick block for $\nu = 0.3$

A Von Mises Material

Slightly different results are obtained with the Von Mises yield criterion, Eqn. 8.4.11, which for this problem reads

$$\sigma_{xx}^2 - \sigma_{xx}\sigma_{zz} + \sigma_{zz}^2 = Y^2 \tag{8.4.16}$$

The Prandtl-Reuss equations can be solved by making the substitution

$$\sigma_{xx} = -\frac{2Y}{\sqrt{3}}\cos\theta \tag{8.4.17}$$

in the plastic region. In what follows, use is made of the trigonometric relations

$$\begin{aligned} \sin\left(\frac{\pi}{6} - \theta\right) &= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \\ \cos\left(\frac{\pi}{6} - \theta\right) &= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta \end{aligned} \tag{8.4.18}$$

From Eqn. 8.4.16,

$$\sigma_{zz} = -\frac{2Y}{\sqrt{3}}\sin\left(\frac{\pi}{6} - \theta\right) \tag{8.4.19}$$

Substituting into 8.4.10 then leads to

$$\begin{aligned} \frac{E}{Y}d\varepsilon_{xx} &= \frac{2}{\sqrt{3}}\left\{\left[\sin\theta - \nu\cos\left(\frac{\pi}{6} - \theta\right)\right]d\theta - \frac{1}{\sqrt{3}}Ed\lambda\cos\left(\frac{\pi}{6} - \theta\right)\right\} \\ \frac{E}{Y}d\varepsilon_{yy} &= \frac{2}{\sqrt{3}}\left\{-\nu\left(\sin\theta + \cos\left(\frac{\pi}{6} - \theta\right)\right)d\theta - \frac{1}{3}Ed\lambda\left(-\cos\theta - \sin\left(\frac{\pi}{6} - \theta\right)\right)\right\} \\ \frac{E}{Y}d\varepsilon_{zz} &= \frac{2}{\sqrt{3}}\left\{\left[\cos\left(\frac{\pi}{6} - \theta\right) - \nu\sin\theta\right]d\theta + \frac{1}{\sqrt{3}}Ed\lambda\sin\theta\right\} \end{aligned} \tag{8.4.20}$$

Using $d\varepsilon_{zz} = 0$ leads to

$$d\lambda = -\frac{\sqrt{3}}{E}\frac{\cos\left(\frac{\pi}{6} - \theta\right) - \nu\sin\theta}{\sin\theta}d\theta \tag{8.4.21}$$

and

$$\frac{E}{Y}d\varepsilon_{xx} = \frac{2}{\sqrt{3}}\left\{(1 - 2\nu)\cos\left(\frac{\pi}{6} - \theta\right) + \frac{3}{4}\operatorname{cosec}\theta\right\}d\theta \tag{8.4.22}$$

An integration gives

$$\frac{E}{Y}\varepsilon_{xx} = -\frac{2}{\sqrt{3}}(1 - 2\nu)\sin\left(\frac{\pi}{6} - \theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta}{2}\right| + C \tag{8.4.23}$$

To determine the constant of integration, consider again the conditions at first yield. Suppose the block first yields when σ_{xx} reaches σ_{xx}^Y. Then $\sigma_{zz}^Y = \nu\sigma_{xx}^Y$ and

$$\sigma_{xx}^Y = -\frac{Y}{\sqrt{1-\nu+\nu^2}} \tag{8.4.24}$$

Note that in this case it is predicted that first yield occurs when $\sigma_{xx} < -Y$. From Eqn. 8.4.17, the value of θ at first yield is

$$\cos\theta^Y = \frac{\sqrt{3}}{2\sqrt{1-\nu+\nu^2}} \quad \text{or} \quad \tan\theta^Y = \frac{1-2\nu}{\sqrt{3}} \tag{8.4.25}$$

Thus, with $\varepsilon_{xx} = \sigma_{xx}^Y\left(1-\nu^2\right)/E$ at yield,

$$C = -\sqrt{1-\nu+\nu^2} - \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^Y}{2}\right| \tag{8.4.26}$$

and so

$$-\frac{E}{Y}\varepsilon_{xx} = \frac{2}{\sqrt{3}}(1-2\nu)\sin\left(\frac{\pi}{6}-\theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^Y}{2}\cot\frac{\theta}{2}\right| + \sqrt{1-\nu+\nu^2} \tag{8.4.27}$$

This leads to a similar stress-strain curve as for the Tresca criterion, only now the limiting value of σ_{zz} is $-Y/\sqrt{3} \approx -0.58Y$.

8.4.4 Application: Combined Tension/Torsion of a thin walled tube

Consider now the combined tension/torsion of a thin-walled tube as in the Taylor/Quinney tests. The only stresses in the tube are $\sigma_{xx} = \sigma$ due to the tension along the axial direction and $\sigma_{xy} = \tau$ due to the torsion. The Prandtl-Reuss equations reduce to

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \end{aligned} \tag{8.4.28}$$

Consider the case where the tube is twisted up to the yield point. Torsion is then halted and tension is applied, holding the angle of twist constant. In that case, during the tension, $d\varepsilon_{xy} = 0$ and so {▲Problem 3}

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma - \frac{2}{3}\frac{1+\nu}{E}\frac{d\tau}{\tau}\sigma \tag{8.4.29}$$

If one takes the Von Mises criterion, then $\sigma^2 + 3\tau^2 = Y^2$ (see Eqn. 8.3.17). Assuming perfect plasticity, one has {▲Problem 4},

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma + \frac{2}{3}\frac{1+\nu}{E}\frac{\sigma^2 d\sigma}{Y^2 - \sigma^2} \tag{8.4.30}$$

Using the relation

$$\int \frac{x^2}{a^2 - x^2}dx = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right), \tag{8.4.31}$$

an integration leads to {▲Problem 5}

$$\frac{E}{Y}\varepsilon_{xx} = \frac{1}{3}\left\{(1-2\nu)\frac{\sigma}{Y} + (1+\nu)\ln\left(\frac{1+\sigma/Y}{1-\sigma/Y}\right)\right\} \tag{8.4.32}$$

This result is plotted in Fig. 8.4.4. Note that, with $\sigma^2 + 3\tau^2 = Y^2$, as σ increases (rapidly) to its limiting value Y, τ decreases from its yield value of $Y/\sqrt{3}$ to zero.

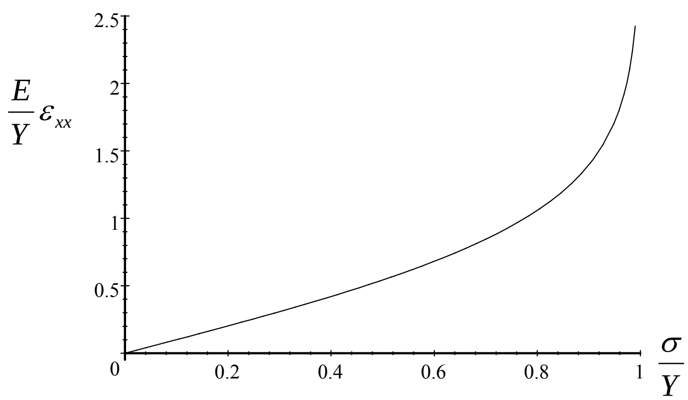

Figure 8.4.4: Stress-strain results for combined tension/torsion of a thin walled tube for $\nu = 0.3$

8.4.5 The Tresca Flow Rule

The flow rule used in the preceding applications was the Prandtl-Reuss rule 8.4.7. Many other flow rules have been proposed. For example, the Tresca flow rule is simply (for $\sigma_1 > \sigma_2 > \sigma_3$)

$$\begin{aligned} d\varepsilon_1^p &= +d\lambda \\ d\varepsilon_2^p &= 0 \\ d\varepsilon_3^p &= -d\lambda \end{aligned} \qquad (8.4.33)$$

This flow rule will be used in the next section, which details the classic solution for the plastic deformation and failure of a thick cylinder under internal pressure.

A unifying theory of flow rules will be presented in a later section, in which the reason for the name "Treca flow rule" will become clear.

8.4.6 Problems

1. Derive the elastic strains for the plane strain compression of a thick block, Eqns. 8.4.8.

2. Derive Eqns. 8.4.13 and 8.4.15

3. Derive Eqn. 8.4.29

4. Use Eqn. 8.3.17 to show that $\sigma d\tau/\tau = -\sigma^2 d\sigma/\left(Y^2 - \sigma^2\right)$ and hence derive Eqn. 8.4.30

5. Derive Eqns. 8.4.32

6. Does the axial stress-stress curve of Fig. 8.4.4 differ when the Tresca criterion is used?

7. Consider the uniaxial straining of a perfectly plastic isotropic Von Mises metallic block. There is only one non-zero strain, ε_{xx}. One only need consider two stresses, σ_{xx}, σ_{yy} since $\sigma_{zz} = \sigma_{yy}$ by isotropy.
   (i) Write down the two relevant Prandtl-Reuss equations
   (ii) Evaluate the stresses and strains at first yield
   (iii) For plastic flow, show that $d\sigma_{xx} = d\sigma_{yy}$ and that the plastic modulus is

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}} = \frac{E}{3(1-2\nu)}$$

8. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of a perfectly plastic Von Mises metal and Y is the uniaxial yield strength in tension. The only stresses are $\sigma_{xx} = \sigma$ and $\sigma_{xy} = \tau$ and the Prandtl-Reuss equations reduce to

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx}$$
$$d\varepsilon_{yy} = d\varepsilon_{zz} = -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx}$$
$$d\varepsilon_{xy} = \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy}$$

The axial strain is increased from zero until yielding occurs (with $\varepsilon_{xy} = 0$). From first yield, the axial strain is held constant and the shear strain is increased up to its final value of $(1+\nu)Y/\sqrt{3}E$

(i) Write down the yield criterion in terms of σ and τ only and sketch the yield locus in $\sigma - \tau$ space
(ii) Evaluate the stresses and strains at first yield
(iii) Evaluate $d\lambda$ in terms of $\sigma, d\sigma$
(iv) Relate $\sigma, d\sigma$ to $\tau, d\tau$ and hence derive a differential equation for shear strain in terms of τ only
(v) Solve the differential equation and evaluate any constant of integration
(vi) Evaluate the shear stress when ε_{xy} reaches its final value of $(1+\nu)Y/\sqrt{3}E$. Taking $\nu = 1/2$, put in the form $\tau = \alpha Y$ with α to 3 d.p.