7 3D Elasticity

7.1 Vectors, Tensors and the Index Notation

The equations governing three dimensional mechanics problems can be quite lengthy. For this reason, it is essential to use a short-hand notation called the **index notation**[1]. Consider first the notation used for vectors.

7.1.1 Vectors

Vectors are used to describe physical quantities which have both a magnitude and a direction associated with them. Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow. Analytically, in what follows, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **b**.

The **dot product** of two vectors **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \tag{7.1.1}$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$.

Cartesian Coordinate System

So far the short discussion has been in **symbolic notation**[2], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. Vectors exist independently of any coordinate system. The symbolic notation is very useful, but there are many circumstances in which use of the component forms of vectors is more helpful – or essential. To this end, introduce the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \tag{7.1.2}$$

so that they are mutually perpendicular, and

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \tag{7.1.3}$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 7.1.1. This set of vectors forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{7.1.4}$$

where a_1, a_2 and a_3 are scalars, called the **Cartesian components** or **coordinates** of **a** along the given three directions. The unit vectors are called **base vectors** when used for

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[1] or **indicial** or **subscript** or **suffix** notation

[2] or **absolute** or **invariant** or **direct** or **vector** notation

this purpose. The components a_1, a_2 and a_3 are measured along lines called the x_1, x_2 and x_3 **axes**, drawn through the base vectors.

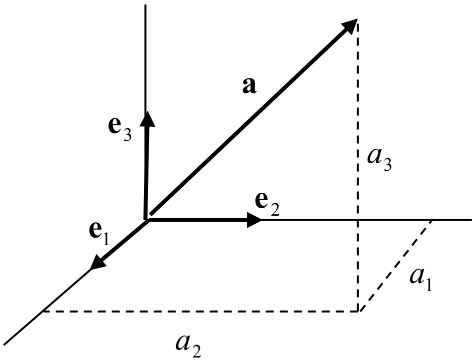

Figure 7.1.1: an orthonormal set of base vectors and Cartesian coordinates

Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \tag{7.1.5}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (*i* in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i\mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors **u** and **v**, referred to this coordinate system, is

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)\\ &= u_1v_1(\mathbf{e}_1\cdot\mathbf{e}_1) + u_1v_2(\mathbf{e}_1\cdot\mathbf{e}_2) + u_1v_3(\mathbf{e}_1\cdot\mathbf{e}_3)\\ &\quad + u_2v_1(\mathbf{e}_2\cdot\mathbf{e}_1) + u_2v_2(\mathbf{e}_2\cdot\mathbf{e}_2) + u_2v_3(\mathbf{e}_2\cdot\mathbf{e}_3)\\ &\quad\quad + u_3v_1(\mathbf{e}_3\cdot\mathbf{e}_1) + u_3v_2(\mathbf{e}_3\cdot\mathbf{e}_2) + u_3v_3(\mathbf{e}_3\cdot\mathbf{e}_3)\\ &= u_1v_1 + u_2v_2 + u_3v_3\end{aligned} \tag{7.1.6}$$

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_iv_i} \quad \textbf{Dot Product} \tag{7.1.7}$$

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[3] 2 in the case of a two-dimensional space/analysis

The repeated index *i* is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u} \cdot \mathbf{v} = u_j v_j$ or $u_k v_k$.

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{7.1.8}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (7.1.2, 7.1.3) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \quad \textbf{Orthonormal Basis Rule} \tag{7.1.9}$$

Example

Recall the equations of motion, Eqns. 1.1.9, which in full read

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= \rho a_1 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= \rho a_2 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \rho a_3 \end{aligned} \tag{7.1.10}$$

The index notation for these equations is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.1.11}$$

Note the dummy index *j*. The index *i* is called a **free index**; if one term has a free index *i*, then, to be consistent, all terms must have it. One free index, as here, indicates three separate equations.

7.1.2 Matrix Notation

The symbolic notation $\mathbf{v}$ and index notation $v_i \mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector **v** can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the index notation v_i.

Note the distinction between a vector and a 3×1 matrix: the former is a mathematical object independent of any coordinate system, the latter is a representation of the vector in a particular coordinate system – matrix notation, as with the index notation, relies on a particular coordinate system.

As an example, the dot product can be written in the matrix notation as

$$[\mathbf{u}^{\mathrm{T}}][\mathbf{v}] = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

“short”
matrix notation

“full”
matrix notation

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, $u_i v_i$.

The matrix notation for the Kronecker delta δ_{ij} is the identity matrix

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, for example, in both index and matrix notation:

$$\delta_{ij} u_j = u_i \qquad [\mathbf{I}][\mathbf{u}] = [\mathbf{u}] \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{7.1.12}$$

Matrix – Matrix Multiplication

When discussing vector transformation equations further below, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in the short-hand notation.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$ or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

This operation is called the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u} \otimes \mathbf{v}$ (or simply **uv**).

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij} u_j$. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which can be expressed as $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 4} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ → the vector is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector → the vector is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the *j* in $u_j Q_{ji}$ or $Q_{ij} u_j$
   → the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the *j* in $u_j Q_{ij}$
   → the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 8} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik} B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki} B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \qquad (7.1.13)$$

Note also the following:

(i) if there is no free index, as in $u_i v_i$, there is one element

(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix

(iii) if there are two free indices, as in $A_{ki} B_{kj}$, it is a 3×3 matrix

7.1.3 Vector Transformation Rule

Introduce two Cartesian coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ and common origin *o*, Fig. 7.1.2. The vector **u** can then be expressed in two ways:

$$\mathbf{u} = u_i \mathbf{e}_i = u'_i \mathbf{e}'_i \tag{7.1.14}$$

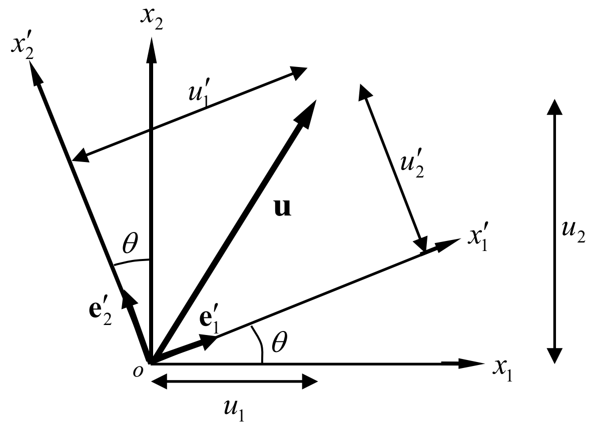

Figure 7.1.2: a vector represented using two different coordinate systems

Note that the x'_i coordinate system is obtained from the x_i system by a *rotation* of the base vectors. Fig. 7.1.2 shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 7.1.3),

$$\begin{aligned} \mathbf{u} &= u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \\ &= \left[|OB| - |AB| \right] \mathbf{e}_1 + \left[|BD| + |CP| \right] \mathbf{e}_2 \\ &= \left[\cos\theta\, u'_1 - \sin\theta\, u'_2 \right] \mathbf{e}_1 + \left[\sin\theta\, u'_1 + \cos\theta\, u'_2 \right] \mathbf{e}_2 \end{aligned} \tag{7.1.15}$$

and so

$$\begin{aligned} u_1 &= \cos\theta\, u'_1 - \sin\theta\, u'_2 \\ u_2 &= \sin\theta\, u'_1 + \cos\theta\, u'_2 \end{aligned} \tag{7.1.16}$$

vector components in first coordinate system

vector components in second coordinate system

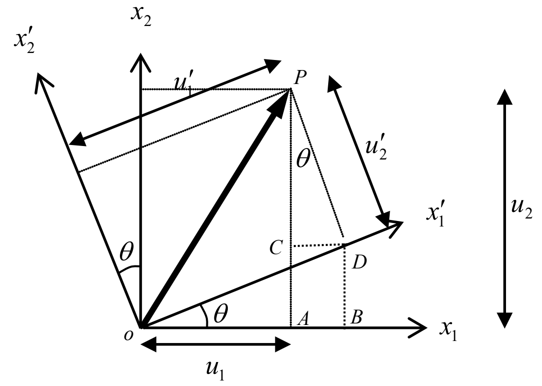

Figure 7.1.3: geometry of the 2D coordinate transformation

In matrix form, these transformation equations can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \tag{7.1.17}$$

The 2×2 matrix is called the **transformation matrix** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.1.18}$$

It can be seen that the components of $[\mathbf{Q}]$ are the **directions cosines**, i.e. the cosines of the angles between the coordinate directions:

$$Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{7.1.19}$$

It is straight forward to show that, in the full three dimensions, Fig. 7.1.4, the components in the two coordinate systems are also related through

$$\boxed{\begin{array}{ll} u_i = Q_{ij} u_j' & \cdots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \\ u_i' = Q_{ji} u_j & \cdots \quad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{array}} \quad \textbf{Vector Transformation Rule} \tag{7.1.20}$$

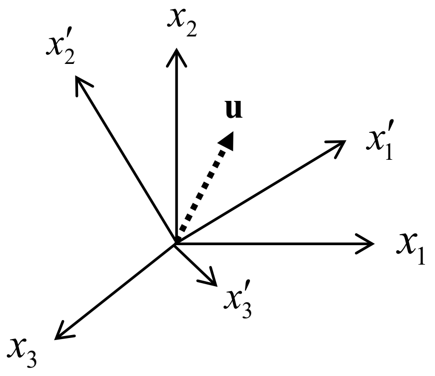

Figure 7.1.4: two different coordinate systems in a 3D space

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

From 7.1.20, it follows that

$$\begin{aligned} u_i &= Q_{ij} u_j' && \cdots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \\ &= Q_{ij} Q_{kj} u_k && \cdots \quad\quad = [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{aligned} \tag{7.1.21}$$

and so

$$Q_{ij} Q_{kj} = \delta_{ik} \qquad \cdots \qquad [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}] = [\mathbf{I}] \tag{7.1.22}$$

A matrix such as this for which $[\mathbf{Q}^{\mathrm{T}}] = [\mathbf{Q}^{-1}]$ is called an **orthogonal matrix**.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\begin{aligned} \mathbf{e}_1' &= a_1^{(1)} \mathbf{e}_1 + a_2^{(1)} \mathbf{e}_2 + a_3^{(1)} \mathbf{e}_3 \\ \mathbf{e}_2' &= a_1^{(2)} \mathbf{e}_1 + a_2^{(2)} \mathbf{e}_2 + a_3^{(2)} \mathbf{e}_3 \\ \mathbf{e}_3' &= a_1^{(3)} \mathbf{e}_1 + a_2^{(3)} \mathbf{e}_2 + a_3^{(3)} \mathbf{e}_3 \end{aligned}$$

What is the transformation matrix?

Solution

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j'$, so

$$[\mathbf{Q}] = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} \end{bmatrix}$$

■

7.1.4 Tensors

The concept of the **tensor** is discussed in detail in Book III, where it is indispensable for the description of large-strain deformations. For small deformations, it is not so necessary; the main purpose for introducing the tensor here (in a rather non-rigorous way) is that it helps to deepen one's understanding of the concept of stress.

A **second-order tensor**[4] **A** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or

$$\boxed{\mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (7.1.23)$$

The second-order tensor **T** is a **linear operator**, by which is meant

$$\begin{aligned} &\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb} && \ldots \text{ distributive} \\ &\mathbf{T}(\alpha\mathbf{a}) = \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

for scalar α. In a Cartesian coordinate system, the tensor **T** has nine components and can be represented in the matrix form

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The rule 7.1.23, which is expressed in symbolic notation, can be expressed in the index and matrix notation when **T** is referred to particular axes:

$$u_i = T_{ij} v_j \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad (7.1.24)$$

Again, one should be careful to distinguish between a tensor such as **T** and particular matrix representations of that tensor. The relation 7.1.23 is a **tensor relation**, relating vectors and a tensor and is valid in all coordinate systems; the matrix representation of this tensor relation, Eqn. 7.1.24, is to be sure valid in all coordinate systems, but the entries in the matrices of 7.1.24 depend on the coordinate system chosen.

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[4] to be called simply a tensor in what follows

Note also that the transformation formulae for vectors, Eqn. 7.1.20, is not a tensor relation; although 7.1.20 looks similar to the tensor relation 7.1.24, the former relates the components of a vector to the components of the *same* vector in different coordinate systems, whereas (by definition of a tensor) the relation 7.1.24 relates the components of a vector to those of a different vector in the same coordinate system.

For these reasons, the notation $u_i = Q_{ij}u'$ in Eqn. 7.1.20 is more formally called **element form**, the Q_{ij} being elements of a matrix rather than components of a tensor. This distinction between element form and index notation should be noted, but the term "index notation" is used for both tensor and matrix-specific manipulations in these notes.

Example

Recall the strain-displacement relations, Eqns. 1.2.19, which in full read

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ &\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \end{aligned} \tag{7.1.25}$$

The index notation for these equations is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.1.26}$$

This expression has two free indices and as such indicates nine separate equations. Further, with its two subscripts, ε_{ij}, the strain, is a tensor. It can be expressed in the matrix notation

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \partial u_1/\partial x_1 & \frac{1}{2}(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) & \frac{1}{2}(\partial u_1/\partial x_3 + \partial u_3/\partial x_1) \\ \frac{1}{2}(\partial u_2/\partial x_1 + \partial u_1/\partial x_2) & \partial u_2/\partial x_2 & \frac{1}{2}(\partial u_2/\partial x_3 + \partial u_3/\partial x_2) \\ \frac{1}{2}(\partial u_3/\partial x_1 + \partial u_1/\partial x_3) & \frac{1}{2}(\partial u_3/\partial x_2 + \partial u_2/\partial x_3) & \partial u_3/\partial x_3 \end{bmatrix}$$

7.1.5 Tensor Transformation Rule

Consider now the tensor definition 7.1.23 expressed in two different coordinate systems:

$$\begin{aligned} u_i &= T_{ij}v_j \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad \text{in } \{x_i\} \\ u'_i &= T'_{ij}v'_j \qquad [\mathbf{u}'] = [\mathbf{T}'][\mathbf{v}'] \qquad \text{in } \{x'_i\} \end{aligned} \tag{7.1.27}$$

From the vector transformation rule 7.1.20,

$$\begin{aligned} u_i' &= Q_{ji}u_j & [\mathbf{u}'] &= [\mathbf{Q}^\mathrm{T}][\mathbf{u}] \\ v_i' &= Q_{ji}v_j & [\mathbf{v}'] &= [\mathbf{Q}^\mathrm{T}][\mathbf{v}] \end{aligned} \tag{7.1.28}$$

Combining 7.1.27-28,

$$Q_{ji}u_j = T_{ij}'Q_{kj}v_k \qquad [\mathbf{Q}^\mathrm{T}][\mathbf{u}] = [\mathbf{T}'][\mathbf{Q}^\mathrm{T}][\mathbf{v}] \tag{7.1.29}$$

and so

$$Q_{mi}Q_{ji}u_j = Q_{mi}T_{ij}'Q_{kj}v_k \qquad [\mathbf{u}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^\mathrm{T}][\mathbf{v}] \tag{7.1.30}$$

(Note that $Q_{mi}Q_{ji}u_j = \delta_{mj}u_j = u_m$.) Comparing with 7.1.24, it follows that

$$\boxed{\begin{aligned} T_{ij} &= Q_{ip}Q_{jq}T_{pq}' & \ldots\quad [\mathbf{T}] &= [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^\mathrm{T}] \\ T_{ij}' &= Q_{pi}Q_{qj}T_{pq} & \ldots\quad [\mathbf{T}'] &= [\mathbf{Q}^\mathrm{T}][\mathbf{T}][\mathbf{Q}] \end{aligned}} \quad \textbf{Tensor Transformation Rule} \tag{7.1.31}$$

7.1.6 Problems

1. Write the following in index notation: $|\mathbf{v}|$, $\mathbf{v}\cdot\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{e}_k$.
2. Show that $\delta_{ij}a_ib_j$ is equivalent to $\mathbf{a}\cdot\mathbf{b}$.
3. Evaluate or simplify the following expressions:
   (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$
4. Show that $[\mathbf{u}^\mathrm{T}][\mathbf{Q}]$ is a 1×3 matrix with elements u_jQ_{ji} (write the matrices out in full)
5. Show that $([\mathbf{Q}][\mathbf{u}])^\mathrm{T} = [\mathbf{u}^\mathrm{T}][\mathbf{Q}^\mathrm{T}]$
6. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^\mathrm{T}][\mathbf{Q}]$?
7. What is the index notation for $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$?
8. Write out the 3×3 matrices $[\mathbf{A}]$ and $[\mathbf{B}]$ in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik}B_{kj}$ for $i=2,\ j=1$.
9. What is the index notation for
   (a) $[\mathbf{A}][\mathbf{B}^\mathrm{T}]$
   (b) $[\mathbf{v}^\mathrm{T}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^\mathrm{T}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^\mathrm{T}]([\mathbf{A}][\mathbf{v}])$)
   (c) $[\mathbf{B}^\mathrm{T}][\mathbf{A}][\mathbf{B}]$
10. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.

11. Consider a two-dimensional problem. If the components of a vector **u** in one coordinate system are

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

what are they in a second coordinate system, obtained from the first by a positive rotation of 30°? Sketch the two coordinate systems and the vector to see if your answer makes sense.

12. Consider again a two-dimensional problem with the same change in coordinates as in Problem 11. The components of a 2D tensor in the first system are

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

What are they in the second coordinate system?

7.2 Analysis of Three Dimensional Stress and Strain

The concept of traction and stress was introduced and discussed in Book I, §3. For the most part, the discussion was confined to two-dimensional states of stress. Here, the fully three dimensional stress state is examined. There will be some repetition of the earlier analyses.

7.2.1 The Traction Vector and Stress Components

Consider a traction vector **t** acting on a surface element, Fig. 7.2.1. Introduce a Cartesian coordinate system with base vectors $\mathbf{e}_i$ so that one of the base vectors is a normal to the surface and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 7.1.1, the $\mathbf{e}_3$ direction is taken to be normal to the plane, and a superscript on **t** denotes this normal:

$$\mathbf{t}^{(\mathbf{e}_3)} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3 \tag{7.2.1}$$

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the direction of the normal and the second denotes the direction of the component to the plane. Thus the three components of the traction vector shown in Fig. 7.2.1 are $\sigma_{31}, \sigma_{32}, \sigma_{33}$:

$$\mathbf{t}^{(\mathbf{e}_3)} = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \tag{7.2.2}$$

The first two stresses, the components acting tangential to the surface, are shear stresses whereas σ_{33}, acting normal to the plane, is a normal stress.

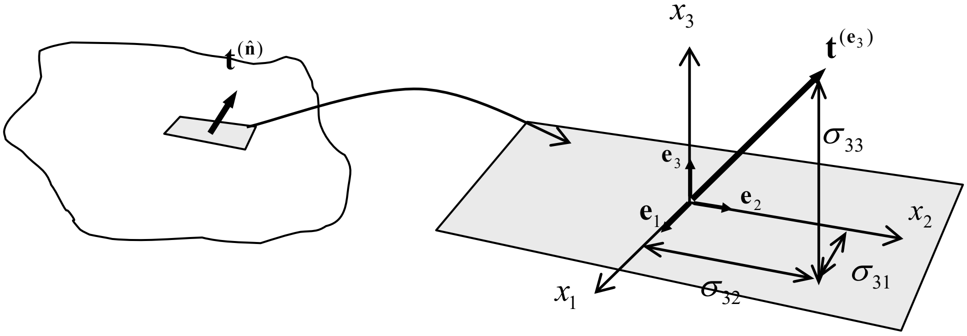


Figure 7.2.1: components of the traction vector

Consider the three traction vectors $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}, \mathbf{t}^{(\mathbf{e}_3)}$ acting on the surface elements whose outward normals are aligned with the three base vectors $\mathbf{e}_j$, Fig. 7.2.2a. The three (or six) surfaces can be amalgamated into one diagram as in Fig. 7.2.2b.

In terms of stresses, the traction vectors are

$$\begin{aligned}\mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \qquad \text{or} \qquad \mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3\end{aligned} \tag{7.2.3}$$

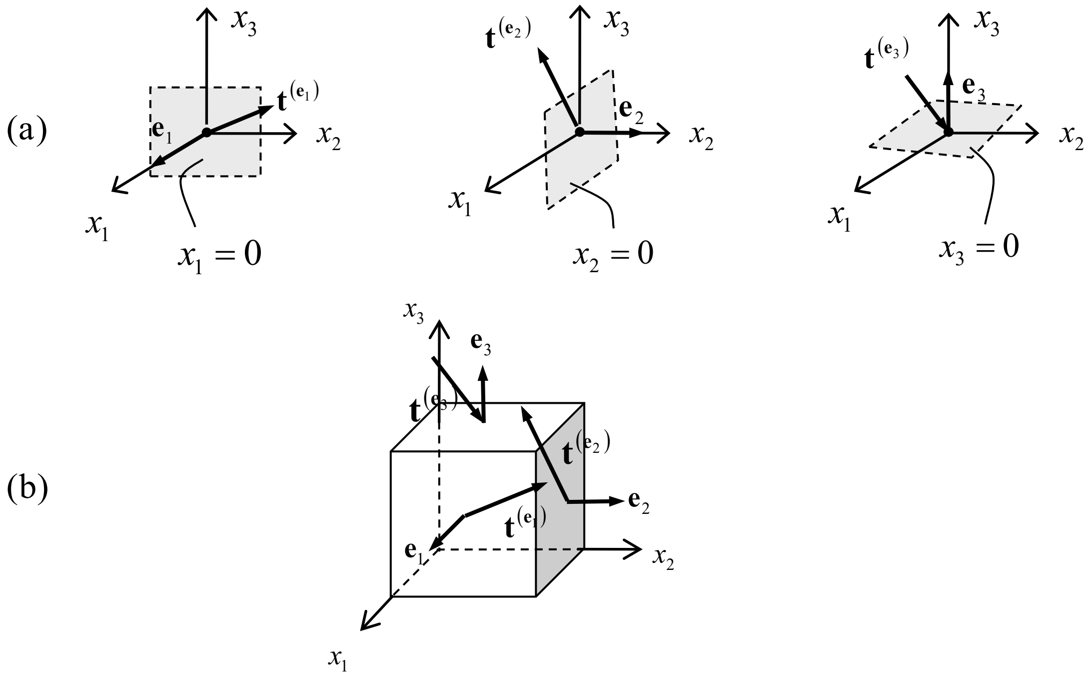


Figure 7.2.2: the three traction vectors acting at a point; (a) on mutually orthogonal planes, (b) the traction vectors illustrated on a box element

The components of the three traction vectors, i.e. the stress components, can now be displayed on a box element as in Fig. 7.2.3. Note that the stress components will vary slightly over the surfaces of an elemental box of finite size. However, it is assumed that the element in Fig. 7.2.3 is small enough that the stresses can be treated as constant, so that they are the stresses acting *at* the origin.

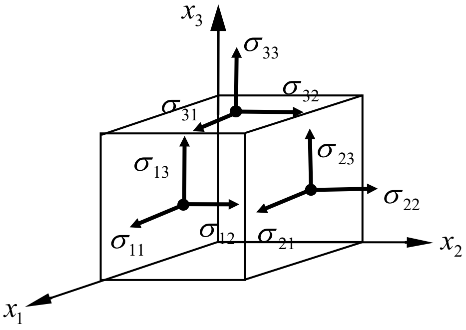


Figure 7.2.3: the nine stress components with respect to a Cartesian coordinate system

The nine stresses can be conveniently displayed in 3×3 matrix form:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{7.2.4}$$

It is important to realise that, if one were to take an element at some different orientation to the element in Fig. 7.2.3, but at the *same material particle*, for example aligned with the axes x_1', x_2', x_3' shown in Fig. 7.2.4, one would then have different tractions acting and the nine stresses would be different also. The stresses acting in this new orientation can be represented by a new matrix:

$$[\sigma_{ij}'] = \begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} \tag{7.2.5}$$

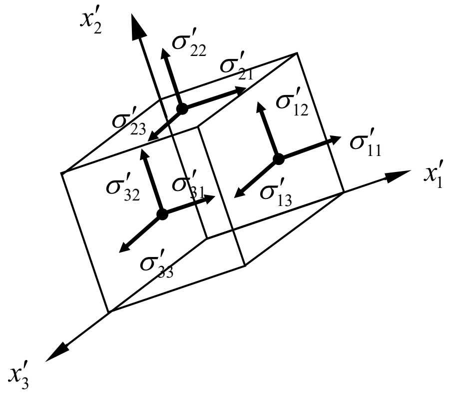


Figure 7.2.4: the stress components with respect to a Cartesian coordinate system different to that in Fig. 7.2.3

7.2.2 Cauchy's Law

Cauchy's Law, which will be proved below, states that the normal to a surface, $\mathbf{n} = n_i \mathbf{e}_i$, is related to the traction vector $\mathbf{t}^{(\mathbf{n})} = t_i \mathbf{e}_i$ acting on that surface, according to

$$t_i = \sigma_{ji} n_j \tag{7.2.6}$$

Writing the traction and normal in vector form and the stress in 3×3 matrix form,

$$[t_i^{(\mathbf{n})}] = \begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix}, \quad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad [n_i] = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.7}$$

and Cauchy's law in matrix notation reads

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.8}$$

Note that it is the transpose stress matrix which is used in Cauchy's law. Since the stress matrix is symmetric, one can express Cauchy's law in the form

$$\boxed{t_i = \sigma_{ij} n_j} \qquad \textbf{Cauchy's Law} \tag{7.2.9}$$

Cauchy's law is illustrated in Fig. 7.2.5; in this figure, positive stresses σ_{ij} are shown.

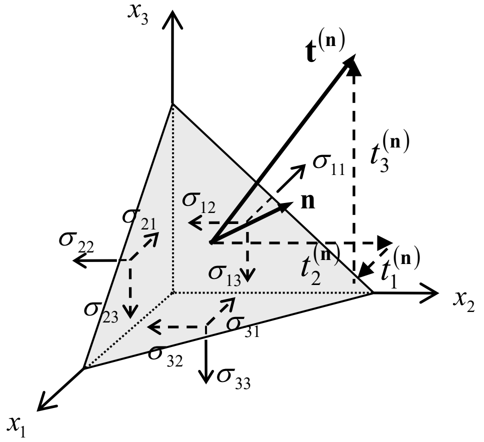


Figure 7.2.5: Cauchy's Law; given the stresses and the normal to a plane, the traction vector acting on the plane can be determined

Normal and Shear Stress

It is useful to be able to evaluate the normal stress σ_N and shear stress σ_S acting on any plane, Fig. 7.2.6. For this purpose, note that the stress acting normal to a plane is the projection of $\mathbf{t}^{(\mathbf{n})}$ in the direction of $\mathbf{n}$,

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(\mathbf{n})} \tag{7.2.10}$$

The magnitude of the shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \tag{7.2.11}$$

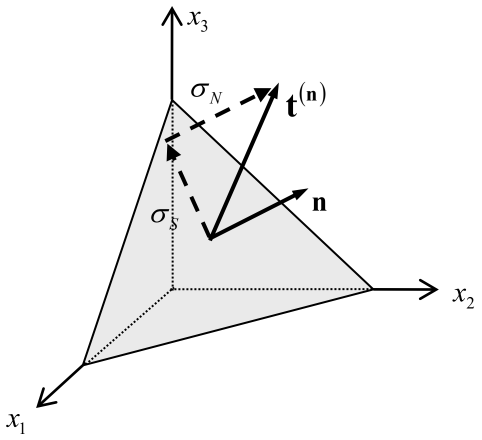


Figure 7.2.6: the normal and shear stress acting on an arbitrary plane through a point

Example

The state of stress at a point with respect to a Cartesian coordinates system $0x_1x_2x_3$ is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine:

(a) the traction vector acting on a plane through the point whose unit normal is $\mathbf{n} = (1/3)\mathbf{e}_1 + (2/3)\mathbf{e}_2 - (2/3)\mathbf{e}_3$

(b) the component of this traction acting perpendicular to the plane

(c) the shear component of traction on the plane

Solution

(a) From Cauchy's law,

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (-2/3)\mathbf{e}_1 + 3\mathbf{e}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \left\{\left[\left(-\tfrac{2}{3}\right)^2 + (3)^2 + (-1)^2\right] - \left[\left(\tfrac{22}{9}\right)^2\right]\right\}^{1/2} \approx 2.1$$

■

Proof of Cauchy’s Law

Cauchy’s law can be proved using force equilibrium of material elements. First, consider a tetrahedral free-body, with vertex at the origin, Fig. 7.2.7. It is required to determine the traction **t** in terms of the nine stress components (which are all shown positive in the diagram).

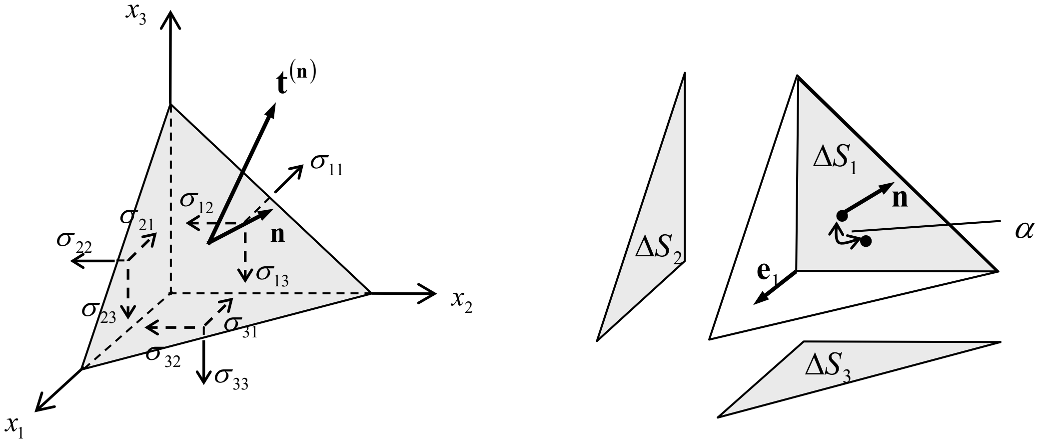


Figure 7.2.7: proof of Cauchy’s Law

The components of the unit normal, n_i, are the direction cosines of the normal vector, i.e. the cosines of the angles between the normal and each of the coordinate directions:

$$\cos(\mathbf{n}, \mathbf{e}_i) = \mathbf{n} \cdot \mathbf{e}_i = n_i \tag{7.2.12}$$

Let the area of the base of the tetrahedron, with normal **n**, be ΔS. The area ΔS_1 is then $\Delta S \cos\alpha$, where α is the angle between the planes, as shown to the right of Fig. 7.2.7; this angle is the same as that between the vectors **n** and $\mathbf{e}_1$, so $\Delta S_1 = n_1 \Delta S$, and similarly for the other surfaces:

$$\Delta S_i = n_i \Delta S \tag{7.2.13}$$

The resultant surface force on the body, acting in the x_i direction, is then

$$\sum F_i = t_i \Delta S - \sigma_{ji} \Delta S_j = t_i \Delta S - \sigma_{ji} n_j \Delta S \tag{7.2.14}$$

For equilibrium, this expression must be zero, and one arrives at Cauchy’s law.

Note:
As proved in Book III, this result holds also in the general case of accelerating material elements in the presences of body forces.

7.2.3 The Stress Tensor

Cauchy's law 7.2.9 is of the same form as 7.1.24 and so by definition the stress is a tensor. Denote the stress tensor in symbolic notation by $\boldsymbol{\sigma}$. Cauchy's law in symbolic form then reads

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \qquad (7.2.15)$$

Further, the transformation rule for stress follows the general tensor transformation rule 7.1.31:

$$\boxed{\begin{array}{ll} \sigma_{ij} = Q_{ip}Q_{jq}\sigma'_{pq} & \ldots\quad [\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}] \\ \sigma'_{ij} = Q_{pi}Q_{qj}\sigma_{pq} & \ldots\quad [\boldsymbol{\sigma}'] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}][\mathbf{Q}] \end{array}} \quad \textbf{Stress Transformation Rule} \qquad (7.2.16)$$

As with the normal and traction vectors, the components and hence matrix representation of the stress changes with coordinate system, as with the two different matrix representations 7.2.4 and 7.2.5. However, there is only one stress tensor $\boldsymbol{\sigma}$ at a point. Another way of looking at this is to note that an infinite number of planes pass through a point, and on each of these planes acts a traction vector, and each of these traction vectors has three (stress) components. *All* of these traction vectors taken together define the complete **state of stress** at a point.

Example

The state of stress at a point with respect to an $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) What are the stress components with respect to axes $0x'_1x'_2x'_3$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_2 axis, Fig. 7.2.8?

(b) Use Cauchy's law to evaluate the normal and shear stress on a plane with normal $\mathbf{n} = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_3$ and relate your result with that from (a)

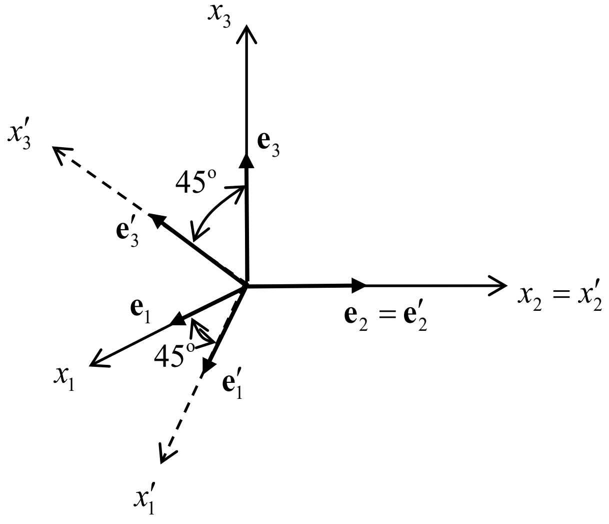


Figure 7.2.8: two different coordinate systems at a point

Solution

(a) The transformation matrix is

$$[Q_{ij}] = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ as expected. The rotated stress components are therefore

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{\sqrt{2}} & 3 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{3}{2} \end{bmatrix}$$

and the new stress matrix is symmetric as expected.

(b) From Cauchy's law, the traction vector is

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (\sqrt{2})\mathbf{e}_1 - (1/\sqrt{2})\mathbf{e}_2 + (1/\sqrt{2})\mathbf{e}_3$. The normal and shear stress on the plane are

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = 3/2$$

and

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \sqrt{3 - (3/2)^2} = \sqrt{3}/2$$

The normal to the plane is equal to $\mathbf{e}_3'$ and so σ_N should be the same as σ_{33}' and it is. The stress σ_S should be equal to $\sqrt{(\sigma_{31}')^2 + (\sigma_{32}')^2}$ and it is. The results are displayed in Fig. 7.2.9, in which the traction is represented in different ways, with components $\left(t_1^{(\mathbf{n})}, t_2^{(\mathbf{n})}, t_3^{(\mathbf{n})}\right)$ and $\left(\sigma'_{31}, \sigma'_{32}, \sigma'_{33}\right)$.

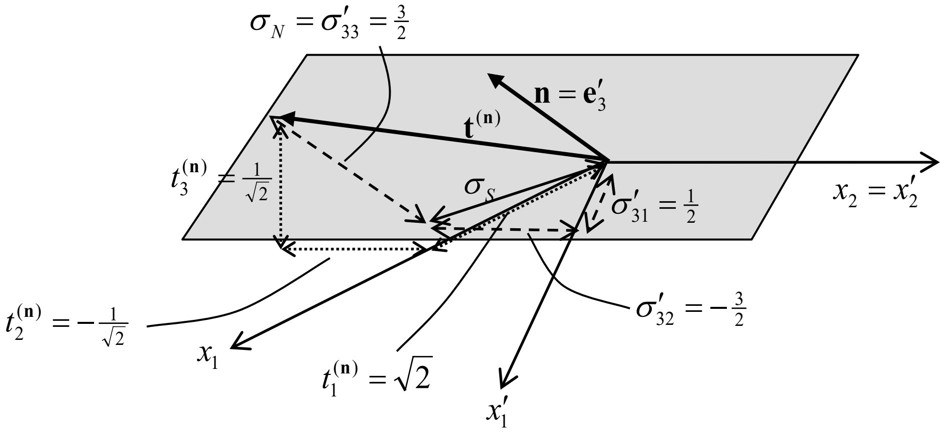


Figure 7.2.9: traction and stresses acting on a plane

Isotropic State of Stress

Suppose the state of stress in a body is

$$\sigma_{ij} = \sigma_0 \delta_{ij} \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \tag{7.2.17}$$

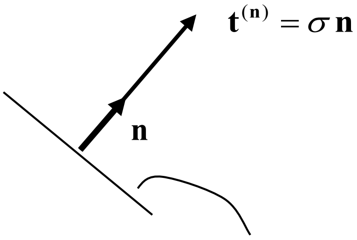
One finds that the application of the stress tensor transformation rule yields the very same components no matter what the new coordinate system {▲Problem 3}. In other words, no shear stresses act, no matter what the orientation of the plane through the point. This is termed an **isotropic state of stress**, or a **spherical state of stress**. One example of isotropic stress is the stress arising in a fluid at rest, which cannot support shear stress, in which case

$$[\boldsymbol{\sigma}] = -p[\mathbf{I}] \tag{7.2.18}$$

where the scalar p is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic state of stress**.

7.2.4 Principal Stresses

For certain planes through a material particle, there are traction vectors which act normal to the plane, as in Fig. 7.2.10. In this case the traction can be expressed as a scalar multiple of the normal vector, $\mathbf{t}^{(\mathbf{n})} = \sigma \mathbf{n}$.



no shear stress – only a normal component to the traction

Figure 7.2.10: a purely normal traction vector

From Cauchy's law then, for these planes,

$$\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n}, \quad \sigma_{ij}n_j = \sigma n_i, \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad (7.2.19)$$

This is a standard **eigenvalue problem** from Linear Algebra: given a matrix $\lfloor \sigma_{ij} \rfloor$, find the **eigenvalues** σ and associated **eigenvectors n** such that Eqn. 7.2.19 holds.
To solve the problem, first re-write the equation in the form

$$(\boldsymbol{\sigma} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}, \quad (\sigma_{ij} - \sigma\delta_{ij})n_j = 0, \quad \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7.2.20)$$

or

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7.2.21)$$

This is a set of three homogeneous equations in three unknowns (if one treats σ as known). From basic linear algebra, this system has a solution (apart from $n_i = 0$) if and only if the determinant of the coefficient matrix is zero, i.e. if

$$\det(\boldsymbol{\sigma} - \sigma\mathbf{I}) = \det \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} = 0 \qquad (7.2.22)$$

Evaluating the determinant, one has the following cubic **characteristic equation** of the stress tensor $\boldsymbol{\sigma}$,

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \qquad \textbf{Characteristic Equation} \qquad (7.2.23)$$

and the **principal scalar invariants** of the stress tensor are

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{31}
\end{aligned}
\qquad (7.2.24)
$$

(I_3 is the determinant of the stress matrix.) The characteristic equation 7.2.23 can now be solved for the eigenvalues σ and then Eqn. 7.2.21 can be used to solve for the eigenvectors **n**.

Now another theorem of linear algebra states that the eigenvalues of a real (that is, the components are real), symmetric matrix (such as the stress matrix) are all real and further that the associated eigenvectors are mutually orthogonal. This means that the three roots of the characteristic equation are real and that the three associated eigenvectors form a mutually orthogonal system. This is illustrated in Fig. 7.2.11; the eigenvalues are called **principal stresses** and are labelled $\sigma_1, \sigma_2, \sigma_3$ and the three corresponding eigenvectors are called **principal directions**, the directions in which the principal stresses act. The planes on which the principal stresses act (to which the principal directions are normal) are called the **principal planes**.

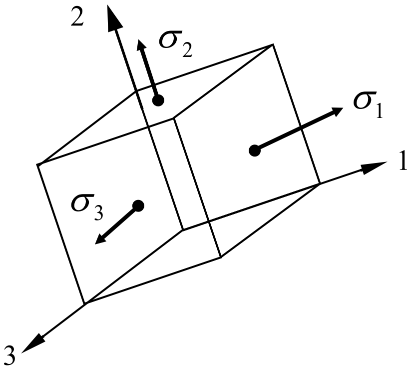


Figure 7.2.11: the three principal stresses acting at a point and the three associated principal directions 1, 2 and 3

Once the principal stresses are found, as mentioned, the principal directions can be found by solving Eqn. 7.2.21, which can be expressed as

$$
\begin{aligned}
(\sigma_{11} - \sigma)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 &= 0 \\
\sigma_{21}n_1 + (\sigma_{22} - \sigma)n_2 + \sigma_{23}n_3 &= 0 \\
\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \sigma)n_3 &= 0
\end{aligned}
\qquad (7.2.25)
$$

Each principal stress value in this equation gives rise to the three components of the associated principal direction vector, n_1, n_2, n_3. The solution also requires that the magnitude of the normal be specified: for a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. The directions of the normals are also chosen so that they form a right-handed set.

Example

The stress at a point is given with respect to the axes $Ox_1x_2x_3$ by the values

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}.$$

Determine (a) the principal values, (b) the principal directions (and sketch them).

Solution:

(a)

The principal values are the solution to the characteristic equation

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & -6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = (-10+\sigma)(5-\sigma)(15+\sigma) = 0$$

which yields the three principal values $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$.

(b)

The eigenvectors are now obtained from Eqn. 7.2.25. First, for $\sigma_1 = 10$,

$$\begin{aligned} -5n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 16n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 9n_3 &= 0 \end{aligned}$$

and using also the equation $n_1^2 + n_2^2 + n_3^2 = 1$ leads to $\mathbf{n}_1 = -(3/5)\mathbf{e}_2 + (4/5)\mathbf{e}_3$. Similarly, for $\sigma_2 = 5$ and $\sigma_3 = -15$, one has, respectively,

$$\begin{aligned} 0n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 11n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 4n_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 20n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 + 9n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 + 16n_3 &= 0 \end{aligned}$$

which yield $\mathbf{n}_2 = \mathbf{e}_1$ and $\mathbf{n}_3 = (4/5)\mathbf{e}_2 + (3/5)\mathbf{e}_3$. The principal directions are sketched in Fig. 7.2.12. Note that the three components of each principal direction, n_1, n_2, n_3, are the direction cosines: the cosines of the angles between that principal direction and the three coordinate axes. For example, for σ_1 with $n_1 = 0, n_2 = -3/5, n_3 = 4/5$, the angles made with the coordinate axes x_1, x_2, x_3 are, respectively, 90, 126.87° and 36.87°.

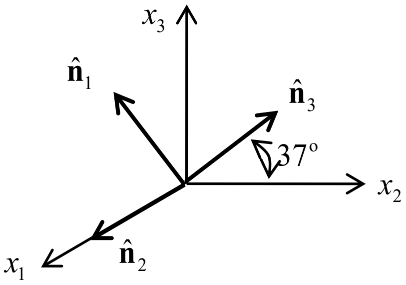


Figure 7.2.12: principal directions

■

Invariants

The principal stresses σ_1, σ_2, σ_3 are independent of any coordinate system; the $0x_1x_2x_3$ axes to which the stress matrix in Eqn. 7.2.19 is referred can have any orientation – the same principal stresses will be found from the eigenvalue analysis. This is expressed by using the symbolic notation for the problem: $\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n}$, which is independent of any coordinate system. Thus the principal stresses are intrinsic properties of the stress state at a point. It follows that the functions I_1, I_2, I_3 in the characteristic equation Eqn. 7.2.23 are also independent of any coordinate system, and hence the name principal scalar invariants (or simply **invariants**) of the stress.

The stress invariants can also be written neatly in terms of the principal stresses:

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{7.2.26}$$

Also, if one chooses a coordinate system to coincide with the principal directions, Fig. 7.2.12, the stress matrix takes the simple form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{7.2.27}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions – the stress matrix has the form 7.2.27 no matter what orientation the planes through the point.

Example

The two stress matrices from the Example of §7.2.3, describing the stress state at a point with respect to different coordinate systems, are

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad [\sigma'_{ij}] = \begin{bmatrix} 3/2 & 3/\sqrt{2} & 1/2 \\ 3/\sqrt{2} & 3 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 3/2 \end{bmatrix}$$

The first invariant is the sum of the normal stresses, the diagonal terms, and is the same for both as expected:

$$I_1 = 2 + 3 + 1 = \tfrac{3}{2} + 3 + \tfrac{3}{2} = 6$$

The other invariants can also be obtained from either matrix, and are

$$I_2 = 6, \; I_3 = -3$$

■

7.2.5 Maximum and Minimum Stress Values

Normal Stresses

The three principal stresses include the maximum and minimum normal stress components acting at a point. To prove this, first let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions*. Consider next an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. From Cauchy's law (see Fig. 7.2.13 – the stress matrix in Cauchy's law is now with respect to the principal directions 1, 2 and 3), the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \sigma_N = \sigma_{ij} n_j n_i \tag{7.2.28}$$

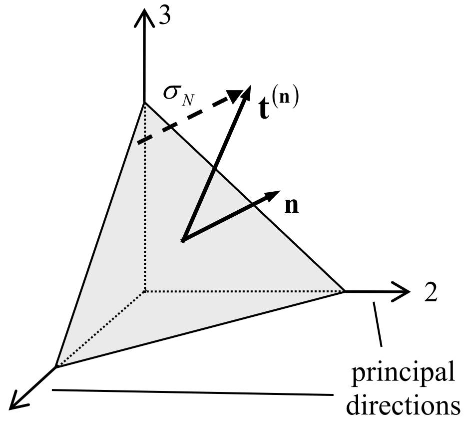


Figure 7.2.13: normal stress acting on a plane defined by the unit normal n

Thus

$$\sigma_N = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.29}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{7.2.30}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_3 \tag{7.2.31}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses. First, again, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors in the principal directions and consider an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. The normal stress is given by Eqn. 7.2.29,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.32}$$

Cauchy's law gives the components of the traction vector as

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{bmatrix} \tag{7.2.33}$$

and so the shear stress on the plane is, from Eqn. 7.2.11,

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{7.2.34}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right)n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right)n_2^2 + \sigma_3^2 - \left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2 + \sigma_3\right]^2 \tag{7.2.35}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned} \frac{\partial\left(\sigma_S^2\right)}{\partial n_1} &= n_1(\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2\right]\right\} = 0 \\ \frac{\partial\left(\sigma_S^2\right)}{\partial n_2} &= n_2(\sigma_2 - \sigma_3)\left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2\right]\right\} = 0 \end{aligned} \tag{7.2.36}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 7.2.36 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|
\end{aligned} \tag{7.2.37}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) \tag{7.2.38}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 7.2.14.

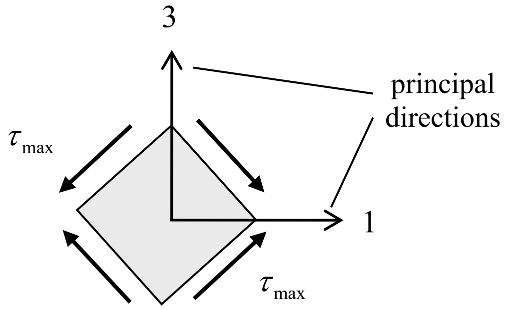


Figure 7.2.14: maximum shear stress at a point

Example

Consider the stress state examined in the Example of §7.2.4:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

The principal stresses were found to be $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}\left(\sigma_1 - \sigma_3\right) = \frac{25}{2}$$

One of the planes upon which they act is shown in Fig. 7.2.15 (see Fig. 7.2.12)

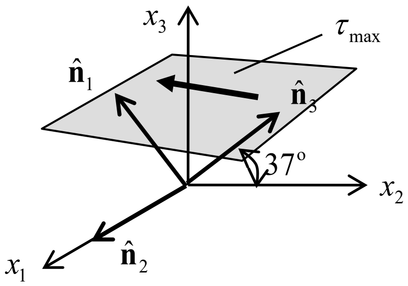


Figure 7.2.15: maximum shear stress

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7.2.6 Mohr's Circles of Stress

The Mohr's circle for 2D stress states was discussed in Book I, §3.5.5. For the 3D case, following on from section 7.2.5, one has the conditions

$$\begin{aligned} \sigma_N &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\ \sigma_S^2 + \sigma_N^2 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned} \tag{7.2.39}$$

Solving these equations gives

$$\begin{aligned} n_1^2 &= \frac{(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\ n_2^2 &= \frac{(\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\ n_3^2 &= \frac{(\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \end{aligned} \tag{7.2.40}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and noting that the squares of the normal components must be positive, one has that

$$\begin{aligned} (\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2 \geq 0 \\ (\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2 \leq 0 \\ (\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2 \geq 0 \end{aligned} \tag{7.2.41}$$

and these can be re-written as

$$\begin{aligned} \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_2 + \sigma_3)\right]^2 \geq \left[\tfrac{1}{2}(\sigma_2 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right]^2 \leq \left[\tfrac{1}{2}(\sigma_1 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right]^2 \geq \left[\tfrac{1}{2}(\sigma_1 - \sigma_2)\right]^2 \end{aligned} \tag{7.2.42}$$

If one takes coordinates (σ_N, σ_S), the equality signs here represent circles in (σ_N, σ_S) stress space, Fig. 7.2.16. Each point (σ_N, σ_S) in this stress space represents the stress on a particular plane through the material particle in question. Admissible (σ_N, σ_S) pairs are given by the conditions Eqns. 7.2.42; they must lie inside a circle of centre $\left(\frac{1}{2}(\sigma_1 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_3)$. This is the large circle in Fig. 7.2.16. The points must lie outside the circle with centre $\left(\frac{1}{2}(\sigma_2 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_2 - \sigma_3)$ and also outside the circle with centre $\left(\frac{1}{2}(\sigma_1 + \sigma_2), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_2)$; these are the two smaller circles in the figure. Thus the admissible points in stress space lie in the shaded region of Fig. 7.2.16.

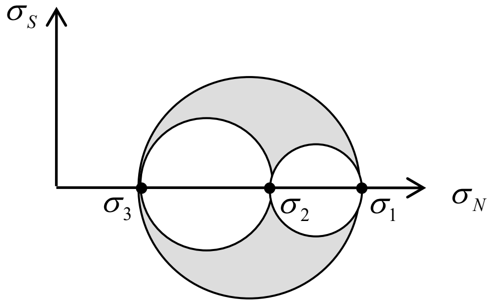

Figure 7.2.16: admissible points in stress space

7.2.7 Three Dimensional Strain

The strain ε_{ij}, in symbolic form $\boldsymbol{\varepsilon}$, is a tensor and as such it follows the same rules as for the stress tensor. In particular, it follows the general tensor transformation rule 7.2.16; it has principal values ε which satisfy the characteristic equation 7.2.23 and these include the maximum and minimum normal strain at a point. There are three principal strain invariants given by 7.2.24 or 7.2.26 and the maximum shear strain occurs on planes oriented at 45° to the principal directions.

7.2.8 Problems

1. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -2 \end{bmatrix}$$

Use Cauchy's law to determine the traction vector acting on a plane trough this point whose unit normal is $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. What is the normal stress acting on the plane? What is the shear stress acting on the plane?

2. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

What are the stress components with respect to axes $0x_1'x_2'x_3'$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_3 axis

3. Show, in both the index and matrix notation, that the components of an isotropic stress state remain unchanged under a coordinate transformation.

4. Consider a two-dimensional problem. The stress transformation formulae are then, in full,

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' \\ \sigma_{21}' & \sigma_{22}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Multiply the right hand side out and use the fact that the stress tensor is symmetric ($\sigma_{12} = \sigma_{21}$ - not true for all tensors). What do you get? Look familiar?

5. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 5/2 & -1/2 & 0 \\ -1/2 & 5/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evaluate the principal stresses and the principal directions. What is the maximum shear stress acting at the point?

7.3 Governing Equations of Three Dimensional Elasticity

7.3.1 Hooke's Law and Lamé's Constants

Linear elasticity was introduced in Book I, §6. The three-dimensional Hooke's law for isotropic linear elastic solids (Book I, Eqns. 6.1.9) can be expressed in index notation as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{7.3.1}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.3.2}$$

are the Lamé constants (μ is the Shear Modulus). Eqns. 7.3.1 can be inverted to obtain (see Book I, Eqns. 6.1.8) {▲Problem 1}

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} \tag{7.3.3}$$

7.3.2 Navier's Equations

The governing equations of elasticity are Hooke's law (Eqn. 7.3.1), the equations of motion, Eqn. 1.1.9 (see Eqns. 7.1.10-11),

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.3.4}$$

and the strain-displacement relations, Eqn. 1.2.19 (see Eqns. 7.1.25-26),

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.3.5}$$

Substituting 7.3.5 into 7.3.1 and then into 7.3.4 leads to the 3D Navier's equations {▲Problem 2}

$$\boxed{(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = \rho a_i} \qquad \textbf{Navier's Equations} \tag{7.3.6}$$

These reduce to the 2D plane strain Navier's equations, Eqns. 3.1.4, by setting $u_3 = 0$ and $\partial / \partial x_3 = 0$. They do not reduce to the plane stress equations since the latter are only an approximate solution to the equations of elasticity which are valid only in the limit as the thickness of the thin plate of plane stress tends to zero.

In terms of the dilatation and the rotation

The dilatation (unit volume change) is the scalar invariant

$$e = \varepsilon_{ii} = \mathrm{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_j}{\partial x_j} \tag{7.3.7}$$

The gradient of this scalar is then the vector

$$\text{grad div}\mathbf{u} \equiv \nabla(\nabla \cdot \mathbf{u}) \equiv \frac{\partial^2 u_j}{\partial x_i \partial x_j}. \tag{7.3.8}$$

The Lapacian of the displacement vector is defined to be the vector

$$\nabla^2 \mathbf{u} \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{7.3.9}$$

Navier's equations can thus be expressed in the vector notation

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{7.3.10}$$

or, in full, and in terms of the dilatation,

$$\begin{aligned}
(\lambda + \mu)\frac{\partial e}{\partial x_1} + \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + b_1 = \rho \frac{\partial u_1}{\partial t^2} \\
(\lambda + \mu)\frac{\partial e}{\partial x_2} + \mu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right) + b_2 = \rho \frac{\partial u_2}{\partial t^2} \\
(\lambda + \mu)\frac{\partial e}{\partial x_3} + \mu\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + b_1 = \rho \frac{\partial u_3}{\partial t^2}
\end{aligned} \tag{7.3.11}$$

Using the vector relation

$$\nabla^2 \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a}), \tag{7.3.12}$$

Naviers equations can also be expressed in the form

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{7.3.13}$$

Now the curl of the displacement is the rotation vector

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u} \tag{7.3.14}$$

In full, these are (see Eqns. 1.2.20):

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \quad \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \quad \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{7.3.15}$$

For example, as illustrated in Fig. 7.3.1, the third component is the (angle of) rotation of material about the x_3 axis.

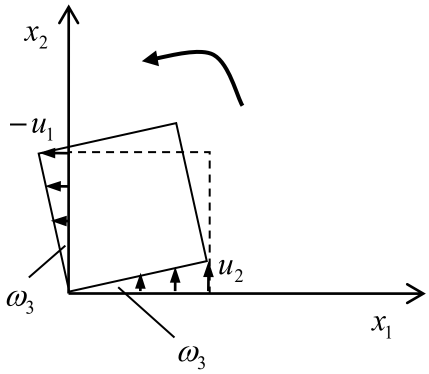

Figure 7.3.1: a rotation

Navier's equations can thus be expressed as

$$(\lambda + 2\mu)\nabla e - 2\mu\nabla \times \boldsymbol{\omega} + \mathbf{b} = \rho\ddot{\mathbf{u}} \tag{7.3.16}$$

7.3.3 Problems

1. Invert Eqns. 7.3.1 to get 7.3.3.

2. Derive the 3D Navier's equations from 7.3.6 from 7.3.1, 7.3.4 and 7.3.5

7.4 Elastodynamics

In this section, solutions to the Navier's equations are derived, which show that two different types of wave travel through elastic solids. The (gravitational) body force will be neglected, as it is a relatively small term in most practical applications of wave propagation through elastic solids.

7.4.1 Solutions to Navier's equations

Navier's equations are given by Eqn. 7.3.6 (and 7.3.10)

$$(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}=\rho a_i$$

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\ddot{\mathbf{u}} \qquad (7.4.1)$$

It was seen that Navier's equations could also be expressed in terms of the dilataion and rotation, Eqn. 7.3.16:

$$\begin{gathered}(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u})=\rho\ddot{\mathbf{u}}\\ (\lambda+2\mu)\nabla e-2\mu\nabla\times\boldsymbol{\omega}=\rho\ddot{\mathbf{u}}\end{gathered} \qquad (7.4.2)$$

Navier's equations can be reduced to the three-dimensional wave equation in one of two ways.

P Waves

First, taking the divergence of Eqn. 7.4.2 leads to

$$(\lambda+2\mu)\nabla\cdot\nabla e-\mu\nabla\cdot\left(\nabla\times(\nabla\times\mathbf{u})\right)=\rho\nabla\cdot\ddot{\mathbf{u}} \qquad (7.4.3)$$

But the divergence of the curl of any vector is zero:

$$\begin{aligned}\text{div curl}\mathbf{a}&=\text{div}\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3\\ a_1 & a_2 & a_3\end{vmatrix}\\ &=\frac{\partial}{\partial x_1}\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\right)=0\end{aligned} \qquad (7.4.4)$$

and so one has

$$(\lambda+2\mu)\nabla^2 e=\rho\ddot{e} \qquad (7.4.5)$$

In full, this is the three dimensional wave equation:

$$\frac{\partial^2 e}{\partial x_1^2}+\frac{\partial^2 e}{\partial x_2^2}+\frac{\partial^2 e}{\partial x_3^2}=\frac{1}{c_L^2}\frac{\partial^2 e}{\partial t^2} \tag{7.4.6}$$

where the wave speed is

$$c_L=\sqrt{\frac{\lambda+2\mu}{\rho}}=\sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{7.4.7}$$

This solution predicts that a wave propagates through an elastic solid at wave speed c_L, and the wave excites volume changes within the material it passes, as characterised by the dilatation *e*.

Such waves are called **P-waves** (for reasons which will be explained later).

S Waves

A second wave equation can be derived from Navier's equations; taking the curl of 7.4.2 leads to:

$$(\lambda+2\mu)\nabla\times(\nabla e)-2\mu\nabla\times(\nabla\times\boldsymbol{\omega})=\rho\nabla\times\ddot{\mathbf{u}} \tag{7.4.8}$$

Now the curl of the gradient of any scalar is zero:

$$\text{curl grad}e=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3\\ \partial e/\partial x_1 & \partial e/\partial x_2 & \partial e/\partial x_3\end{vmatrix}=0 \tag{7.4.9}$$

Further, using again the vector identity Eqn. 7.3.12,

$$-\nabla\times(\nabla\times\boldsymbol{\omega})=\nabla^2\boldsymbol{\omega}-\nabla(\nabla\cdot\boldsymbol{\omega}) \tag{7.4.10}$$

From Eqn. 7.4.4, the second term on the right is again zero, so one has the alternative wave equation:

$$\mu\nabla^2\boldsymbol{\omega}=\rho\ddot{\boldsymbol{\omega}} \tag{7.4.11}$$

which in full reads

$$\frac{\partial^2 \omega_i}{\partial x_1^2}+\frac{\partial^2 \omega_i}{\partial x_2^2}+\frac{\partial^2 \omega_i}{\partial x_3^2}=\frac{1}{c_T^2}\frac{\partial^2 \omega_i}{\partial t^2} \tag{7.4.12}$$

where the wave speed is

$$c_T = \sqrt{\frac{\mu}{\rho}} \tag{7.4.13}$$

This solution predicts that a wave propagates through an elastic solid at wave speed c_T, and the wave excites rotation within the material it passes, as characterised by the rotation vector $\boldsymbol{\omega}$.

Such waves are called **S-waves**.

7.4.2 Longitudinal and Transverse Waves

Consider now a sinusoidal wave of the general form

$$\begin{aligned}\mathbf{u} &= \mathbf{a}\sin\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\\ &= \mathbf{a}\sin\left[\mathbf{k}\cdot\left(\mathbf{x} - \mathbf{c}t\right)\right]\end{aligned} \tag{7.4.14}$$

This is a displacement with amplitude **a**, wave vector **k** and angular frequency ω. The wave vector indicates the direction of the propagation of the wave, through the wave velocity vector:

$$\mathbf{c} = \frac{\omega}{|\mathbf{k}|^2}\mathbf{k} \tag{7.4.15}$$

In substituting Eqn. 7.4.14 into Navier's equations 7.4.1, make the substitution

$$\begin{aligned}\xi &= k_m x_m - \omega t\\ \frac{\partial \xi}{\partial x_i} &= k_m \frac{\partial x_m}{\partial x_i} = k_m \delta_{mi} = k_i\\ \frac{\partial u_i}{\partial x_j} &= \frac{\partial}{\partial x_j} a_i \sin\xi = k_j a_i \cos\xi\\ \frac{\partial^2 u_i}{\partial x_j \partial x_j} &= \frac{\partial}{\partial x_j} k_j a_i \cos\xi = -k_j k_j a_i \sin\xi\\ \frac{\partial^2 u_i}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i} k_j a_i \cos\xi = -k_i k_j a_i \sin\xi\\ \frac{\partial^2 u_i}{\partial t^2} &= \frac{\partial}{\partial t^2} a_i \sin\xi = a_i \sin\xi\end{aligned} \tag{7.4.1}$$

and using the relations

$$
\begin{gathered}
(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}=\rho a_i \\
(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\ddot{\mathbf{u}}
\end{gathered}
\qquad (7.4.1)
$$

7.4.3 Description in terms of Potentials

The above analysis can be expressed in terms of vector potentials, as described next.

It can be shown that any vector **u** can be written in the form[1]

$$\mathbf{u}=\nabla\phi+\mathrm{curl}\,\mathbf{a} \qquad (7.4.17)$$

where ϕ is a **scalar potential** and **a** is a **vector potential**. These two terms in the displacement field can be examined separately. The most general displacement field can be obtained by adding both solutions together.

P Waves (Irrotational Waves)

First looking at the scalar potential term, suppose that the displacement is given by $\mathbf{u}=\nabla\phi$. Now, as in Eqn. 7.4.9, the curl of $\nabla\phi$ is zero, and so $\mathrm{curl}\,\mathbf{u}=\mathbf{0}$.Thus the rotation $\boldsymbol{\omega}=\mathbf{0}$. Thus there is *no rotation* of material particles.

Taking the displacement field $\mathbf{u}=\nabla\phi$, writing it in index notation, $u_j=\partial\phi/\partial x_j$, and substituting into Navier's equations, leads to the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k}=\frac{1}{c_L^2}\frac{\partial^2 u_i}{\partial t^2} \qquad (7.4.4)$$

This displacement field thus corresponds to stress waves travelling at speed c_L, causing material to strain but not to rotate. These **irrotational waves** are also called **waves of dilatation**.

Equivoluminal Waves

Consider now the displacement field $\mathbf{u}=\mathrm{curl}\,\mathbf{a}$. If one can find a vector **a** such that $\mathbf{u}=\mathrm{curl}\,\mathbf{a}$, then it follows that $e=\nabla\cdot\mathbf{u}=0$. Thus the condition that the displacement field be divergence-free implies that there is *no volume change*. There can be normal strains only so long as their sum is zero.

Taking $\varepsilon_{kk}=\partial u_k/\partial x_k$ and substituting into Navier's equations then leads immediately to

---

[1] from the **Helmholtz theory**

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{7.4.6}$$

or the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{1}{c_T^2} \frac{\partial^2 u_i}{\partial t^2}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{7.4.7}$$

This displacement field thus corresponds to stress waves travelling at speed c_T, causing material to shear. These equivoluminal waves are also called **shear waves** or **waves of distortion**.

In summary, when an event such as an explosion occurs, two different types of wave emerge, irrotational waves which result in irrotational displacement fields, and equivoluminal waves which result in equivoluminal displacements. These waves travel at different speeds.

7.4.4 Plane Waves

At a sufficient distance from any initial disturbance, a stress wave will travel in a plane. It can be assumed that all material particles will displace either parallel to the direction of wave propagation (**longitudinal waves**) or perpendicular to this direction (**transverse waves**).

Let the wave travel in the x_1 direction.

Irrotational (p / longitudinal) Plane Waves

Consider particles which displace in the direction of wave propagation according to $\mathbf{u} = u_1(x_1, t)\mathbf{e}_1$. This is an irrotational wave since $\text{curl}\mathbf{u} = \mathbf{0}$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_L^2} \frac{\partial^2 u_1}{\partial t^2} \tag{7.4.8}$$

These longitudinal plane waves are also called **p-waves**[2].

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[2] p stands for “primary”

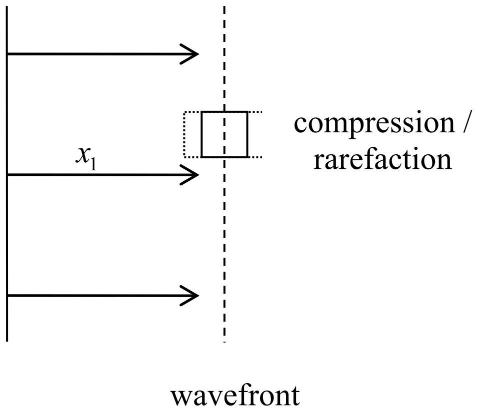

Figure 7.4.2: a longitudinal wave

Equivoluminal (s / transverse / shear) Plane Waves

Consider particles which displace according to $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$. This is an equivoluminal wave since $\nabla \cdot \mathbf{u} = 0$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{c_T^2}\frac{\partial^2 u_2}{\partial t^2} \tag{7.4.9}$$

These transverse/shear waves are also called **s-waves**[3].

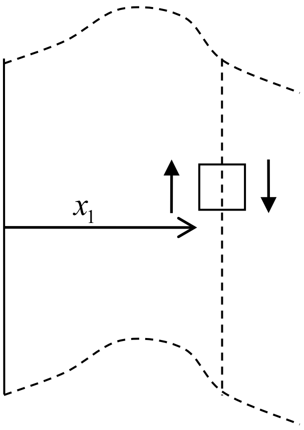

Figure 7.4.3: a transverse wave

7.4.5 Vibration Analysis

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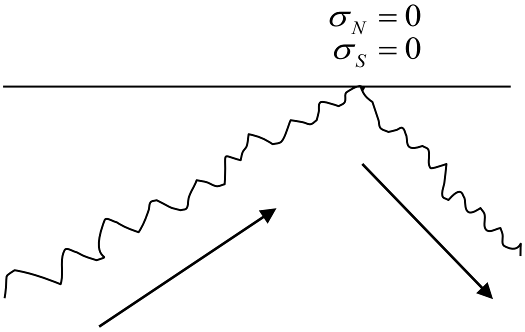
[3] s stands for "secondary"

A vibration analysis can be carried out in exactly the same way as in Chapter 2, only the wave speeds in the 1D wave equations 7.4.8 and 7.4.9 are now different from the 1D speed $\sqrt{E/\rho}$. The particular solutions, forced vibration and resonance theory of Chapter 2 can again be applied here. The analysis here is appropriate for thin plates "infinitely wide" in the x_2, x_3 directions, Fig. 7.4.4. The figure shows longitudinal vibration, but one can also have transverse vibration where the particles displace perpendicular to the x_1 axis.



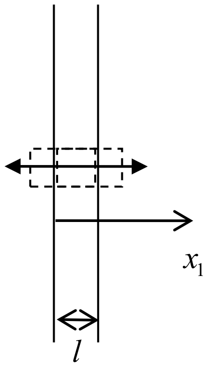


Figure 7.4.4: stretch vibration of a plate

7.4.6 Waves at Boundaries

Plane waves exist in unbounded elastic continua. In a finite body, a plane wave will be *reflected* when it hits a free surface. In this case, one needs to solve Navier's equations subject to the boundary conditions of zero normal and shear stress at the free surface. Waves of both types will in general be reflected for any single type of incident wave.

Similarly, when a wave meets an interface between two different materials, there will be reflection and refraction. The boundary conditions are that the displacements are continuous and the normal and shear stresses are continuous, Fig. 7.4.5

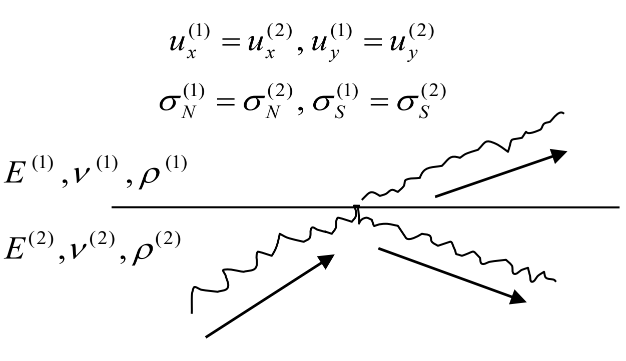


Figure 7.4.5: reflection and refraction of a wave at an interface

7.4.7 Waves at Boundaries

The waves discussed thus far are **body waves**. When a free surface exists, for example the surface of the earth, another type of wave motion is possible; these are the **Rayleigh waves** and travel along the surface very much like water waves. It can be shown that the speed of Rayleigh waves is between 90% and 95% of c_T, depending on the value of Poisson's ratio. Similar types of waves can propagate along the interface between two different materials.

7.4.8 Problems

1. Consider the motion

$$u_1 = \bar{u} \sin \frac{2\pi}{l}(x_1 - ct), u_2 = 0, u_3 = 0,$$

What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

2. Consider the motion

$$u_1 = 0, u_2 = \bar{u} \sin \frac{2\pi}{l}(x_1 - ct), u_3 = 0,$$

What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

3. Derive an expression for the ratio c_L / c_T in terms of the material's Poisson's ratio only. Which is the faster, the longitudinal or transverse wave?

4. Show that the motion

$$u_1 = 0, u_2 = 0, u_3 = \bar{u} \cos(px_2) \cos \frac{2\pi}{l}(x_1 - ct),$$

is equivoluminal.

5. Consider the motion

$$u_1 = \bar{u}[\sin \beta(x_3 - ct) + \alpha \sin \beta(x_3 + ct)], u_2 = 0, u_3 = 0$$

(i) what kind of elastic stress wave does this involve ? (Sketch the plane of the wave and its direction of propagation.)

(ii) what are the strains and stresses.

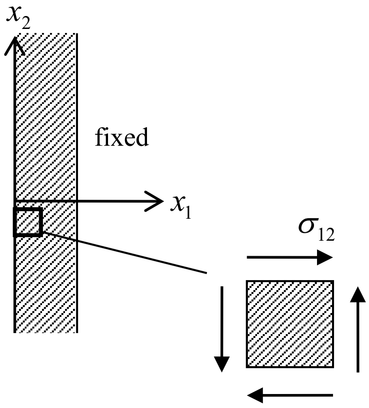
(iii) use the equations of motion to determine the wave speed. Is it what you expected?

(iv) Suppose that the plane $x_3 = 0$ is a free surface. Determine α.

(v) Suppose also that $x_3 = h$ is a free surface. Determine β.

6. Consider a plate with left face $\left(x_1 = 0\right)$ subjected to a forced displacement $\mathbf{u} = \alpha \sin \Omega t \mathbf{e}_1$ and the right face $\left(x_1 = l\right)$ free.
   (i) find the "thickness-stretch" vibration of the plate. What are the natural frequencies?
   (ii) When does resonance occur?

7. Consider a plate with left face $\left(x_1 = 0\right)$ subjected to a traction $\mathbf{t} = -\alpha \cos \Omega t \mathbf{e}_2$ and the right face $\left(x_1 = l\right)$ fixed, as shown in the figure below.
   (i) find the "thickness-shear" vibration of the plate. What are the natural frequencies?
   (ii) When does resonance occur?



[note: assume a displacement $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$; as with transverse waves, this will satisfy the 1-d wave equation with *c* being the transverse wave speed. Use the traction to obtain an expression for the shear stress σ_{12} over the left hand face. When applying the stress boundary condition, you will need the strain-displacement expression and stress-strain law,

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \varepsilon_{12} = \frac{1}{2\mu}\sigma_{12}$$

8. Consider the case of $\mathbf{u} = \alpha\left(\cos \Omega t \mathbf{e}_2 + \sin \Omega t \mathbf{e}_3\right)$ over the left face $\left(x_1 = 0\right)$ with the right face $\left(x_1 = l\right)$ fixed. Derive an expression for the particular solution and show that it represents circular motion of the particles in the $x_2 - x_3$ plane.
   [hint: evaluate the particular solutions for u_2 and u_3 separately and then show that $u_2^2 + u_3^2 = r^2$ for some *r* (independent of time)]