8.2 Stress Analysis for Plasticity

This section follows on from the analysis of three dimensional stress carried out in §7.2. The plastic behaviour of materials is often independent of a **hydrostatic stress** and this feature necessitates the study of the **deviatoric stress**.

8.2.1 Deviatoric Stress

Any state of stress can be decomposed into a **hydrostatic** (or **mean**) **stress** $\sigma_m \mathbf{I}$ and a **deviatoric stress s**, according to

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{8.2.1}$$

where

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \tag{8.2.2}$$

and

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix} \tag{8.2.3}$$

In index notation,

$$\sigma_{ij} = \sigma_m \delta_{ij} + s_{ij} \tag{8.2.4}$$

In a completely analogous manner to the derivation of the principal stresses and the principal scalar invariants of the stress matrix, §7.2.4, one can determine the principal stresses and principal scalar invariants of the deviatoric stress matrix. The former are denoted s_1, s_2, s_3 and the latter are denoted by J_1, J_2, J_3. The characteristic equation analogous to Eqn. 7.2.23 is

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \tag{8.2.5}$$

and the deviatoric invariants are (compare with 7.2.24, 7.2.26)[1]

[1] unfortunately, there is a convention (adhered to by most authors) to write the characteristic equation for stress with a $+I_2\sigma$ term and that for deviatoric stress with a $-J_2 s$ term; this means that the formulae for J_2 in Eqn. 8.2.5 are the negative of those for I_2 in Eqn. 7.2.24

$$\begin{aligned} J_1 &= s_{11} + s_{22} + s_{33} \\ &= s_1 + s_2 + s_3 \\ J_2 &= -\left(s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2\right) \\ &= -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right) \\ J_3 &= s_{11}s_{22}s_{33} - s_{11}s_{23}^2 - s_{22}s_{31}^2 - s_{33}s_{12}^2 + 2s_{12}s_{23}s_{31} \\ &= s_1 s_2 s_3 \end{aligned} \tag{8.2.6}$$

Since the hydrostatic stress remains unchanged with a change of coordinate system, the principal directions of stress coincide with the principal directions of the deviatoric stress, and the decomposition can be expressed with respect to the principal directions as

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \tag{8.2.7}$$

Note that, from the definition Eqn. 8.2.3, the first invariant of the deviatoric stress, the sum of the normal stresses, is zero:

$$J_1 = 0 \tag{8.2.8}$$

The second invariant can also be expressed in the useful forms {▲Problem 3}

$$J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right), \tag{8.2.9}$$

and, in terms of the principal stresses, {▲Problem 4}

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \tag{8.2.10}$$

Further, the deviatoric invariants are related to the stress tensor invariants through {▲Problem 5}

$$J_2 = \tfrac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \tfrac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27I_3\right) \tag{8.2.11}$$

A State of Pure Shear

The stress state at a point is one of **pure shear** if for *any* one coordinate axes through the point one has only shear stress acting, i.e. the stress matrix is of the form

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix} \tag{8.2.12}$$

Applying the stress transformation rule 7.2.16 to this stress matrix and using the fact that the transformation matrix $\mathbf{Q}$ is orthogonal, i.e. $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, one finds that the first invariant is zero, $\sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$. Hence the deviatoric stress is one of pure shear.

8.2.2 The Octahedral Stresses

Examine now a material element subjected to principal stresses $\sigma_1, \sigma_2, \sigma_3$ as shown in Fig. 8.2.1. By definition, no shear stresses act on the planes shown.

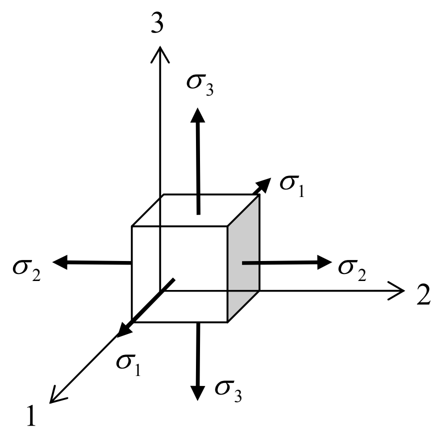


Figure 8.2.1: stresses acting on a material element

Consider next the **octahedral plane**; this is the plane shown shaded in Fig. 8.2.2, whose normal $\mathbf{n}_a$ makes equal angles with the principal directions. It is so-called because it cuts a cubic material element (with faces perpendicular to the principal directions) into a triangular plane and eight of these triangles around the origin form an octahedron.

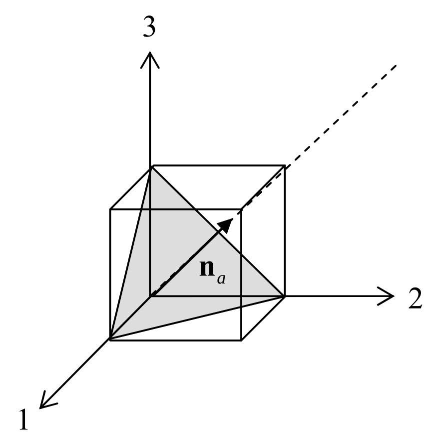


Figure 8.2.2: the octahedral plane

Next, a new Cartesian coordinate system is constructed with axes parallel and perpendicular to the octahedral plane, Fig. 8.2.3. One axis runs along the unit normal $\mathbf{n}_a$;

this normal has components $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$ with respect to the principal axes. The angle θ_0 the normal direction makes with the 1 direction can be obtained from $\mathbf{n}_a \cdot \mathbf{e}_1 = \cos\theta_0$, where $\mathbf{e}_1 = (1,0,0)$ is a unit vector in the 1 direction, Fig. 8.2.3. To complete the new coordinate system, any two perpendicular unit vectors which lie in (parallel to) the octahedral plane can be chosen. Choose one which is along the projection of the 1 axis down onto the octahedral plane. The components of this vector are {▲Problem 6} $\mathbf{n}_c = \left(\sqrt{2/3}, -1/\sqrt{6}, -1/\sqrt{6}\right)$. The final unit vector $\mathbf{n}_b$ is chosen so that it forms a right hand Cartesian coordinate system with $\mathbf{n}_a$ and $\mathbf{n}_c$, i.e. $\mathbf{n}_a \times \mathbf{n}_b = \mathbf{n}_c$. In summary,

$$\mathbf{n}_a = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \quad \mathbf{n}_c = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\-1\\-1\end{bmatrix} \tag{8.2.13}$$

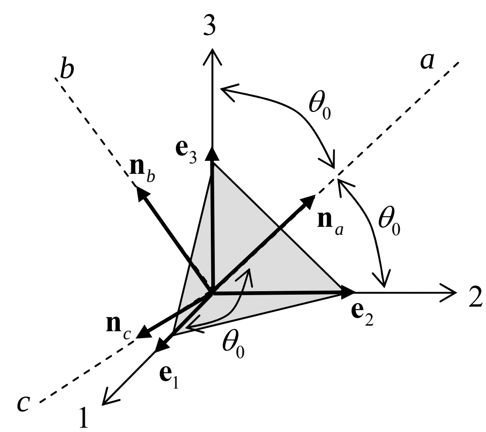


Figure 8.2.3: a new Cartesian coordinate system

To express the stress state in terms of components in the a, b, c directions, construct the stress transformation matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{n}_a & \mathbf{e}_1 \cdot \mathbf{n}_b & \mathbf{e}_1 \cdot \mathbf{n}_c \\ \mathbf{e}_2 \cdot \mathbf{n}_a & \mathbf{e}_2 \cdot \mathbf{n}_b & \mathbf{e}_2 \cdot \mathbf{n}_c \\ \mathbf{e}_3 \cdot \mathbf{n}_a & \mathbf{e}_3 \cdot \mathbf{n}_b & \mathbf{e}_3 \cdot \mathbf{n}_c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \tag{8.2.14}$$

and the new stress components are

$$\begin{aligned}\begin{bmatrix} \sigma_{aa} & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ba} & \sigma_{bb} & \sigma_{bc} \\ \sigma_{ca} & \sigma_{cb} & \sigma_{cc} \end{bmatrix} &= \mathbf{Q}^T \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \mathbf{Q} \\ &= \begin{bmatrix} \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) & -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{2}(\sigma_2 + \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) & \frac{1}{6}(4\sigma_1 + \sigma_2 + \sigma_3) \end{bmatrix}\end{aligned} \tag{8.2.15}$$

Now consider the stress components acting on the octahedral plane, $\sigma_{aa}, \sigma_{ab}, \sigma_{ac}$, Fig. 8.2.4. Recall from Cauchy's law, Eqn. 7.2.9, that these are the components of the traction vector $\mathbf{t}^{(\mathbf{n}_a)}$ acting on the octahedral plane, with respect to the (a,b,c) axes:

$$\mathbf{t}^{(\mathbf{n}_a)} = \sigma_{aa}\mathbf{n}_a + \sigma_{ab}\mathbf{n}_b + \sigma_{ac}\mathbf{n}_c \tag{8.2.16}$$

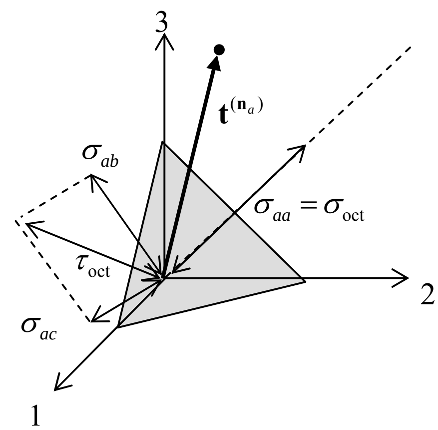


Figure 8.2.4: the stress vector σ and its components

The magnitudes of the normal and shear stresses acting on the octahedral plane are called the **octahedral normal stress** σ_{oct} and the **octahedral shear stress** τ_{oct}. Referring to Fig. 8.2.4, these can be expressed as {▲Problem 7}

$$\begin{aligned} \sigma_{oct} &= \sigma_{aa} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1 \\ \tau_{oct} &= \sqrt{\sigma_{ab}^2 + \sigma_{ac}^2} \\ &= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\frac{2J_2}{3}} \end{aligned} \tag{8.2.17}$$

The octahedral normal and shear stresses on all 8 octahedral planes around the origin are the same.

Note that the octahedral normal stress is simply the hydrostatic stress. This implies that the deviatoric stress has no normal component in the direction $\mathbf{n}_a$ and only contributes to shearing on the octahedral plane. Indeed, from Eqn. 8.2.15,

$$\begin{bmatrix} s_{aa} & s_{ab} & s_{ac} \\ s_{ba} & s_{bb} & s_{bc} \\ s_{ca} & s_{cb} & s_{cc} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & -\frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) & \frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) \end{bmatrix} \tag{8.2.18}$$

The σ 's on the right here can be replaced with s 's since $\sigma_i - \sigma_j = s_i - s_j$.

8.2.3 Problems

1. What are the hydrostatic and deviatoric stresses for the uniaxial stress $\sigma_{11} = \sigma_0$? What are the hydrostatic and deviatoric stresses for the state of pure shear $\sigma_{12} = \tau$? In both cases, verify that the first invariant of the deviatoric stress is zero: $J_1 = 0$.

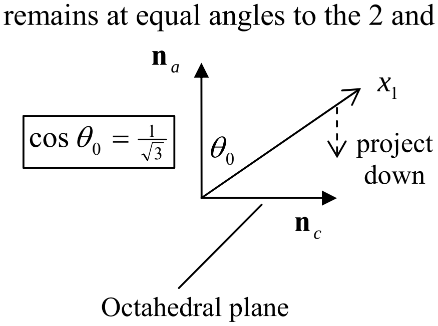
2. For the stress state $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$, calculate
   (a) the hydrostatic stress
   (b) the deviatoric stresses
   (c) the deviatoric invariants

3. The second invariant of the deviatoric stress is given by Eqn. 8.2.6,
$$J_2 = -(s_1 s_2 + s_2 s_3 + s_3 s_1)$$
By squaring the relation $J_1 = s_1 + s_2 + s_3 = 0$, derive Eqn. 8.2.9,
$$J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right)$$

4. Use Eqns. 8.2.9 (and your work from Problem 3) and the fact that $\sigma_1 - \sigma_2 = s_1 - s_2$, etc. to derive 8.2.10,
$$J_2 = \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

5. Use the fact that $J_1 = s_1 + s_2 + s_3 = 0$ to show that
$$\begin{aligned} I_1 &= 3\sigma_m \\ I_2 &= (s_1 s_2 + s_2 s_3 + s_3 s_1) + 3\sigma_m^2 \\ I_3 &= s_1 s_2 s_3 + \sigma_m (s_1 s_2 + s_2 s_3 + s_3 s_1) + \sigma_m^3 \end{aligned}$$
Hence derive Eqns. 8.2.11,
$$J_2 = \tfrac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \tfrac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27 I_3\right)$$

6. Show that a unit normal $\mathbf{n}_c$ in the octahedral plane in the direction of the projection of the 1 axis down onto the octahedral plane has coordinates $\left(\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$, Fig. 8.2.3. To do this, note the geometry shown below and the fact that when the 1 axis is projected down, it remains at equal angles to the 2 and 3 axes.



7. Use Eqns. 8.2.15 to derive Eqns. 8.2.17.

8. For the stress state of problem 2, calculate the octahedral normal stress and the octahedral shear stress