7.3 Governing Equations of Three Dimensional Elasticity

7.3.1 Hooke's Law and Lamé's Constants

Linear elasticity was introduced in Part I, §4.2. The three-dimensional Hooke's law for isotropic linear elastic solids (Part I, Eqns. 4.2.9) can be expressed in index notation as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{7.3.1}$$

where (see also Part I, Eqns. 6.2.21)

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.3.2}$$

are the Lamé constants (μ is the Shear Modulus). Eqns. 7.3.1 can be inverted to obtain {▲Problem 1}

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} \tag{7.3.3}$$

7.3.2 Navier's Equations

The governing equations of elasticity are Hooke's law (Eqn. 7.3.1), the equations of motion, Eqn. 1.1.9 (see Eqns. 7.1.10-11),

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.3.4}$$

and the strain-displacement relations, Eqn. 1.2.19 (see Eqns. 7.1.25-26),

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.3.5}$$

Substituting 7.3.5 into 7.3.1 and then into 7.3.4 leads to the 3D Navier's equations {▲Problem 2}

$$\boxed{(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = \rho a_i} \qquad \textbf{Navier's Equations} \tag{7.3.6}$$

These reduce to the 2D plane strain Navier's equations, Eqns. 3.1.4, by setting $u_3 = 0$ and $\partial / \partial x_3 = 0$. They do not reduce to the plane stress equations since the latter are only an approximate solution to the equations of elasticity which are valid only in the limit as the thickness of the thin plate of plane stress tends to zero.

7.3.3 Problems

1. Invert Eqns. 7.3.1 to get 7.3.3.

2. Derive the 3D Navier's equations from 7.3.6 from 7.3.1, 7.3.4 and 7.3.5