7.4 Elastodynamics

7.4.1 Propagation of Waves in Elastic Solids

When a stress wave travels through a material, it causes material particles to displace by $\mathbf{u}$. It can be shown that any vector $\mathbf{u}$ can be written in the form[1]

$$\mathbf{u} = \nabla\phi + \mathrm{curl}\,\mathbf{a} \tag{7.4.1}$$

where ϕ is a **scalar potential** and $\mathbf{a}$ is a vector. These two terms in the displacement field can be examined separately. The most general displacement field can be obtained by adding both solutions together.

Irrotational Waves

First looking at the scalar potential term, suppose that the displacement is given by $\mathbf{u} = \nabla\phi$. If one can find a scalar ϕ such that $\mathbf{u} = \nabla\phi$, then it follows that $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$, or

$$\begin{aligned} \mathrm{curl}\,\mathbf{u} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{e}_3 = \mathbf{0} \end{aligned} \tag{7.4.2}$$

Thus each of the terms inside the brackets is zero. But these terms represent rotations of material particles (see Eqns. 1.1.20). For example, as illustrated in Fig. 7.4.1,

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{7.4.3}$$

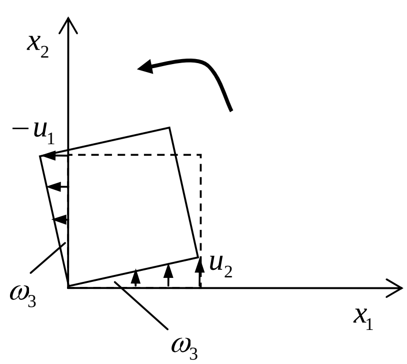


Figure 7.4.1: a rotation

[1] from the **Helmholtz theory**

Thus $\text{curl}\,\mathbf{u} = \mathbf{0}$ can be interpreted as *no rotation* of material particles. A small element of material can still undergo normal and shear strain, but the element will not rotate as a rigid body in space.

Taking the displacement field $\mathbf{u} = \nabla\phi$, writing it in index notation, $u_j = \partial\phi / \partial x_j$, and substituting into Navier's equations, leads to the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{1}{c_L^2}\frac{\partial^2 u_i}{\partial t^2}, \qquad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{7.4.4}$$

This displacement field thus corresponds to stress waves travelling at speed c_L, causing material to strain but not to rotate. These **irrotational waves** are also called **waves of dilatation**.

Equivoluminal Waves

Consider now the displacement field $\mathbf{u} = \text{curl}\,\mathbf{a}$. If one can find a vector $\mathbf{a}$ such that $\mathbf{u} = \text{curl}\,\mathbf{a}$, then it follows that $\nabla \cdot \mathbf{u} = 0$, or

$$\begin{aligned} \nabla \cdot \mathbf{u} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\ &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ &= \frac{\Delta V}{V} \end{aligned} \tag{7.4.5}$$

Thus the condition that the displacement field be divergence-free implies that there is *no volume change*. There can be normal strains only so long as their sum is zero.

Taking $\varepsilon_{kk} = \partial u_k / \partial x_k$ and substituting into Navier's equations then leads immediately to

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{7.4.6}$$

or the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{1}{c_T^2}\frac{\partial^2 u_i}{\partial t^2}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{7.4.7}$$

This displacement field thus corresponds to stress waves travelling at speed c_T, causing material to shear. These equivoluminal waves are also called **shear waves** or **waves of distortion**.

In summary, when an event such as an explosion occurs, two different types of wave emerge, irrotational waves which result in irrotational displacement fields, and equivoluminal waves which result in equivoluminal displacements. These waves travel at different speeds.

7.4.2 Plane Waves

At a sufficient distance from any initial disturbance, a stress wave will travel in a plane. It can be assumed that all material particles will displace either parallel to the direction of wave propagation (**longitudinal waves**) or perpendicular to this direction (**transverse waves**).

Let the wave travel in the x_1 direction.

Irrotational (p / longitudinal) Plane Waves

Consider particles which displace in the direction of wave propagation according to $\mathbf{u} = u_1(x_1, t)\mathbf{e}_1$. This is an irrotational wave since $\mathrm{curl}\mathbf{u} = \mathbf{0}$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_L^2}\frac{\partial^2 u_1}{\partial t^2} \qquad (7.4.8)$$

These longitudinal plane waves are also called **p-waves**[2].

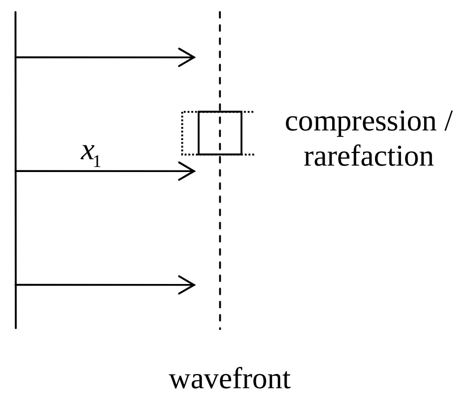


Figure 7.4.2: a longitudinal wave

Equivoluminal (s / transverse / shear) Plane Waves

Consider particles which displace according to $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$. This is an equivoluminal wave since $\nabla \cdot \mathbf{u} = 0$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{c_T^2}\frac{\partial^2 u_2}{\partial t^2} \qquad (7.4.9)$$

[2] p stands for "primary"

These transverse/shear waves are also called **s-waves**[3].

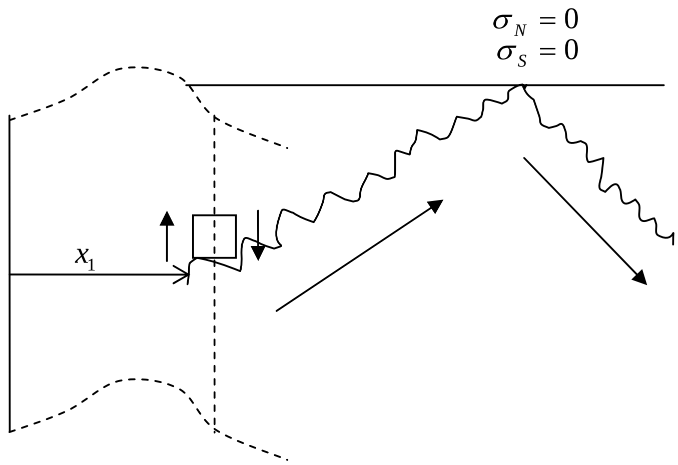


Figure 7.4.3: a transverse wave

7.4.3 Vibration Analysis

A vibration analysis can be carried out in exactly the same way as in Chapter 2, only the wave speeds in the 1D wave equations 7.4.8 and 7.4.9 are now different from the 1D speed $\sqrt{E/\rho}$. The particular solutions, forced vibration and resonance theory of Chapter 2 can again be applied here. The analysis here is appropriate for thin plates "infinitely wide" in the x_2, x_3 directions, Fig. 7.4.4. The figure shows longitudinal vibration, but one can also have transverse vibration where the particles displace perpendicular to the x_1 axis.

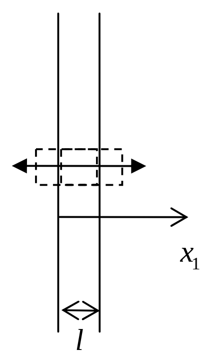


Figure 7.4.4: stretch vibration of a plate

7.4.4 Waves at Boundaries

Plane waves exist in unbounded elastic continua. In a finite body, a plane wave will be *reflected* when it hits a free surface. In this case, one needs to solve Navier's equations

[3] s stands for "secondary"

subject to the boundary conditions of zero normal and shear stress at the free surface. Waves of both types will in general be reflected for any single type of incident wave.

Similarly, when a wave meets an interface between two different materials, there will be reflection and refraction. The boundary conditions are that the displacements are continuous and the normal and shear stresses are continuous, Fig. 7.4.5

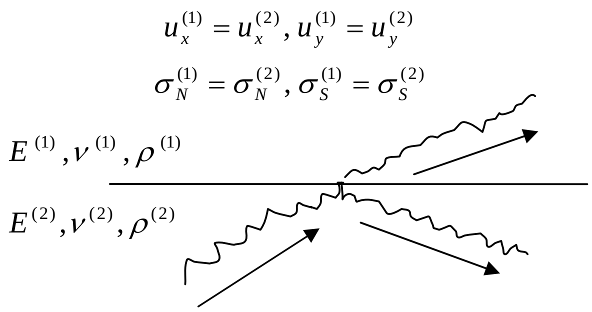


Figure 7.4.5: reflection and refraction of a wave at an interface

7.4.5 Waves at Boundaries

The waves discussed thus far are **body waves**. When a free surface exists, for example the surface of the earth, another type of wave motion is possible; these are the **Rayleigh waves** and travel along the surface very much like water waves. It can be shown that the speed of Rayleigh waves is between 90% and 95% of c_T, depending on the value of Poisson's ratio. Similar types of waves can propagate along the interface between two different materials.

7.4.6 Problems

1. Consider the motion

$$u_1 = \bar{u}\sin\frac{2\pi}{l}(x_1 - ct),\; u_2 = 0,\; u_3 = 0\,,$$

   What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

2. Consider the motion

$$u_1 = 0,\; u_2 = \bar{u}\sin\frac{2\pi}{l}(x_1 - ct),\; u_3 = 0\,,$$

   What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

3. Derive an expression for the ratio c_L / c_T in terms of the material's Poisson's ratio only. Which is the faster, the longitudinal or transverse wave?

4. Show that the motion

$$u_1 = 0,\ u_2 = 0,\ u_3 = \bar{u}\cos(px_2)\cos\frac{2\pi}{l}(x_1 - ct),$$

is equivoluminal.

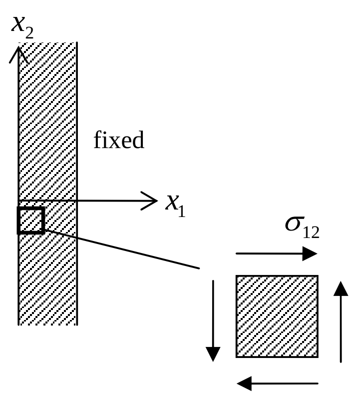
5. Consider the motion

$$u_1 = \bar{u}[\sin\beta(x_3 - ct) + \alpha\sin\beta(x_3 + ct)],\ u_2 = 0,\ u_3 = 0$$

(i) what kind of elastic stress wave does this involve ? (Sketch the plane of the wave and its direction of propagation.)
(ii) what are the strains and stresses.
(iii) use the equations of motion to determine the wave speed. Is it what you expected?
(iv) Suppose that the plane $x_3 = 0$ is a free surface. Determine α.
(v) Suppose also that $x_3 = h$ is a free surface. Determine β.

6. Consider a plate with left face $(x_1 = 0)$ subjected to a forced displacement $\mathbf{u} = \alpha\sin\Omega t\mathbf{e}_1$ and the right face $(x_1 = l)$ free.
(i) find the "thickness-stretch" vibration of the plate. What are the natural frequencies?
(ii) When does resonance occur?

7. Consider a plate with left face $(x_1 = 0)$ subjected to a traction $\mathbf{t} = -\alpha\cos\Omega t\mathbf{e}_2$ and the right face $(x_1 = l)$ fixed, as shown in the figure below.
(i) find the "thickness-shear" vibration of the plate. What are the natural frequencies?
(ii) When does resonance occur?



[note: assume a displacement $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$; as with transverse waves, this will satisfy the 1-d wave equation with c being the transverse wave speed. Use the traction to obtain an expression for the shear stress σ_{12} over the left hand face. When applying the stress boundary condition, you will need the strain-displacement expression and stress-strain law,

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \varepsilon_{12} = \frac{1}{2\mu}\sigma_{12}$$

8. Consider the case of $\mathbf{u} = \alpha\left(\cos\Omega t\mathbf{e}_2 + \sin\Omega t\mathbf{e}_3\right)$ over the left face $\left(x_1 = 0\right)$ with the right face $\left(x_1 = l\right)$ fixed. Derive an expression for the particular solution and show that it represents circular motion of the particles in the $x_2 - x_3$ plane.
[hint: evaluate the particular solutions for u_2 and u_3 separately and then show that $u_2^2 + u_3^2 = r^2$ for some *r* (independent of time)]