# 8.3 Yield Criteria in Three Dimensional Plasticity

The question now arises: a material yields at a stress level *Y* in a uniaxial tension test, but when does it yield when subjected to a complex three-dimensional stress state?

Let us begin with a very general case: an anisotropic material with different yield strengths in different directions. For example, consider the material shown in Fig. 8.3.1. This is a composite material with long fibres along the $x_1$ direction, giving it extra strength in that direction – it will yield at a higher tension when pulled in the $x_1$ direction than when pulled in other directions.

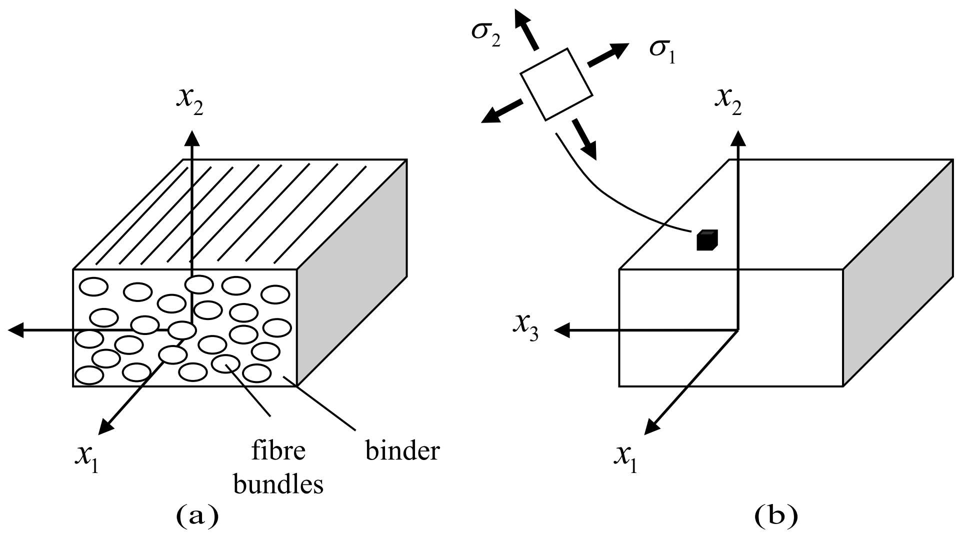


**Figure 8.3.1: an anisotropic material; (a) microstructural detail, (b) continuum model**

We can assume that yield will occur at a particle when some combination of the stress components reaches some critical value, say when

$$F(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}) = k \ . \tag{8.3.1}$$

Here, *F* is some function of the 6 independent components of the stress tensor and *k* is some material property which can be determined experimentally[1].

Alternatively, it is very convenient to express yield criteria in terms of principal stresses. Let us suppose that we know the principal stresses everywhere, $(\sigma_1, \sigma_2, \sigma_3)$, Fig. 8.3.1. Yield must depend somehow on the microstructure – on the orientation of the axes $x_1, x_2, x_3$, but this information is not contained in the three numbers $(\sigma_1, \sigma_2, \sigma_3)$. Thus we express the yield criterion in terms of principal stresses in the form

$$F(\sigma_1, \sigma_2, \sigma_3, \mathbf{n}_i) = k \tag{8.3.2}$$

[1] *F* will no doubt also contain other parameters which need to be determined experimentally

where $\mathbf{n}_i$ represent the principal directions – these give the orientation of the principal stresses relative to the material directions $x_1, x_2, x_3$.

If the material is isotropic, the response is independent of any material direction – independent of any "direction" the stress acts in, and so the yield criterion can be expressed in the simple form

$$F(\sigma_1, \sigma_2, \sigma_3) = k \tag{8.3.3}$$

Further, since it should not matter which direction is labelled '1', which '2' and which '3', *F* must be a symmetric function of the three principal stresses.

Alternatively, since the three principal invariants of stress are independent of material orientation, one can write

$$F(I_1, I_2, I_3) = k \tag{8.3.4}$$

or, more usually,

$$F(I_1, J_2, J_3) = k \tag{8.3.5}$$

where $J_2, J_3$ are the non-zero principal invariants of the deviatoric stress. With the further restriction that the yield stress is independent of the hydrostatic stress, one has

$$F(J_2, J_3) = k \tag{8.3.6}$$

## 8.3.1 The Tresca and Von Mises Yield Conditions

The two most commonly used and successful yield criteria for isotropic metallic materials are the **Tresca** and **Von Mises** criteria.

### The Tresca Yield Condition

The Tresca yield criterion states that a material will yield if the maximum shear stress reaches some critical value, that is, Eqn. 8.3.3 takes the form

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} = k \tag{8.3.7}$$

The value of *k* can be obtained from a simple experiment. For example, in a tension test, $\sigma_1 = \sigma_0, \sigma_2 = \sigma_3 = 0$, and failure occurs when $\sigma_0$ reaches *Y*, the yield stress in tension. It follows that

$$k = \frac{Y}{2}. \tag{8.3.8}$$

In a shear test, $\sigma_1 = \tau, \sigma_2 = 0, \sigma_3 = -\tau$, and failure occurs when $\tau$ reaches $\tau_Y$, the yield stress of a material in pure shear, so that $k = \tau_Y$.

### The Von Mises Yield Condition

The Von Mises criterion states that yield occurs when the principal stresses satisfy the relation

$$\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} = k \tag{8.3.9}$$

Again, from a uniaxial tension test, one finds that the *k* in Eqn. 8.3.9 is

$$k = \frac{Y}{\sqrt{3}}. \tag{8.3.10}$$

Writing the Von Mises condition in terms of *Y*, one has

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = Y \tag{8.3.11}$$

The quantity on the left is called the **Von Mises Stress**, sometimes denoted by $\sigma_{VM}$. When it reaches the yield stress in pure tension, the material begins to deform plastically. In the shear test, one again finds that $k = \tau_Y$, the yield stress in pure shear.

Sometimes it is preferable to work with arbitrary stress components; for this purpose, the Von Mises condition can be expressed as {▲Problem 2}

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right) = 6k^2 \tag{8.3.12}$$

The piecewise linear nature of the Tresca yield condition is sometimes a theoretical advantage over the quadratic Mises condition. However, the fact that in many problems one often does not know which principal stress is the maximum and which is the minimum causes difficulties when working with the Tresca criterion.

### The Tresca and Von Mises Yield Criteria in terms of Invariants

From Eqn. 8.2.10 and 8.3.9, the Von Mises criterion can be expressed as

$$f(J_2) \equiv J_2 - k^2 = 0 \tag{8.3.13}$$

Note the relationship between $\sqrt{J_2}$ and the octahedral shear stress, Eqn. 8.2.17; the Von Mises criterion can be interpreted as predicting yield when the octahedral shear stress reaches a critical value.

With $\sigma_1 \ge \sigma_2 \ge \sigma_3$, the Tresca condition can be expressed as

$$f(J_2, J_3) \equiv 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0 \tag{8.3.14}$$

but this expression is too cumbersome to be of much use.

## Experiments of Taylor and Quinney

In order to test whether the Von Mises or Tresca criteria best modelled the real behaviour of metals, G I Taylor & Quinney (1931), in a series of classic experiments, subjected a number of thin-walled cylinders made of copper and steel to combined tension and torsion, Fig. 8.3.2.

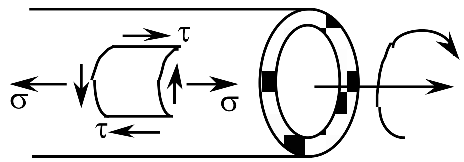


**Figure 8.3.2: combined tension and torsion of a thin-walled tube**

The cylinder wall is in a state of plane stress, with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and all other stress components zero. The principal stresses corresponding to such a stress-state are (zero and) {▲Problem 3}

$$\tfrac{1}{2}\sigma \pm \sqrt{\tfrac{1}{4}\sigma^2 + \tau^2} \tag{8.3.15}$$

and so Tresca's condition reduces to

$$\sigma^2 + 4\tau^2 = 4k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/2}\right)^2 = 1 \tag{8.3.16}$$

The Mises condition reduces to {▲Problem 4}

$$\sigma^2 + 3\tau^2 = 3k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1 \tag{8.3.17}$$

Thus both models predict an elliptical **yield locus** in $(\sigma, \tau)$ **stress space**, but with different ratios of principal axes, Fig. 8.3.3. The origin in Fig. 8.3.3 corresponds to an unstressed state. The horizontal axes refer to uniaxial tension in the absence of shear, whereas the vertical axis refers to pure torsion in the absence of tension. When there is a combination of $\sigma$ and $\tau$, one is off-axes. If the combination remains "inside" the yield locus, the material remains elastic; if the combination is such that one reaches anywhere along the locus, then plasticity ensues.

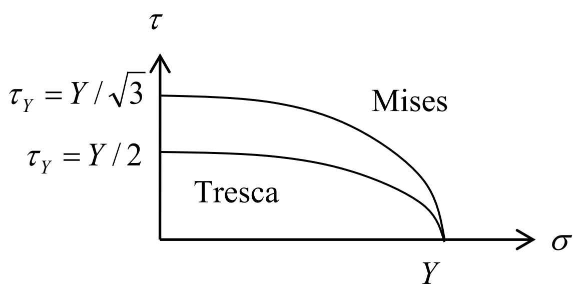


**Figure 8.3.3: the yield locus for a thin-walled tube in combined tension and torsion**

Taylor and Quinney, by varying the amount of tension and torsion, found that their measurements were closer to the Mises ellipse than the Tresca locus, a result which has been repeatedly confirmed by other workers[2].

## 2D Principal Stress Space

Fig. 8.3.3 gives a geometric interpretation of the Tresca and Von Mises yield criteria in $(\sigma, \tau)$ space. It is more usual to interpret yield criteria geometrically in a **principal stress space**. The Taylor and Quinney tests are an example of plane stress, where one principal stress is zero. Following the convention for plane stress, label now the two non-zero principal stresses $\sigma_1$ and $\sigma_2$, so that $\sigma_3 = 0$ (even if it is not the minimum principal stress). The criteria can then be displayed in $(\sigma_1, \sigma_2)$ 2D principal stress space. With $\sigma_3 = 0$, one has

$$\text{Tresca:} \qquad \max\left\{ |\sigma_1 - \sigma_2|, |\sigma_2|, |\sigma_1| \right\} = Y$$

$$\text{Von Mises:} \qquad \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2 \qquad (8.3.18)$$

These are plotted in Fig. 8.3.4. The Tresca criterion is a hexagon. The Von Mises criterion is an ellipse with axes inclined at $45^0$ to the principal axes, which can be seen by expressing Eqn. 8.3.18b in the canonical form for an ellipse:

[2] the maximum difference between the predicted stresses from the two criteria is about 15%. The two criteria can therefore be made to agree to within ± 7.5% by choosing *k* to be half-way between $Y/2$ and $Y/\sqrt{3}$

$$\begin{aligned}
\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 &= \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1 & -1/2 \\ -1/2 & 1\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix} \\
&= \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/2 & 0 \\ 0 & 3/2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix} \\
&= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{\sigma_2-\sigma_1}{\sqrt{2}}\right)^2 = Y^2 \\
&\rightarrow \quad \left(\frac{\sigma_1'}{\sqrt{2}Y}\right)^2 + \left(\frac{\sigma_2'}{\sqrt{2/3}Y}\right)^2 = 1
\end{aligned}$$

where $\sigma_1', \sigma_2'$ are coordinates along the new axes; the major axis is thus $\sqrt{2}Y$ and the minor axis is $\sqrt{2/3}Y$.

Some stress states are shown in the stress space: point *A* corresponds to a uniaxial tension, *B* to a equi-biaxial tension and *C* to a pure shear $\tau$.

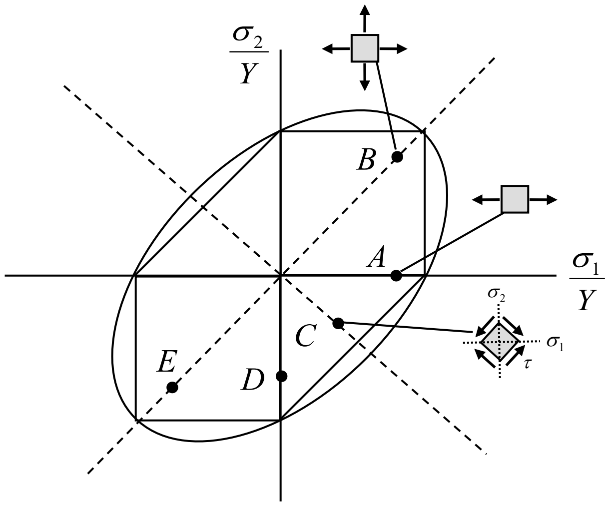


**Figure 8.3.4: yield loci in 2D principal stress space**

Again, points inside these loci represent an elastic stress state. Any combination of principal stresses which push the point out to the yield loci results in plastic deformation.

## 8.3.2 Three Dimensional Principal Stress Space

The 2D principal stress space has limited use. For example, a stress state that might start out two dimensional can develop into a fully three dimensional stress state as deformation proceeds.

In three dimensional principal stress space, one has a **yield surface** $f(\sigma_1, \sigma_2, \sigma_3) = 0$, Fig. 8.3.5 [3]. In this case, one can draw a line at equal angles to all three principal stress axes, the **space diagonal**. Along the space diagonal $\sigma_1 = \sigma_2 = \sigma_3$ and so points on it are in a state of hydrostatic stress.

Assume now, for the moment, that *hydrostatic stress does not affect yield* and consider some arbitrary point A, $(\sigma_1, \sigma_2, \sigma_3) = (a, b, c)$, on the yield surface, Fig. 8.3.5. A pure hydrostatic stress $\sigma_h$ can be superimposed on this stress state without affecting yield, so any other point $(\sigma_1, \sigma_2, \sigma_3) = (a + \sigma_h, b + \sigma_h, c + \sigma_h)$ will also be on the yield surface. Examples of such points are shown at B, C and D, which are obtained from A by moving along a line parallel to the space diagonal. The yield behaviour of the material is therefore specified by a yield *locus* on a plane perpendicular to the space diagonal, and the yield surface is generated by sliding this locus up and down the space diagonal.

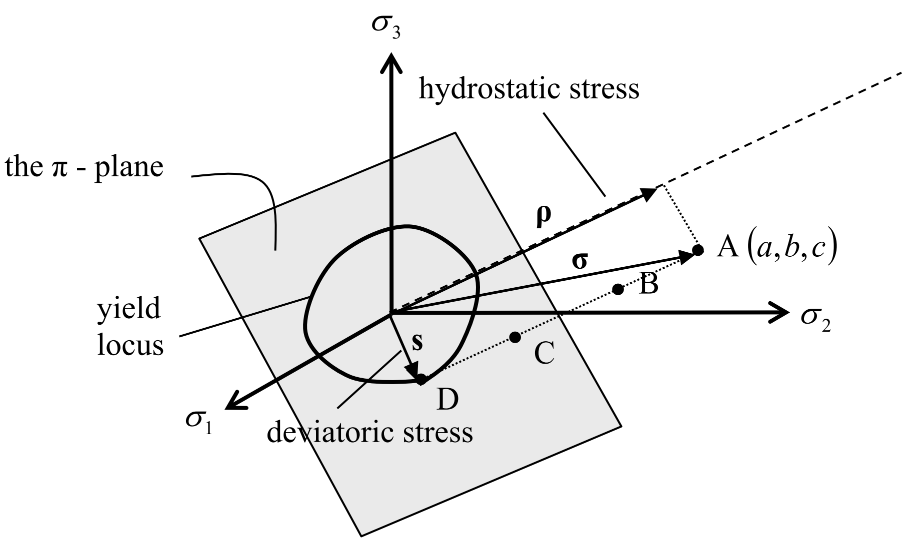


**Figure 8.3.5: Yield locus/surface in three dimensional stress-space**

### The π-plane

Any surface in stress space can be described by an equation of the form

$$f(\sigma_1, \sigma_2, \sigma_3) = \text{const} \tag{8.3.19}$$

and a normal to this surface is the gradient vector

$$\frac{\partial f}{\partial \sigma_1}\mathbf{e}_1 + \frac{\partial f}{\partial \sigma_2}\mathbf{e}_2 + \frac{\partial f}{\partial \sigma_3}\mathbf{e}_3 \tag{8.3.20}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the stress space axes. In particular, any plane perpendicular to the space diagonal is described by the equation

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[3] as mentioned, one has a six dimensional stress space for an anisotropic material and this cannot be visualised

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{const} \tag{8.3.21}$$

Without loss of generality, one can choose as a representative plane the **π – plane**, which is defined by $\sigma_1 + \sigma_2 + \sigma_3 = 0$. For example, the point $(\sigma_1, \sigma_2, \sigma_3) = (1, -1, 0)$ is on the π – plane and, with yielding independent of hydrostatic stress, is equivalent to points in principal stress space which differ by a hydrostatic stress, e.g. the points $(2,0,1)$, $(0,-2,-1)$, etc.

The stress state at any point A represented by the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ can be regarded as the sum of the stress state at the corresponding point on the π – plane, D, represented by the vector $\mathbf{s} = (s_1, s_2, s_3)$ together with a hydrostatic stress represented by the vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$:

$$(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_m, \sigma_2 - \sigma_m, \sigma_3 - \sigma_m) + (\sigma_m, \sigma_m, \sigma_m) \tag{8.3.22}$$

The components of the first term/vector on the right here sum to zero since it lies on the π – plane, and this is the deviatoric stress, whilst the hydrostatic stress is $\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$.

### Projected view of the π-plane

Fig. 8.3.6a shows principal stress space and Fig. 8.3.6b shows the π – plane. The heavy lines $\sigma_1', \sigma_2', \sigma_3'$ in Fig. 8.3.6b represent the projections of the principal axes down onto the $\pi$ – plane (so one is "looking down" the space diagonal). Some points, $A$, $B$, $C$ in stress space and their projections onto the $\pi$ – plane are also shown. Also shown is some point $D$ *on* the $\pi$ – plane. It should be kept in mind that the deviatoric stress vector **s** in the projected view of Fig. 8.3.6b is in reality a three dimensional vector (see the corresponding vector in Fig. 8.3.6a).

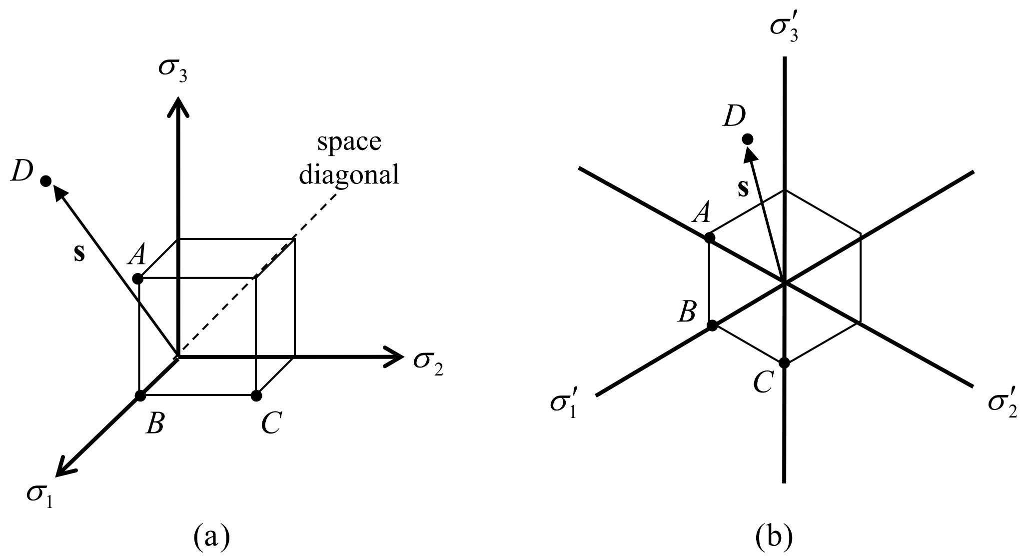


**Figure 8.3.6: Stress space; (a) principal stress space, (b) the π – plane**

Consider the more detailed Fig. 8.3.7 below. Point *A* here represents the stress state $(2,-1,0)$, as indicated by the arrows in the figure. It can also be "reached" in different ways, for example it represents $(3,0,1)$ and $(1,-2,-1)$. These three stress states of course differ by a hydrostatic stress. The actual $\pi-$plane value for *A* is the one for which $\sigma_1+\sigma_2+\sigma_3=0$, i.e. $(\sigma_1,\sigma_2,\sigma_3)=(s_1,s_2,s_3)=\left(\frac{5}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Points *B* and *C* also represent multiple stress states {▲Problem 7}.

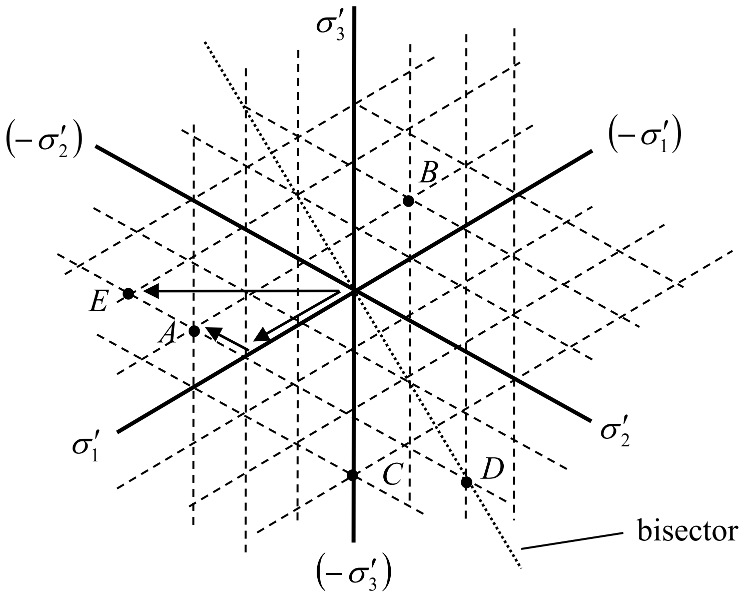


**Figure 8.3.7: the π-plane**

The bisectors of the principal plane projections, such as the dotted line in Fig. 8.3.7, represent states of pure shear. For example, the $\pi-$plane value for point *D* is $(0,2,-2)$, corresponding to a pure shear in the $\sigma_2-\sigma_3$ plane.

The dashed lines in Fig. 8.3.7 are helpful in that they allow us to plot and visualise stress states easily. The distance between each dashed line along the directions of the projected axes represents one unit of principal stress. Note, however, that these "units" are not consistent with the *actual* magnitudes of the deviatoric vectors in the $\pi-$plane. To create a more complete picture, note first that a unit vector along the space diagonal is $\mathbf{n}_\rho=\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, Fig. 8.3.8. The components of this normal are the direction cosines; for example, a unit normal along the '1' principal axis is $\mathbf{e}_1=[1,0,0]$ and so the angle $\theta_0$ between the '1' axis and the space diagonal is given by $\mathbf{n}_\rho\cdot\mathbf{e}_1=\cos\theta_0=\frac{1}{\sqrt{3}}$. From Fig. 8.3.8, the angle $\theta$ between the '1' axis and the $\pi-$plane is given by $\cos\theta=\sqrt{\frac{2}{3}}$, and so a length of $\sigma_1$ units gets projected down to a length $s=|\mathbf{s}|=\sqrt{\frac{2}{3}}\sigma_1$.

For example, point *E* in Fig. 8.3.7 represents a pure shear $(\sigma_1,\sigma_2,\sigma_3)=(2,-2,0)$, which is on the $\pi-$plane. The length of the vector out to *E* in Fig. 8.3.7 is $2\sqrt{3}$ "units". To convert to actual magnitudes, multiply by $\sqrt{\frac{2}{3}}$ to get $s = 2\sqrt{2}$, which agrees with $s = |\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

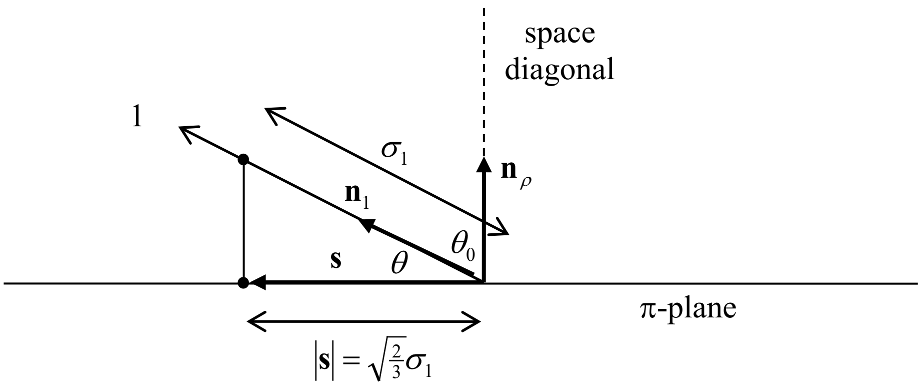


**Figure 8.3.8: principal stress projected onto the π-plane**

## Typical π-plane Yield Loci

Consider next an arbitrary point $(a,b,c)$ on the $\pi-$plane *yield locus*. If the material is isotropic, the points $(a,c,b)$, $(b,a,c)$, $(b,c,a)$, $(c,a,b)$ and $(c,b,a)$ are also on the yield locus. If one assumes the same yield behaviour in tension as in compression, e.g. neglecting the Bauschinger effect, then so also are the points $(-a,-b,-c)$, $(-a,-c,-b)$, etc. Thus 1 point becomes 12 and one need only consider the yield locus in one 30° sector of the $\pi-$plane, the rest of the locus being generated through symmetry. One such sector is shown in Fig. 8.3.9, the axes of symmetry being the three projected principal axes and their (pure shear) bisectors.

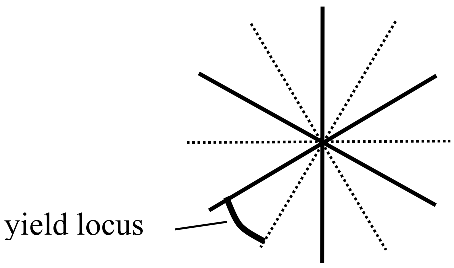


**Figure 8.3.9: A typical sector of the yield locus**

## The Tresca and Von Mises Yield Loci in the π-plane

The Tresca criterion, Eqn. 8.3.7, is a regular hexagon in the $\pi-$plane as illustrated in Fig. 8.3.10. Which of the six sides of the locus is relevant depends on which of $\sigma_1$, $\sigma_2$, $\sigma_3$ is the maximum and which is the minimum, and whether they are tensile or compressive.

For example, yield at the pure shear $\sigma_1 = Y/2, \sigma_2 = 0, \sigma_3 = -Y/2$ is indicated by point $A$ in the figure.

Point $B$ represents yield under uniaxial tension, $\sigma_1 = Y$. The distance $oB$, the "magnitude" of the hexagon, is therefore $\sqrt{\frac{2}{3}}Y$; the corresponding point on the $\pi-$plane is $(s_1, s_2 s_3) = \left(\frac{2}{3}Y, -\frac{1}{3}Y, -\frac{1}{3}Y\right)$.

A criticism of the Tresca criterion is that there is a sudden change in the planes upon which failure occurs upon a small change in stress at the sharp corners of the hexagon.

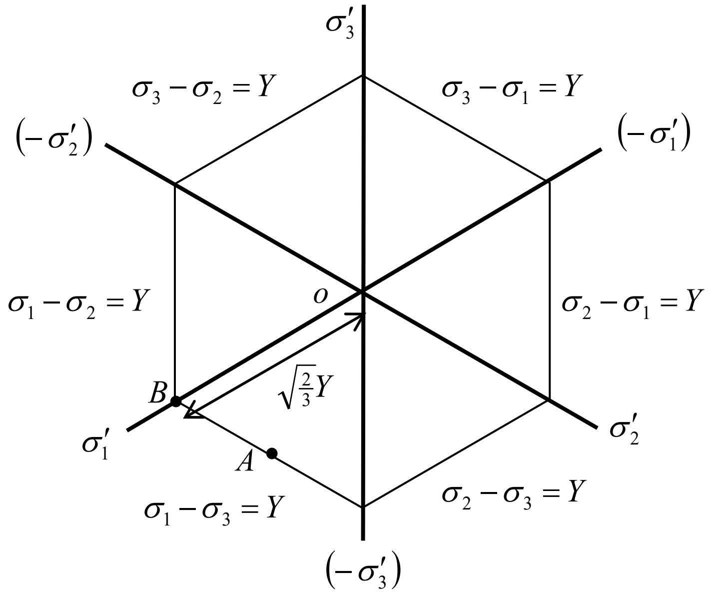


**Figure 8.3.10: The Tresca criterion in the π-plane**

Consider now the Von Mises criterion. From Eqns. 8.3.10, 8.3.13, the criterion is $\sqrt{J_2} = Y/\sqrt{3}$. From Eqn. 8.2.9, this can be re-written as

$$\sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{\frac{2}{3}}Y \tag{8.3.23}$$

Thus, the magnitude of the deviatoric stress vector is constant and one has a circular yield locus with radius $\sqrt{\frac{2}{3}}Y = \sqrt{2}k$, which circumscribes the Tresca hexagon, as illustrated in Fig. 8.3.11.

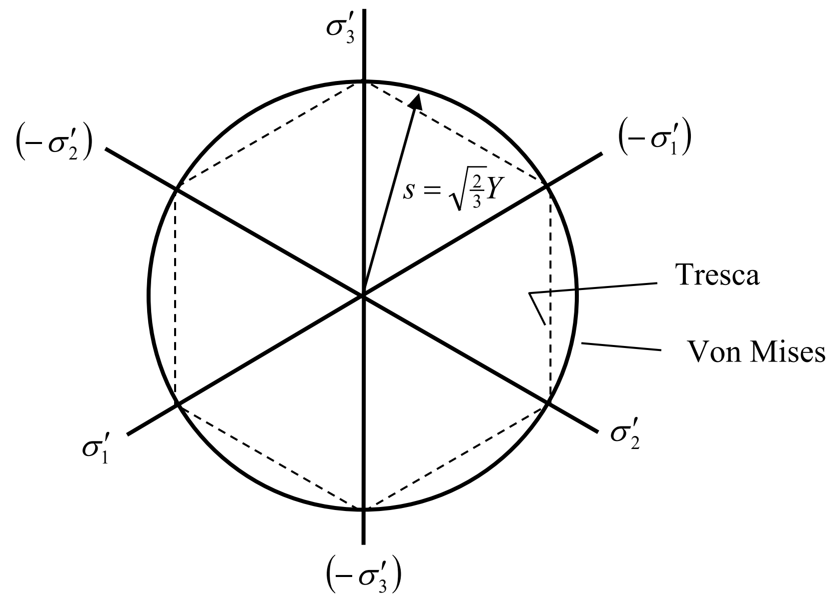


**Figure 8.3.11: The Von Mises criterion in the π-plane**

The yield surface is a circular cylinder with axis along the space diagonal, Fig. 8.3.12. The Tresca surface is a similar hexagonal cylinder.

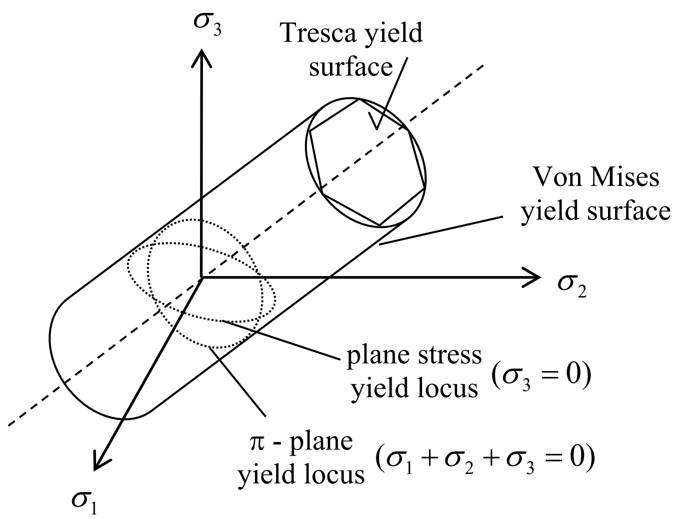


**Figure 8.3.12: The Von Mises and Tresca yield surfaces**

## 8.3.3 Haigh-Westergaard Stress Space

Thus far, yield criteria have been described in terms of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. It is often convenient to work with $(\rho, s, \theta)$ coordinates, Fig. 8.3.13; these cylindrical coordinates are called **Haigh-Westergaard coordinates**. They are particularly useful for describing and visualising geometrically pressure-dependent yield-criteria.

The coordinates $(\rho, s)$ are simply the magnitudes of, respectively, the hydrostatic stress vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$ and the deviatoric stress vector $\mathbf{s} = (s_1, s_2, s_3)$. These are given by ($|\mathbf{s}|$ can be obtained from Eqn. 8.2.9)

$$\rho = |\boldsymbol{\rho}| = \sqrt{3}\sigma_m = I_1 / \sqrt{3}, \qquad s = |\mathbf{s}| = \sqrt{2J_2} \tag{8.3.24}$$

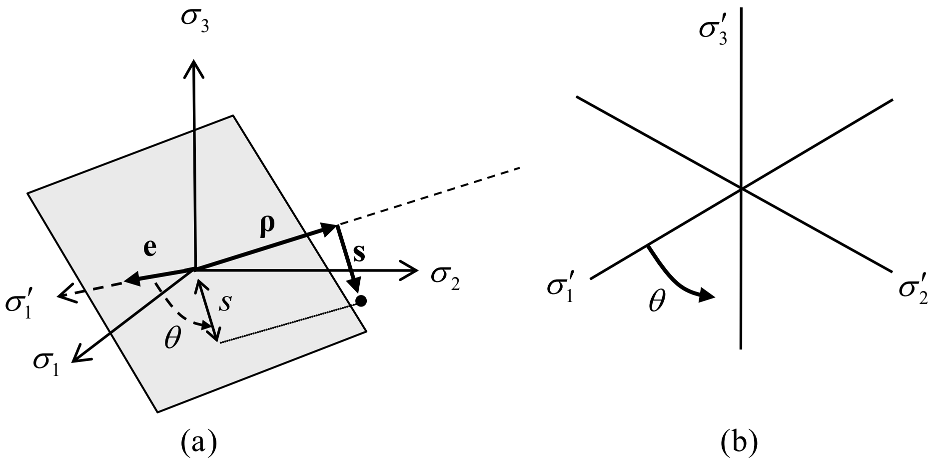


**Figure 8.3.13: A point in stress space**

$\theta$ is measured from the $\sigma_1'$ $(s_1)$ axis in the $\pi$ – plane. To express $\theta$ in terms of invariants, consider a unit vector $\mathbf{e}$ in the $\pi$ – plane in the direction of the $\sigma_1'$ axis; this is the same vector $\mathbf{n}_c$ considered in Fig. 8.2.4 in connection with the octahedral shear stress, and it has coordinates $(\sigma_1, \sigma_2, \sigma_2) = \left(\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$, Fig. 8.3.13. The angle $\theta$ can now be obtained from $\mathbf{s} \cdot \mathbf{e} = s\cos\theta$ {▲Problem 9}:

$$\cos\theta = \frac{\sqrt{3}s_1}{2\sqrt{J_2}} \tag{8.3.25}$$

Further manipulation leads to the relation {▲Problem 10}

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \tag{8.3.26}$$

Since $J_2$ and $J_3$ are invariant, it follows that $\cos 3\theta$ is also. Note that $J_3$ enters through $\cos 3\theta$, and does not appear in $\rho$ or $s$; it is $J_3$ which makes the yield locus in the $\pi$-plane non-circular.

From Eqn. 8.3.25 and Fig. 8.3.13b, the deviatoric stresses can be expressed in terms of the Haigh-Westergaard coordinates through

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \frac{2}{\sqrt{3}}\sqrt{J_2}\begin{bmatrix} \cos\theta \\ \cos(2\pi/3-\theta) \\ \cos(2\pi/3+\theta) \end{bmatrix} \tag{8.3.27}$$

The principal stresses and the Haigh-Westergaard coordinates can then be related through {▲Problem 12}

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} \rho \\ \rho \\ \rho \end{bmatrix} + \sqrt{\frac{2}{3}}s\begin{bmatrix} \cos\theta \\ \cos(\theta-2\pi/3) \\ \cos(\theta+2\pi/3) \end{bmatrix} \tag{8.3.28}$$

In terms of the Haigh-Westergaard coordinates, the yield criteria are

$$\begin{aligned} &\text{Von Mises:} \quad f(s) = \tfrac{1}{2}s^2 - k^2 = 0 \\ &\text{Tresca} \quad f(s,\theta) = \sqrt{2}s\sin\left(\theta + \tfrac{\pi}{3}\right) - Y = 0 \end{aligned} \tag{8.3.29}$$

## 8.3.4 Pressure Dependent Yield Criteria

The Tresca and Von Mises criteria are independent of hydrostatic pressure and are suitable for the modelling of plasticity in metals. For materials such as rock, soils and concrete, however, there is a strong dependence on the hydrostatic pressure.

### The Drucker-Prager Criteria

The **Drucker-Prager** criterion is a simple modification of the Von Mises criterion, whereby the hydrostatic-dependent first invariant $I_1$ is introduced to the Von Mises Eqn. 8.3.13:

$$f(I_1, J_2) \equiv \alpha I_1 + \sqrt{J_2} - k = 0 \tag{8.3.30}$$

with $\alpha$ is a new material parameter. On the $\pi$ – plane, $I_1 = 0$, and so the yield locus there is as for the Von Mises criterion, a circle of radius $\sqrt{2}k$, Fig. 8.3.14a. Off the $\pi$ – plane, the yield locus remains circular but the radius changes. When there is a state of pure hydrostatic stress, the magnitude of the hydrostatic stress vector is {▲Problem 13} $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$, with $s = |\mathbf{s}| = 0$. For large pressures, $\sigma_1 = \sigma_2 = \sigma_3 < 0$, the $I_1$ term in Eqn. 8.3.30 allows for large deviatoric stresses. This effect is shown in the **meridian plane** in Fig. 8.3.14b, that is, the $(\rho, s)$ plane which includes the $\sigma_1$ axis.

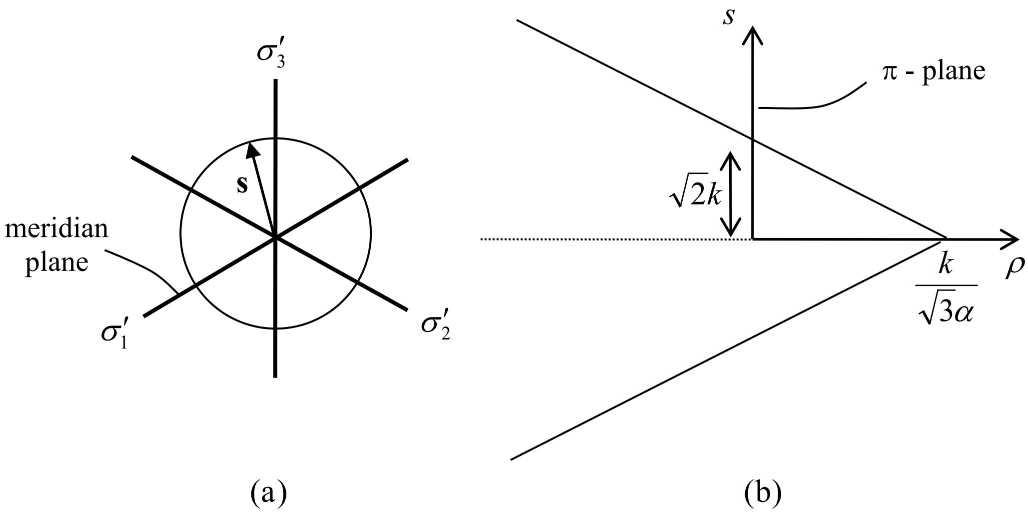


**Figure 8.3.14: The Drucker-Prager criterion; (a) the π-plane, (b) the Meridian Plane**

The Drucker-Prager surface is a right-circular cone with apex at $\rho = k/\sqrt{3}\alpha$ , Fig. 8.3.15. Note that the plane stress locus, where the cone intersects the $\sigma_3 = 0$ plane, is an ellipse, but whose centre is off-axis, at some $(\sigma_1 < 0, \sigma_2 < 0)$ .

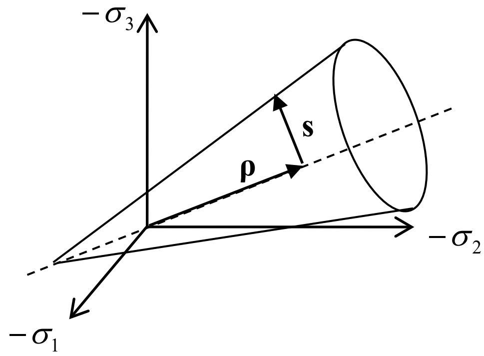


**Figure 8.3.15: The Drucker-Prager yield surface**

In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho, s) = \sqrt{6}\alpha\rho + s - \sqrt{2}k = 0 \tag{8.3.31}$$

## The Mohr Coulomb Criteria

The Mohr-Coulomb criterion is based on Coulomb's 1773 friction equation, which can be expressed in the form

$$|\tau| = c - \sigma_n \tan\phi \tag{8.3.32}$$

where $c, \phi$ are material constants; $c$ is called the **cohesion**[4] and $\phi$ is called the **angle of internal friction**. $\tau$ and $\sigma_n$ are the shear and normal stresses acting on the plane where failure occurs (through a shearing effect), Fig. 8.3.16, with $\tan\phi$ playing the role of a coefficient of friction. The criterion states that the larger the pressure $-\sigma_n$, the more shear the material can sustain. Note that the Mohr-Coulomb criterion can be considered to be a generalised version of the Tresca criterion, since it reduces to Tresca's when $\phi = 0$ with $c = k$.

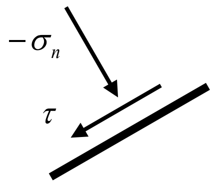


**Figure 8.3.16: Coulomb friction over a plane**

This criterion not only includes a hydrostatic pressure effect, but also allows for different yield behaviours in tension and in compression. Maintaining isotropy, there will now be three lines of symmetry in any deviatoric plane, and a typical sector of the yield locus is as shown in Fig. 8.3.17 (compare with Fig. 8.3.9)

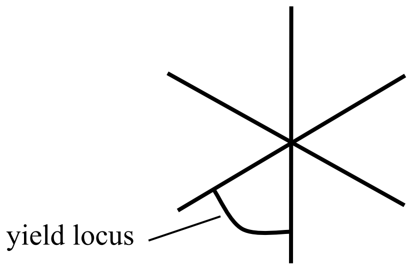


**Figure 8.3.17: A typical sector of the yield locus for an isotropic material with different yield behaviour in tension and compression**

Given values of $c$ and $\phi$, one can draw the failure locus (lines) of the Mohr-Coulomb criterion in $(\sigma_n, \tau)$ stress space, with intercepts $\tau = \pm c$ and slopes $\mp \tan\phi$, Fig. 8.3.18. Given some stress state $\sigma_1 \geq \sigma_2 \geq \sigma_3$, a Mohr stress circle can be drawn also in $(\sigma_n, \tau)$ space (see §7.2.6). When the stress state is such that this circle reaches out and touches the failure lines, yield occurs.

[4] $c = 0$ corresponds to a cohesionless material such as sand or gravel, which has no strength in tension

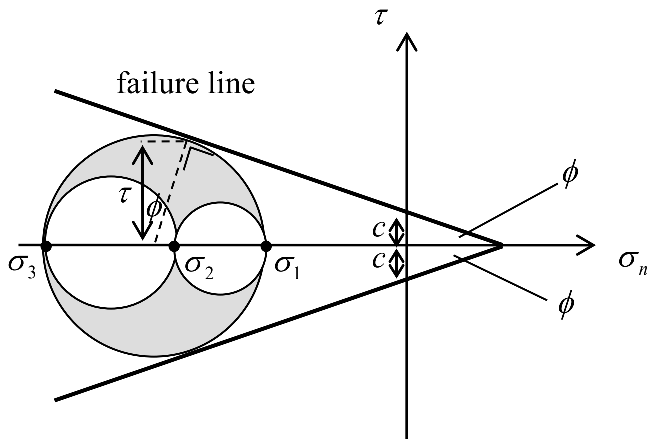


**Figure 8.3.18: Mohr-Coulomb failure criterion**

From Fig. 8.3.18, and noting that the large Mohr circle has centre $\left(\frac{1}{2}(\sigma_1 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_3)$, one has

$$\begin{aligned}\tau &= \frac{\sigma_1 - \sigma_3}{2}\cos\phi \\ \sigma_n &= \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi\end{aligned} \tag{8.3.33}$$

Thus the Mohr-Coulomb criterion in terms of principal stresses is

$$(\sigma_1 - \sigma_3) = 2c\cos\phi - (\sigma_1 + \sigma_3)\sin\phi \tag{8.3.34}$$

The strength of the Mohr-Coulomb material in uniaxial tension, $f_{Yt}$, and in uniaxial compression, $f_{Yc}$, are thus

$$f_{Yt} = \frac{2c\cos\phi}{1+\sin\phi}, \quad f_{Yc} = \frac{2c\cos\phi}{1-\sin\phi} \tag{8.3.35}$$

In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho, s, \theta) = \sqrt{2}\rho\sin\phi + \sqrt{3}s\sin\left(\theta + \tfrac{\pi}{3}\right) + s\cos\left(\theta + \tfrac{\pi}{3}\right)\sin\phi - \sqrt{6}c\cos\phi = 0 \tag{8.3.36}$$

The Mohr-Coulomb yield surface in the $\pi$ – plane and meridian plane are displayed in Fig. 8.3.19. In the $\pi$ – plane one has an irregular hexagon which can be constructed from two lengths: the magnitude of the deviatoric stress in uniaxial tension at yield, $s_{t0}$, and the corresponding (larger) value in compression, $s_{c0}$; these are given by:

$$s_{t0} = \frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3+\sin\phi}, \quad s_{c0} = \frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3-\sin\phi} \tag{8.3.37}$$

In the meridian plane, the failure surface cuts the $s = 0$ axis at $\rho = \sqrt{3} c \cot\phi$ {▲Problem 14}.

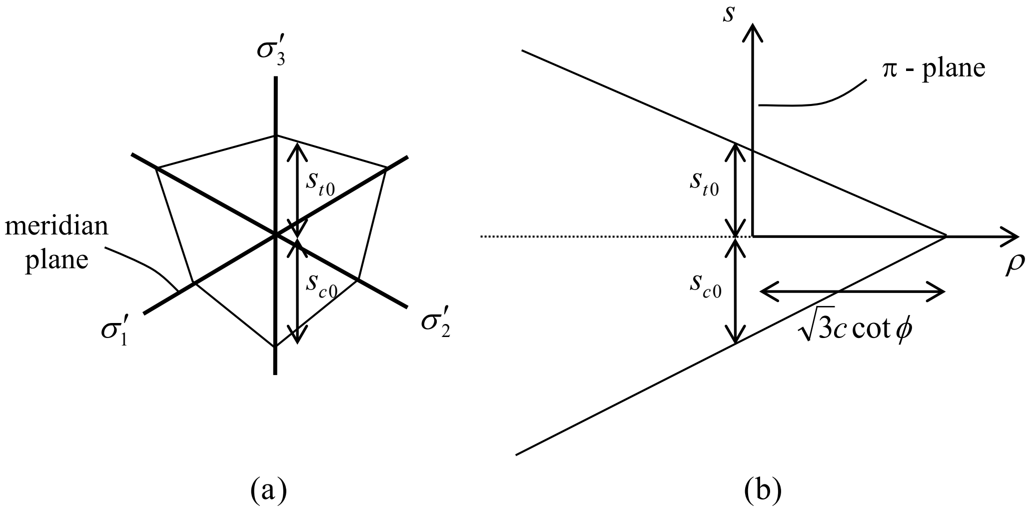


**Figure 8.3.19: The Mohr-Coulomb criterion; (a) the π-plane, (b) the Meridian Plane**

The Mohr-Coulomb surface is thus an irregular hexagonal pyramid, Fig. 8.3.20.

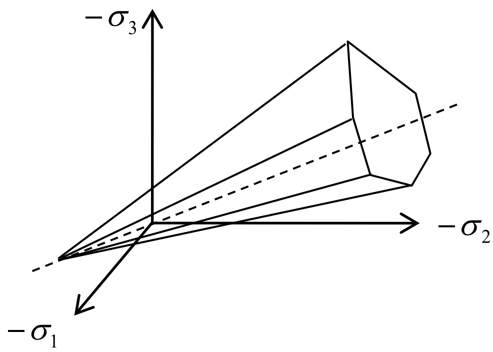


**Figure 8.3.20: The Mohr-Coulomb yield surface**

By adjusting the material parameters $\alpha, k, c, \phi$, the Drucker-Prager cone can be made to match the Mohr-Coulomb hexagon, either inscribing it at the minor vertices, or circumscribing it at the major vertices, Fig. 8.3.21.

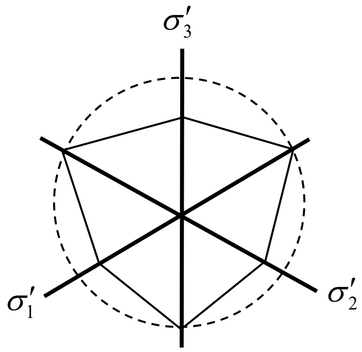


**Figure 8.3.21: The Mohr-Coulomb and Drucker-Prager criteria matched in the π-plane**

## Capped Yield Surfaces

The Mohr-Coulomb and Drucker-Prager surfaces are open in that a pure hydrostatic pressure can be applied without affecting yield. For many geomaterials, however, for example soils, a large enough hydrostatic pressure will induce permanent deformation. In these cases, a **closed (capped) yield surface** is more appropriate, for example the one illustrated in Fig. 8.3.22.

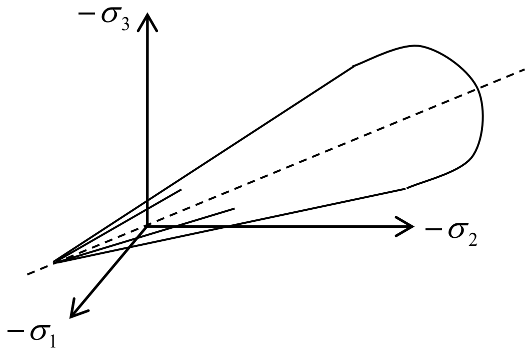


**Figure 8.3.22: a capped yield surface**

An example is the **modified Cam-Clay** criterion:

$$3J_2 = -\tfrac{1}{3}I_1 M^2\left(2p_c + \tfrac{1}{3}I_1\right) \text{ or } s = \sqrt{-\tfrac{2}{3\sqrt{3}}\rho M^2\left(2p_c + \tfrac{1}{\sqrt{3}}\rho\right)}, \quad \rho < 0 \tag{8.3.38}$$

with $M$ and $p_c$ material constants. In terms of the standard geomechanics notation, it reads

$$q^2 = M^2 p\left(2p_c - p\right) \tag{8.3.39}$$

where

$$p = -\frac{1}{3} I_1 = -\frac{1}{\sqrt{3}}\rho, \qquad q = \sqrt{3J_2} = \sqrt{\frac{3}{2}} s \tag{8.3.40}$$

The modified Cam-Clay locus in the meridian plane is shown in Fig. 8.3.23. Since $s$ is constant for any given $\rho$, the locus in planes parallel to the π - plane are circles. The material parameter $p_c$ is called the **critical state pressure**, and is the pressure which carries the maximum deviatoric stress. M is the slope of the dotted line shown in Fig. 8.3.23, known as the **critical state line**.

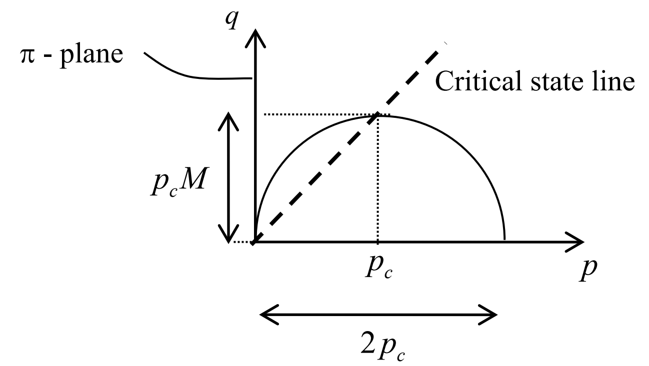


**Figure 8.3.23: The modified Cam-Clay criterion in the Meridian Plane**

## 8.3.5 Anisotropy

Many materials will display anisotropy. For example metals which have been processed by rolling will have characteristic material directions, the tensile yield stress in the direction of rolling being typically 15% greater than that in the transverse direction. The form of anisotropy exhibited by rolled sheets is such that the material properties are symmetric about three mutually orthogonal planes. The lines of intersection of these planes form an orthogonal set of axes known as the **principal axes of anisotropy**. The axes are (a) in the rolling direction, (b) normal to the sheet, (c) in the plane of the sheet but normal to rolling direction. This form of anisotropy is called **orthotropy** (see Part I, §6.2.2). Hill (1948) proposed a yield condition for such a material which is a natural generalisation of the Mises condition:

$$\begin{aligned} f(\sigma_{ij}) = F\left(\sigma_{22}-\sigma_{33}\right)^2 + G\left(\sigma_{33}-\sigma_{11}\right)^2 + H\left(\sigma_{11}-\sigma_{22}\right)^2 \\ + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 - 1 = 0 \end{aligned} \tag{8.3.41}$$

where $F, G, H, L, M, N$ are material constants. One needs to carry out 6 tests: uniaxial tests in the three coordinate directions to find the uniaxial yield strengths $(\sigma_Y)_x, (\sigma_Y)_y, (\sigma_Y)_z$, and shear tests to find the shear strengths $(\tau_Y)_{xy}, (\tau_Y)_{yz}, (\tau_Y)_{zx}$. For a uniaxial test in the $x$ direction, Eqn. 8.3.41 reduces to $G + H = 1/(\sigma_Y)_x^2$. By considering the other simple uniaxial and shear tests, one can solve for the material parameters:

$$F = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_y^2} + \frac{1}{(\sigma_Y)_z^2} - \frac{1}{(\sigma_Y)_x^2}\right] \qquad L = \frac{1}{2}\frac{1}{(\tau_Y)_{yz}^2}$$

$$G = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_z^2} + \frac{1}{(\sigma_Y)_x^2} - \frac{1}{(\sigma_Y)_y^2}\right] \qquad M = \frac{1}{2}\frac{1}{(\tau_Y)_{zx}^2} \qquad (8.3.42)$$

$$H = \frac{1}{2}\left[\frac{1}{(\sigma_Y)_x^2} + \frac{1}{(\sigma_Y)_y^2} - \frac{1}{(\sigma_Y)_z^2}\right] \qquad N = \frac{1}{2}\frac{1}{(\tau_Y)_{xy}^2}$$

The criterion reduces to the Mises condition 8.3.12 when

$$F = G = H = \frac{L}{3} = \frac{M}{3} = \frac{N}{3} = \frac{1}{6k^2} \qquad (8.3.43)$$

The 1, 2, 3 axes of reference in Eqn. 8.3.41 are the principal axes of anisotropy. The form appropriate for a general choice of axes can be derived by using the usual stress transformation formulae. It is complicated and involves cross-terms such as $\sigma_{11}\sigma_{23}$, etc.

## 8.3.6 Problems

1. A material is to be loaded to a stress state
$$[\sigma_{ij}] = \begin{bmatrix} 50 & -30 & 0 \\ -30 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$
What should be the minimum uniaxial yield stress of the material so that it does not fail, according to the
(a) Tresca criterian
(b) Von Mises criterion
What do the theories predict when the yield stress of the material is 80MPa?

2. Use Eqn. 8.2.6, $J_2 = s_{12}^2 + s_{23}^2 + s_{31}^2 - (s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11})$ to derive Eqn. 8.3.12, $(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 6k^2$, for the Von Mises criterion.

3. Use the plane stress principal stress formula $\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$ to derive Eqn. 8.3.15 for the Taylor-Quinney tests.

4. Derive Eqn. 8.3.17 for the Taylor-Quinney tests.

5. Describe the states of stress represented by the points D and E in Fig. 8.3.4. (The complete stress states can be visualised with the help of Mohr's circles of stress, Fig. 7.2.17.)

6. Suppose that, in the Taylor and Quinney tension-torsion tests, one has $\sigma = Y/2$ and $\tau = \sqrt{3}Y/4$. Plot this stress state in the 2D principal stress state, Fig. 8.3.4. (Use Eqn. 8.3.15 to evaluate the principal stresses.) Keeping now the normal stress at $\sigma = Y/2$, what value can the shear stress be increased to before the material yields, according to the von Mises criterion?

7. What are the $\pi-$plane principal stress values for the points *B* and *C* in Fig. 8.3.7?

8. Sketch on the $\pi-$plane Fig. 8.3.7 a line corresponding to $\sigma_1 = \sigma_2$ and also a region corresponding to $\sigma_1 > \sigma_2 > 0 > \sigma_3$

9. Using the relation $\mathbf{s}\cdot\mathbf{e} = s\cos\theta$ and $s_2 + s_3 = -s_1$, derive Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$.

10. Using the trigonometric relation $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$, show that $\cos 3\theta = \dfrac{3\sqrt{3}s_1}{2J_2^{3/2}}\left(s_1^2 - J_2\right)$. Then using the relations 8.2.6, $J_2 = -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right)$, with $J_1 = 0$, derive Eqn. 8.3.26, $\cos 3\theta = \dfrac{3\sqrt{3}}{2}\dfrac{J_3}{J_2^{3/2}}$

11. Consider the following stress states. For each one, evaluate the space coordinates $(\rho, s, \theta)$ and plot in the $\pi-$plane (see Fig. 8.3.13b):
    (a) triaxial tension: $\sigma_1 = T_1 > \sigma_2 = \sigma_3 = T_2 > 0$
    (b) triaxial compression: $\sigma_3 = -p_1 < \sigma_1 = \sigma_2 = -p_2 < 0$ (this is an important test for geomaterials, which are dependent on the hydrostatic pressure)
    (c) a pure shear $\sigma_{xy} = \tau$ : $\sigma_1 = +\tau,\ \sigma_2 = 0,\ \sigma_3 = -\tau$
    (d) a pure shear $\sigma_{xy} = \tau$ in the presence of hydrostatic pressure *p*:
    $\sigma_1 = -p + \tau,\ \sigma_2 = -p,\ \sigma_3 = -p - \tau$, i.e. $\sigma_1 - \sigma_2 = \sigma_2 - \sigma_3$

12. Use relations 8.3.24, $\rho = I_1/\sqrt{3},\ s = \sqrt{2J_2}$ and Eqns. 8.3.27 to derive Eqns. 8.3.28.

13. Show that the magnitude of the hydrostatic stress vector is $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$ for the Drucker-Prager yield criterion when the deviatoric stress is zero

14. Show that the magnitude of the hydrostatic stress vector is $\rho = \sqrt{3}c\cot\phi$ for the Mohr-Coulomb yield criterion when the deviatoric stress is zero

15. Show that, for a Mohr-Coulomb material, $\sin\phi = (r-1)/(r+1)$, where $r = f_{Yc}/f_{Yt}$ is the compressive to tensile strength ratio

16. A sample of concrete is subjected to a stress $\sigma_{11} = \sigma_{22} = -p,\ \sigma_{33} = -Ap$ where the constant $A > 1$. Using the Mohr-Coulomb criterion and the result of Problem 15, show that the material will not fail provided $A < f_{Yc}/p + r$